sciendo
Control and Cybernetics
vol. **52** (2023) No. 1
pages: 79-121
DOI: 10.2478/candc-2023-0033

Limits of stabilization of a networked hyperbolic system with a circle*

by

Martin Gugat[1], Xu Huang[2] and Zhiqiang Wang[3]

[1]Department Mathematik, Chair in Dynamics, Control, Numerics and Machine Learning (Alexander von Humboldt-Professorship), Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), Cauerstr. 11, 91058 Erlangen, Germany
martin.gugat@fau.de
[2]School of Mathematical Sciences, Fudan University, Shanghai 200433, China
xuxu.huang@fau.de
[3]School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China
wzq@fudan.edu.cn

Abstract: This paper is devoted to the discussion of the exponential stability of a networked hyperbolic system with a circle. Our analysis extends an example by Bastin and Coron about the limits of boundary stabilizability of hyperbolic systems to the case of a networked system that is defined on a graph which contains a cycle. By spectral analysis, we prove that the system is stabilizable while the length of the arcs is sufficiently small. However, if the length of the arcs is too large, the system is not stabilizable. Our results are robust with respect to small perturbations of the arc lengths. Complementing our analysis, we provide numerical simulations that illustrate our findings.

Keywords: hyperbolic system, exponential stability, circle, network

1. Introduction

In this paper, we discuss the boundary feedback stabilization of a networked hyperbolic system with a circle. This is motivated by applications in engineering,

*Submitted: January 2023; Accepted: August 2023

where networked systems (for example networks of gas pipelines) often contain cycles, see Schmidt et al. (2017). The stabilization of tree-shaped networks has been studied in depth (see Gu and Li, 2011; Li and Dao, 2004). Studies, where cycles are not excluded are scarce (see, for example, Gugat and Weiland, 2021). Figure 1 shows a network with a circle that we study in this paper. At the end L_1 of Arc 1 a boundary feedback control action takes place.

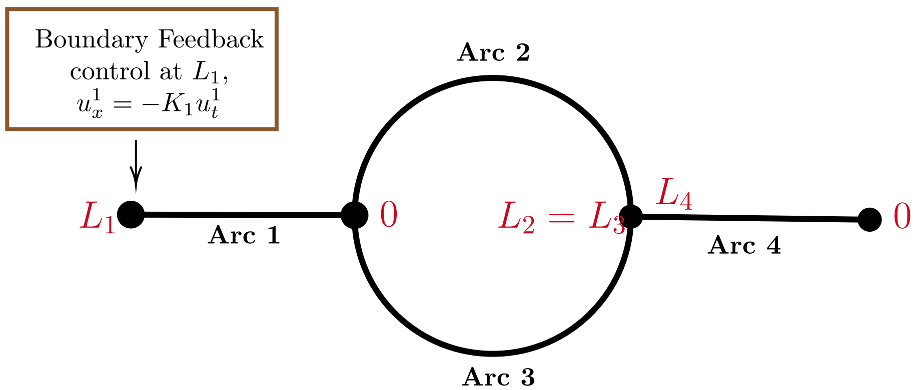


Figure 1: A network with a circle in 4 edges

We consider a network with a circle and two additional edges. At one boundary node, feedback control action takes place. At the other boundary node, a homogeneous Dirichlet condition is prescribed. For $k \in \{1,2,3,4\}$, let real numbers $c_k > 0, \varepsilon_k \geq 0$, be given. We consider the following system:

$$\begin{cases} u_{tt}^k = u_{xx}^k - 2\varepsilon_k u_t^k - (\varepsilon_k^2 - c_k^2)u^k, \\ \qquad t \in (0,+\infty),\ x \in (0, L_k),\ k \in \{1,2,3,4\}, \\ u^1(t,0) = u^2(t,0) = u^3(t,0), \\ u^2(t,L_2) = u^3(t,L_3) = u^4(t,L_4), \\ \Sigma_{k=1,2,3} u_x^k(t,0) = 0, \\ \Sigma_{k=2,3,4} u_x^k(t,L_k) = 0, \\ u^4(t,0) = 0, \\ u_x^1(t,L_1) = -K_1 u_t^1(t,L_1). \end{cases} \tag{1.1}$$

The real number K_1 is the control gain. Besides, the initial state is given:

$$\begin{cases} U(0,x) = (u^1(0,x),...,u^4(0,x)) = (u_0^1(x),...,u_0^4(x)), \\ V(0,x) = (u_t^1(0,x),...,u_t^4(0,x)) = (v_0^1(x),...,v_0^4(x)). \end{cases} \tag{1.2}$$

We give a result about the well-posedness of the solution to the system (1.1) in Lemma 2.1 in Section 2 of the present paper. The result requires the regularity, namely $u_0^k \in H^1(0, L_k), v_0^k \in L^2(0, L_k)$. Then we can investigate the exponential stability of the system. The definition of exponential stability is given below:

Definition 1.1 *The networked hyperbolic system* (1.1) *is said to be* H^1-*exponentially stable if there exist* $\alpha > 0$ *and* $C > 0$ *such that, for every*

$$U(0, \cdot) \in \prod_{k=1}^{4} H^1(0, L_k),\ V(0, \cdot) \in \prod_{k=1}^{4} L^2(0, L_k),$$

that satisfy the compatibility condition (2.1)*, the solution to the system* (1.1) *satisfies:*

$$\|U(t, \cdot)\|_{H^1} \leq Ce^{-\alpha t}\|U(0, \cdot)\|_{H^1},\ t \geq 0,$$

with

$$\|U(t, \cdot)\|_{H^1}^2 = \sum_{k=1}^{4} \int_0^{L_k} u^k(t, x)^2 + u_x^k(t, x)^2 dx.$$

The (not necessarily exponential) stability of the system is defined as follows:

Definition 1.2 *We say that the system* (1.1) (1.2) *is stabilizable if there exists a control parameter* $K_1 \in \mathbb{R}$*, such that for all* $U(0, \cdot) \in \prod_{k=1}^{4} H^1(0, L_k),\ V(0, \cdot) \in \prod_{k=1}^{4} L^2(0, L_k),$ *we have*

$$\lim_{t \to +\infty} \|U(t, \cdot)\|_{H^1} = 0.$$

If $\varepsilon_k > c_k$, the exponential boundary feedback stabilization is possible for arbitrary lengths, because, in this case, the source term is dissipative. Thus, for the examples of the limits of stability, we assume that $\varepsilon_k \in [0, c_k]$.

In Bastin and Coron (2016, Chapter 5.6, pp 197), Coron and Bastin state that for systems of balance laws, there is an intrinsic limit of stabilization under local boundary control. A 2×2 system with stabilizing boundary feedback at one point of the boundary has been discussed, for which, if the space interval is sufficiently long, stabilization is impossible for all control parameters. It is proven that the following system

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + c y_2 = 0, \quad t \in (0, +\infty),\ x \in (0, L), \\ \partial_t y_2 - \partial_x y_2 + c y_1 = 0, \\ y_2(t, L) = y_1(t, L), \\ y_1(t, 0) = k y_2(t, 0). \end{cases} \tag{1.3}$$

cannot be exponentially stable with any control gain $k \in \mathbb{R}$ if $L \geq \frac{\pi}{c}$. Since for $L \in (0, \frac{\pi}{c})$ the system is exponentially stable for suitable k, we obtain the dichotomy value of the interval length of the closed-loop system (1.3), $L_c = \frac{\pi}{c}$, see Huang, Wang and Zhou (2022). This shows that in the boundary control of hyperbolic systems, the relation between the source term and the length of the space interval matters. In this contribution, we want to explore the limits of stabilizability for (1.1).

Gugat and Gerster (2019) discuss the limits of stabilizability for the star-shaped networks of strings, inspired by Coron. In Gugat and Gerster (2019), they show that the system is stabilizable if the lengths of the arcs are sufficiently small and that it is not stabilizable in some other cases. Nakić and Veselić (2020) consider the perturbation of eigenvalues of our discussed operator, although their analysis primarily approached the topic here considered from an abstract operator perspective.

Define the matrices

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; M = \begin{pmatrix} \varepsilon & c \\ c & \varepsilon \end{pmatrix}.$$

Consider the first order 2×2 system

$$Y_t + D\,Y_x + M\,Y = 0. \tag{1.4}$$

For $\varepsilon = 0$, this yields the PDE in system (1.3). If $c^2 = \varepsilon^2$, M is positive semi-definite. Note that twice continuously differentiable solutions of (1.4) also satisfy the wave equation

$$Y_{tt} = Y_{xx} - 2\varepsilon Y_t - (\varepsilon^2 - c^2)\,Y \tag{1.5}$$

that is, both of the components satisfy the wave equation from (1.1). This can be seen as follows: System (1.4) yields

$$Y_{tt} = D^2 Y_{xx} - (DM + MD)(D^{-1}Y_t + D^{-1}MY) + M^2\,Y.$$

Since

$$D^2 = I$$

and

$$(DM + MD)D = 2\varepsilon I \text{ and } M^2 - (DM + MD)DM = (c^2 - \varepsilon^2)I$$

this yields (1.5). Systems of the form (1.4) can occur as the linearization of quasilinear hyperbolic systems that appear in many applications, for example of the isothermal Euler equations that describe gas pipeline flow, see, for example, Gugat and Giesselmann (2021).

In this paper, inspired by Bastin and Coron (2016) and Gugat and Gerster (2019), we study the limits of the stabilizability of a networked hyperbolic system with a circle by spectral analysis. This second-order system is equivalent to the 2×2 first-order system (1.4) under suitable variable substitution. We begin with a simple subcase of the system (1.1). It is an instructive result for the system with circles.

The main result of this article is the following:

THEOREM 1.1 *Assume that $c_k = c_1 = c > 0$, $\varepsilon_k = \varepsilon_1 = \varepsilon > 0$, $L_k = L_1 = L$, that is the length of the arcs in the network and the parameters are the same for all arcs. Assume that for the initial state the compatibility conditions* (2.1) *are satisfied and $\varepsilon \in (0, c)$.*

- *If $L < L_{min} = \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$, the system* (1.1)-(1.2) *is stabilizable (with $|K_1|$ sufficiently small);*
- *If $L > L_{max} = \frac{\pi}{2\sqrt{c^2-\varepsilon^2}}$, the system* (1.1)-(1.2) *is not stabilizable.*

For the proof see Proposition 3.1 and Proposition 4.1.

REMARK 1.1 *Note that if ε is sufficiently close to $c > 0$, the value of L_{min} can become arbitrarily large. If c is sufficiently large, the value of L_{min} can become arbitrarily small. There is still a gap between L_{min} and L_{max}, and the idea of eliminating the gap is to analyze all eigenvalues on the imaginary axis for each interval length L. However, on account of the complexity of the characteristic equation, the result cannot be obtained as of this writing.*

Now we state a result for the special case, where $\varepsilon = 0$. In this case, L_{max} is minimal as a function of ε.

PROPOSITION 1.1 *Under the assumptions of Theorem 1.1, for $c > 0$ and $\varepsilon = 0$ we obtain the statement*

- *If $L > L_{max} = \frac{\pi}{2\,c}$, system* (1.1)-(1.2) *is not stabilizable.*

For the proof see Proposition 4.1. Note that for $\varepsilon = 0$ the spectral analysis for $L < L_{min}$ does not yield eigenvalues with strictly negative real parts. Therefore we do not have a stabilizability result in this case. The structure of this paper is as follows. In Section 2, some preliminary results are presented. Then in Section 3, we use spectral analysis to prove the stability result and we make perturbations on the length of arcs L_k and on the control parameter K_1 and obtain the first statement of Theorem 1.1. In Section 4, we find a real eigenvalue that is bigger than 0 under a certain condition for all discussed control parameters, which means the system is not stabilizable. This demonstrates the second statement of Theorem 1.1. Finally, some numerical results are given in Sections 5 and 6.

2. Preliminaries

In this section, we first introduce the well-posedness of the system and then give some preliminaries that will be used in the proof of our theorem.

The well-posedness issue is fundamental to the control problems. Here we only present the results without proof, which can be derived by classical methods, such as the method of characteristics or by the theory of strongly continuous one-parameter semigroups of linear operators, see Pazy (1983).

Lemma 2.1 *Assume* $u_0^k \in H^1(0, L_k), v_0^k \in L^2(0, L_k)$ *and that they satisfy the compatibility condition* (2.1) *defined below:*

$$\begin{cases} u_0^1(0) = u_0^2(0) = u_0^3(0), \; u_0^2(L_2) = u_0^3(L_3) = u_0^4(L_4), \\ u_0^4(0) = 0. \end{cases} \tag{2.1}$$

Then, for each $T > 0$*, there exists a unique weak solution*

$$u^k(t,x) \in C([0,T]; H^1(0,L_k)) \cap C^1([0,T]; L^2(0,L_k)), \quad k \in \{1,2,3,4\}$$

of the initial boundary value problem (1.1).

Let $\widetilde{u}^k(t,x) = u^k(t,x)e^{\varepsilon t}$. For the functions $\widetilde{u}^k$, from (1.1), we obtain the system

$$\begin{cases} \widetilde{u}^k_{tt} = \widetilde{u}^k_{xx} + c_k^2 \widetilde{u}^k, \quad t \in (0,+\infty), \; x \in (0,L_k), k \in \{1,2,3,4\}, \\ \widetilde{u}^1(t,0) = \widetilde{u}^2(t,0) = \widetilde{u}^3(t,0), \\ \widetilde{u}^2(t,L_2) = \widetilde{u}^3(t,L_3) = \widetilde{u}^4(t,L_4), \\ \Sigma_{k=1,2,3} \widetilde{u}^k_x(t,0) = 0, \\ \Sigma_{k=2,3,4} \widetilde{u}^k_x(t,L_k) = 0, \\ \widetilde{u}^4(t,0) = 0, \\ \widetilde{u}^1_x(t,L_1) = K_1 \left(\widetilde{u}^1_t(t,L_1) - \varepsilon \widetilde{u}^1(t,L_1) \right). \end{cases} \tag{2.2}$$

Remark 2.1 *From the results of Catherine Bandle and Joachim von Below (von Below, 1988), the eigenvalue problem of the system* (2.2) *is a Sturm-Liouville eigenvalue problem on the network with* $K_1 = 0$*. As stated by Joachim von Below and Gilles François at the end of the second section in von Below and François (2005), the eigenvalue problem of the system* (2.2) *is still a Sturm-Liouville eigenvalue problem with* $K_1 \neq 0$*. Although in von Below and François (2005), only the boundary condition with* $\varepsilon = 0$ *is covered explicitly, the corresponding result for* $\varepsilon \neq 0$ *also holds. The corresponding eigenfunctions of the discussed system* (2.2) *form a complete orthonormal system in the solution space*

$\mathcal{H}$, which is the completion with respect to the norm corresponding to the scalar product (2.3) of the set

$$\begin{aligned}&\{\big(\widetilde{f}_1(x),\widetilde{f}_2(x),\widetilde{f}_3(x),\widetilde{f}_4(x),\widetilde{g}_1(x),\widetilde{g}_2(x),\widetilde{g}_3(x),\widetilde{g}_4(x)\big)^T\in\\&\prod_{k=1}^4 C^2[0,L_k]\times\prod_{k=1}^4 C^2[0,L_k]\\&|\widetilde{f}_k(x),\widetilde{g}_k(x)\ (k=1,2,3,4)\ \textit{satisfy}\ (\mathrm{BC})\},\end{aligned}$$

with the condition (BC) *defined as follows:*

$$\begin{cases}\widetilde{f}_1(0)=\widetilde{f}_2(0)=\widetilde{f}_3(0),\ \widetilde{f}_2(L_2)=\widetilde{f}_3(L_3)=\widetilde{f}_4(L_4),\\ \widetilde{f}_1'(0)+\widetilde{f}_2'(0)+\widetilde{f}_3'(0)=0,\ \widetilde{f}_2'(L_2)+\widetilde{f}_3'(L_3)+\widetilde{f}_4'(L_4)=0,\\ \widetilde{f}_4(0)=0,\\ \widetilde{g}_1(0)=\widetilde{g}_2(0)=\widetilde{g}_3(0),\ \widetilde{g}_2(L_2)=\widetilde{g}_3(L_3)=\widetilde{g}_4(L_4),\\ \widetilde{g}_1'(0)+\widetilde{g}_2'(0)+\widetilde{g}_3'(0)=0,\ \widetilde{g}_2'(L_2)+\widetilde{g}_3'(L_3)+\widetilde{g}_4'(L_4)=0,\\ \widetilde{g}_4(0)=0,\\ \widetilde{f}_1'(L_1)=-K_1(\widetilde{g}_1'(L_1)-\varepsilon\widetilde{f}_1(L_1)).\end{cases}\tag{BC}$$

Let $\widetilde{F}=(f_1,f_2,f_3,f_4)^T,\widetilde{G}=(g_1,g_2,g_3,g_4)^T$. Let

$$\mathcal{K}(\widetilde{F},\widetilde{G})=\sum_{j=1}^4\int_0^{L_j} f_j(x)g_j(x)dx.$$

The inner product in the Hilbert space $\mathcal{H}$ is

$$\mathcal{L}\left(\begin{pmatrix}\widetilde{F}_1\\ \widetilde{G}_1\end{pmatrix},\begin{pmatrix}\widetilde{F}_2\\ \widetilde{G}_2\end{pmatrix}\right)=\mathcal{K}(\widetilde{F}_1,\widetilde{F}_2)+\mathcal{K}(\widetilde{G}_1,\widetilde{G}_2).\tag{2.3}$$

Thus, $\mathcal{H}$ is a subspace of $\prod_{k=1}^4 L^2(0,L_k)$.

From Remark 2.1, the spectral properties of the system (1.1) directly determine on the growth of the solution.

We will apply the analytic implicit function theorem in Theorem 3.1. The analytic implicit function theorem is stated as follows:

Lemma 2.2 (Fritzsche and Grauert, 2002) *Let $\mathcal{B}\subset\mathbb{C}^n\times\mathbb{C}^m$ be an open set, $f=(f_1,...,f_m):\mathcal{B}\to\mathbb{C}^m$ a holomorphic mapping, and $(z_0,w_0)\in\mathcal{B}$ a point with $f(z_0,w_0)=0$ and*

$$\det\left(\frac{\partial f_\mu}{\partial z_\mu}(z_0,w_0)\Bigg|\begin{matrix}\mu=1,...,m\\ \nu=n+1,...,n+m\end{matrix}\right)\neq 0.$$

Then there is an open neighborhood $U = U' \times U'' \subset \mathcal{B}$ and a holomorphic map $g : U' \to U''$ such that

$$\{(z,w) \in U' \times U'' : f(z,w) = 0\} = \{(z, g(z)) : z \in U'\}.$$

We also use Rouché's theorem in the following form:

LEMMA 2.3 *Let C be a closed, simple curve (i.e., not self-intersecting). Let $h(z) = f(z) + g(z)$. If f and g are both holomorphic on the interior of C, then h must also be holomorphic on the interior of C. Then, if*

$$|f(z)| > |h(z) - f(z)|,$$

for every z in C, then f and h have the same number of zeros in the interior of C.

3. Stability results

3.1. The essential result

In this section, we discuss the stability of the system using spectral analysis. We suppose that for all arcs the parameters c and ε in the partial differential equation are constants, i.e. $c_k = c_1 = c, \varepsilon_k = \varepsilon_1 = \varepsilon$. First, in this subsection, we can prove that for $L < L_{min} = \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$ the system (1.1) with $K_1 = 0$ and $L_k = L$ $(k = 1,2,3,4)$ is L^2- exponentially stable. Here all arcs have the same length L. Then, in subsection 3.2 we consider small perturbations of the lengths of the arcs. The system is still exponentially stable even though the input edge and the output edge have slightly different lengths. In Section 3.3 we study perturbations of the control gain K_1. For $L < L_{min}$, the system (1.1) with $L_k = L$ $(k = 1,2,3,4)$ is L^2- exponentially stable if $|K_1|$ is sufficiently small.

PROPOSITION 3.1 *Assume that $c > 0$ and $\varepsilon \in (0,c)$.*

The following system (3.1) is L^2- exponentially stable if $L < \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$:

$$\begin{cases} u^k_{tt} = u^k_{xx} - 2\varepsilon u^k_t - (\varepsilon^2 - c^2)u^k, \quad t \in (0,+\infty),\ x \in (0,L), k \in \{1,2,3,4\}, \\ u^1(t,0) = u^2(t,0) = u^3(t,0), \\ u^2(t,L) = u^3(t,L) = u^4(t,L), \\ \Sigma_{k=1,2,3} u^k_x(t,0) = 0, \\ \Sigma_{k=2,3,4} u^k_x(t,L) = 0, \\ u^4(t,0) = 0, \\ u^1_x(t,L) = 0. \end{cases} \tag{3.1}$$

PROOF From Remark 2.1 the eigenvalue problem is a Sturm-Liouville eigenvalue problem. We can make a spectral analysis of the system (3.1).

Let $\lambda \in \mathbb{C}$ be the eigenvalue of the system (3.1); we look for a nontrivial solution

$$U(t,x) = (u^1(t,x), ..., u^4(t,x))$$

of the system (3.1) with the form $u^k(t,x) = e^{\lambda t}\varphi_k(x)$, where $\varphi_k(x)$ are the corresponding eigenfunctions.

Such a $U(t,x)$ is a solution of the system only if for all $k \in \{1,2,3,4\}$

$$\lambda^2 \varphi_k = \varphi_k'' - 2\lambda\varepsilon\varphi_k - (\varepsilon^2 - c^2)\varphi_k, \tag{3.2}$$

which yields:

$$((\lambda+\varepsilon)^2 - c^2)\,\varphi_k = \varphi_k''. \tag{3.3}$$

From (3.3), we have $\varphi_k(x) = R_{1,k}e^{\eta x} + R_{2,k}e^{-\eta x}$ and note that:

$$\eta^2 = (\lambda+\varepsilon)^2 - c^2. \tag{3.4}$$

Using the boundary condition (BC), we have

$$\begin{cases} R_{1,1} + R_{2,1} = R_{1,2} + R_{2,2} = R_{1,3} + R_{2,3}, \\ R_{1,1} + R_{1,2} + R_{1,3} = R_{2,1} + R_{2,2} + R_{2,3}, \\ R_{1,2}e^{\eta L} + R_{2,2}e^{-\eta L} = R_{1,3}e^{\eta L} + R_{2,3}e^{-\eta L} = R_{1,4}e^{\eta L} + R_{2,4}e^{-\eta L}, \\ R_{1,2}e^{\eta L} + R_{1,3}e^{\eta L} + R_{1,4}e^{\eta L} = R_{2,2}e^{-\eta L} + R_{2,3}e^{-\eta L} + R_{2,4}e^{-\eta L}, \\ R_{1,4} + R_{2,4} = 0, \\ R_{1,1}\eta e^{\eta L} + R_{2,1}(-\eta)e^{-\eta L} = 0. \end{cases} \tag{3.5}$$

Using the fifth equation in (3.5), we can take $R_{1,4} = 1, R_{2,4} = -1$.
Then, using the first four equations in (3.5), we obtain:

$$\begin{cases} R_{1,2} = R_{1,3} = e^{-\eta L}(\frac{1}{4}e^{\eta L} - \frac{3}{4}e^{-\eta L}), \\ R_{2,2} = R_{2,3} = e^{\eta L}(\frac{3}{4}e^{\eta L} - \frac{1}{4}e^{-\eta L}), \\ R_{1,1} = -\frac{1}{2} + \frac{9}{8}e^{2\eta L} + \frac{3}{8}e^{-2\eta L}, \\ R_{2,1} = \frac{1}{2} - \frac{3}{8}e^{2\eta L} - \frac{9}{8}e^{-2\eta L}. \end{cases}$$

Finally, we get the characteristic equation by substituting the values of $R_{1,1}$ and $R_{2,1}$ into the last equation in (3.5):

$$\eta(-\cosh(\eta L) + 9\cosh(3\eta L)) = 0. \tag{3.6}$$

Since $\cosh(3x) = 4\cosh^3(x) - 3\cosh(x)$ and $\cosh^2(x) = \frac{1}{2}(\cosh(2x)+1)$, we obtain:

$$\eta\cosh(\eta L)\big(-5+9\cosh(2\eta L)\big)=0. \tag{3.7}$$

We now discuss the solutions of the characteristic equation (3.7):

- If $\eta = 0$, we have $\varphi_k(x) \equiv R_{1,k} + R_{2,k}$. Using boundary condition (3.5), $\varphi^k(x) \equiv 0$, so we cannot obtain an eigenvalue.

- If $\cosh(\eta L) = 0$, let $\eta = \eta^{Re} + \eta^{Im}\mathrm{i}$ ($\eta^{Re}, \eta^{Im} \in \mathbb{R}$), we obtain
$$\cos\big(\eta^{Im}L\big)\cosh\big(\eta^{Re}L\big)+\mathrm{i}\sin\big(\eta^{Im}L\big)\sinh\big(\eta^{Re}L\big)=0.$$
  - ◇ If $\sin\big(\eta^{Im}L\big)=0$, we have $|\cos\big(\eta^{Im}L\big)|=1$, then $\cosh\big(\eta^{Re}L\big)\cos\big(\eta^{Im}L\big)\neq 0$, so the equation cannot hold.
  - ◇ If $\sinh\big(\eta^{Re}L\big)=0$, then we have $\eta^{Re}=0$, so $\cos\big(\eta^{Im}L\big)=0$. We get a sequence of solutions $\{\eta_j = (\frac{j\pi+\frac{\pi}{2}}{L})\mathrm{i}\}_{j\in\mathbb{Z}}$.

- If $-\frac{5}{2}+\frac{9}{2}\cosh(2\eta L)=0$, let $\eta = \eta^{Re} + \eta^{Im}\mathrm{i}$ ($\eta^{Re}, \eta^{Im} \in \mathbb{R}$), we obtain
$$\cos\big(2\eta^{Im}L\big)\cosh\big(2\eta^{Re}L\big)+\mathrm{i}\sin\big(2\eta^{Im}L\big)\sinh\big(2\eta^{Re}L\big)=\frac{5}{9}.$$
  - ◇ If $\sin\big(2\eta^{Im}L\big)=0$, we have $|\cos\big(2\eta^{Im}L\big)|=1$, then $\cosh\big(2\eta^{Re}L\big)=\pm\frac{5}{9}$. Thus, we have no solution for $\eta^{Re}\in\mathbb{R}$.
  - ◇ If $\sinh\big(2\eta^{Re}L\big)=0$, then we have $\eta^{Re}=0$, $\cosh\big(2\eta^{Re}L\big)=1$, so that $\cosh\big(2\eta^{Im}L\big)=\frac{5}{9}$. We get two sequences of solutions
$$\{\eta_j^+|\eta_j^+ = \\ \frac{1}{L}(\arctan\sqrt{\frac{2}{7}}+j\pi)\mathrm{i}\}_{j\in\mathbb{Z}}\cup\{\eta_j^-|\eta_j^-=\frac{1}{L}(-\arctan\sqrt{\frac{2}{7}}-j\pi)\mathrm{i}\}_{j\in\mathbb{Z}}.$$

From Remark 2.1, the corresponding eigenfunctions form a complete orthonormal system in the Hilbert space that is the completion of inner product space $\mathcal{H}$.

The corresponding eigenfunctions $\varphi_k^{j,\pm}(x)$ of $\eta_j^{\pm}$ satisfy
$$\varphi_k^{j,+}(x)=-\varphi_k^{j,-}(x),\quad k\in\{1,2,3,4\},\ j\in\mathbb{Z}.$$

The functions $\varphi_k^{j,+}(x)$ and $\varphi_k^{j,-}(x)$ with the same j and k are linearly dependent. We just need to take one branch of solutions η and without loss of generality, we take
$$\{\eta_{j,1}|\eta_{j,1}=\frac{1}{L}(\arctan\sqrt{\frac{2}{7}}+j\pi)\mathrm{i}\}_{j\in\mathbb{Z}}.$$

The corresponding eigenfunctions $\varphi_k^j(x)$ of η_j also satisfy:

$$\varphi_k^{-j}(x) = -\varphi_k^{j-1}(x), \quad j \in \mathbb{Z}^+.$$

We finally obtain two sequences of $\eta_{j,l}$:

$$\{\eta_{j,1}|\eta_{j,1} = \frac{1}{L}(\arctan\sqrt{\frac{2}{7}} + j\pi)\mathrm{i}\}_{j\in\mathbb{Z}} \cup \{\eta_{j,2}|\eta_{j,2} = (\frac{j\pi + \frac{\pi}{2}}{L})\mathrm{i}\}_{j\in\mathbb{N}}. \quad (3.8)$$

All corresponding eigenvalues $\lambda_{j,l}$ and eigenfunctions $\varphi_k^{j,l}(x)$ satisfy ($j \in \mathbb{Z}$ for $l = 1$ and $j \in \mathbb{N}$ for $l = 2$, $k = 1, 2, 3, 4$)

$$\begin{cases} (\lambda_{j,l}^{\pm} + \varepsilon)^2 = \eta_{j,l}^2 + c^2, \\ \varphi_k^{j,l}(x) = R_{1,k}^{j,l} e^{\eta_{j,l}x} + R_{2,k}^{j,l} e^{-\eta_{j,l}x}. \end{cases} \quad (3.9)$$

$\lambda_{j,l}^{\pm}$ are two roots of the eigenfunction $(\varphi_1^{j,l}, \varphi_2^{j,l}, \varphi_3^{j,l}, \varphi_4^{j,l})$. Thus, for any initial condition

$$\begin{pmatrix} u^1(0,x) \\ u^2(0,x) \\ u^3(0,x) \\ u^4(0,x) \end{pmatrix} = \sum_{m=-\infty}^{\infty} c_{1,m,1} \begin{pmatrix} \varphi_1^{m,1}(x) \\ \varphi_2^{m,1}(x) \\ \varphi_3^{m,1}(x) \\ \varphi_4^{m,1}(x) \end{pmatrix} + \sum_{n=0}^{\infty} c_{1,n,2} \begin{pmatrix} \varphi_1^{n,2}(x) \\ \varphi_2^{n,2}(x) \\ \varphi_3^{n,2}(x) \\ \varphi_4^{n,2}(x) \end{pmatrix},$$

$$\begin{pmatrix} u_t^1(0,x) \\ u_t^2(0,x) \\ u_t^3(0,x) \\ u_t^4(0,x) \end{pmatrix} = \sum_{m=-\infty}^{\infty} c_{2,m,1} \begin{pmatrix} \varphi_1^{m,1}(x) \\ \varphi_2^{m,1}(x) \\ \varphi_3^{m,1}(x) \\ \varphi_4^{m,1}(x) \end{pmatrix} + \sum_{n=0}^{\infty} c_{2,n,2} \begin{pmatrix} \varphi_1^{n,2}(x) \\ \varphi_2^{n,2}(x) \\ \varphi_3^{n,2}(x) \\ \varphi_4^{n,2}(x) \end{pmatrix},$$

We can represent the solution in the form

$$\begin{aligned} u^k(t,x) = &\sum_{m\in\mathbb{Z}} \Big(\frac{c_{2,m,1} - \lambda_{m,1}^- c_{1,m,1}}{\lambda_{m,1}^+ - \lambda_{m,1}^-} e^{\lambda_{m,1}^+ t} - \frac{c_{2,m,1} - \lambda_{m,1}^+ c_{1,m,1}}{\lambda_{m,1}^+ - \lambda_{m,1}^-} e^{\lambda_{m,1}^- t}\Big)\varphi_k^{m,1}(x) \\ &+ \sum_{n\in\mathbb{N}} \Big(\frac{c_{2,n,2} - \lambda_{n,2}^- c_{1,n,2}}{\lambda_{n,2}^+ - \lambda_{n,2}^-} e^{\lambda_{n,2}^+ t} - \frac{c_{2,n,2} - \lambda_{n,2}^+ c_{1,n,2}}{\lambda_{n,2}^+ - \lambda_{n,2}^-} e^{\lambda_{n,2}^- t}\Big)\varphi_k^{n,2}(x). \end{aligned} \quad (3.10)$$

So, if there exists $\lambda_{min} < 0$, such that for all $\lambda_{j,l}^{\pm}$,

$$Re(\lambda_{j,l}^{\pm}) \leq \lambda_{min} < 0$$

the system (2.2) is L^2-exponentially stable. Recalling (3.8) and (3.9),

$$c^2 + \eta_{j,l}^2 \leq c^2 + \eta_{0,1}^2 = c^2 - \frac{1}{L^2}\left(\arctan\sqrt{\frac{2}{7}}\right)^2.$$

- If $|\eta_{j,l}| < c$, we have $c^2 + \eta_{j,l}^2 > 0$,

$$Re(\lambda_{j,l}^{\pm}) = -\varepsilon \pm \sqrt{c^2 + \eta_{j,l}^2} \leq -\varepsilon + \sqrt{c^2 - \frac{1}{L^2}(\arctan\sqrt{\frac{2}{7}})^2} < 0.$$

- If $|\eta_{j,l}| \geq c$, we have $c^2 + \eta_{j,l}^2 \leq 0$, $Re(\lambda_{j,l}^{\pm}) = -\varepsilon < 0$.

While $L < \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$, we have

$$Re(\lambda_{j,l}^{\pm}) < \lambda_{min} = -\varepsilon + \sqrt{c^2 - \frac{1}{L^2}(\arctan\sqrt{\frac{2}{7}})^2} < 0.$$

Thus, we have shown that the system (2.2) is L^2-exponentially stable. ■

3.2. Perturbation of the arcs L_k

We show that the stability result also holds if the input edge and the output edge have slightly different lengths. For technical reasons, we have to assume that the edges in the cycle are of equal length.

THEOREM 3.1 *Assume that* $c > 0$, $\varepsilon \in (0, c)$ *and* $L < L_{min}$. *We consider a small perturbation of the lengths* L_k $(k = 1, 2, 3, 4)$ *of the following form:*

$$\begin{cases} \widetilde{L}_1 = L + d_1 r, \\ \widetilde{L}_2 = \widetilde{L}_3 = L + d_2 r, \\ \widetilde{L}_4 = L + d_4 r. \end{cases} \tag{3.11}$$

Here, d_1, d_2, d_4 *and* r *are real constants. We consider the system:*

$$\begin{cases} u_{tt}^k = u_{xx}^k - 2\varepsilon u_t^k - (\varepsilon^2 - c^2)u^k, \quad t \in (0, +\infty),\ x \in (0, \widetilde{L}_k), k \in \{1, 2, 3, 4\}, \\ u^1(t, 0) = u^2(t, 0) = u^3(t, 0), \\ u^2(t, \widetilde{L}_2) = u^3(t, \widetilde{L}_3) = u^4(t, \widetilde{L}_4), \\ \Sigma_{k=1,2,3} u_x^k(t, 0) = 0, \\ \Sigma_{k=2,3,4} u_x^k(t, \widetilde{L}_k) = 0, \\ u^4(t, 0) = 0, \\ u_x^1(t, \widetilde{L}_1) = 0. \end{cases} \tag{3.12}$$

If $|r|$ *is sufficiently small, the system* (3.12) *is exponentially stable .*

PROOF From Proposition 3.1, system (3.12) is exponentially stable with $r = 0$, $K_1 = 0$. Recall that we obtain all eigenvalues λ and corresponding η of the system (3.12) with $\widetilde{L}_k = L$ $(k = 1, 2, 3, 4)$.

The sequence of η is

$$\{\eta_{j,1} | \eta_{j,1} = \frac{1}{L}(\arctan\sqrt{\frac{2}{7}} + j\pi)\mathrm{i}\}_{j\in\mathbb{Z}} \cup \{\eta_{j,2} | \eta_{j,2} = (\frac{j\pi + \frac{\pi}{2}}{L})\mathrm{i}\}_{j\in\mathbb{N}}. \quad (3.13)$$

The eigenvalues $\lambda^{\pm}_{j,l}$ ($j \in \mathbb{Z}$ for $l = 1$ and $j \in \mathbb{N}$ for $l = 2$, $k = 1, 2, 3, 4$) satisfy

$$(\lambda^{\pm}_{j,l} + \varepsilon)^2 = \eta^2_{j,l} + c^2. \quad (3.14)$$

More precisely,

$$\lambda^{+}_{j,l} = \begin{cases} -\varepsilon + \sqrt{\eta^2_{j,l} + c^2}, & \eta^2_{j,l} + c^2 > 0 \\ -\varepsilon + \sqrt{-(\eta^2_{j,l} + c^2)}\mathrm{i}, & \eta^2_{j,l} + c^2 < 0 \end{cases}$$

$$\lambda^{-}_{j,l} = \begin{cases} -\varepsilon - \sqrt{\eta^2_{j,l} + c^2}, & \eta^2_{j,l} + c^2 > 0 \\ -\varepsilon - \sqrt{-(\eta^2_{j,l} + c^2)}\mathrm{i}, & \eta^2_{j,l} + c^2 < 0 \end{cases}$$

Furthermore, recall that the characteristic equation is:

$$\eta \cosh(\eta L)\big(-5 + 9\cosh(2\eta L)\big) = 0. \quad (3.15)$$

While $\widetilde{L}_1 \neq \widetilde{L}_2 = \widetilde{L}_3 \neq \widetilde{L}_4$, from Remark 2.1 the eigenvalue problem is a Sturm-Liouville eigenvalue problem. We can recalculate the characteristic equation.

Let $\widetilde{\lambda} \in \mathbb{C}$. We look for a nontrivial solution $\widetilde{U}(t,x) = (\widetilde{u}^1(t,x), ..., \widetilde{u}^4(t,x))$ of the system (3.12). The form is $\widetilde{u}^k(t,x) = e^{\widetilde{\lambda}t}\widetilde{\varphi}_k(x)$, with the eigenvalues of the system $\widetilde{\lambda}$. The corresponding eigenfunctions of $\widetilde{\varphi}_k(x)$ are $\widetilde{u}^k(t,x)$.

Such a $\widetilde{U}(t,x)$ can only be a solution of the system if

$$\widetilde{\lambda}^2\widetilde{\varphi}_k = \widetilde{\varphi}''_k + c^2\widetilde{\varphi}_k. \quad (3.16)$$

From (3.16), we have $\widetilde{\varphi}_k(x) = R_{1,k}e^{\widetilde{\eta}x} + R_{2,k}e^{-\widetilde{\eta}x}$ and

$$(\widetilde{\lambda} + \varepsilon)^2 = \widetilde{\eta}^2 + c^2. \quad (3.17)$$

Using the boundary condition, we have

$$\begin{cases} R_{1,1} + R_{2,1} = R_{1,2} + R_{2,2} = R_{1,3} + R_{2,3}, \\ R_{1,1} + R_{1,2} + R_{1,3} = R_{2,1} + R_{2,2} + R_{2,3}, \\ R_{1,2}e^{\widetilde{\eta}\widetilde{L}_2} + R_{2,2}e^{-\widetilde{\eta}\widetilde{L}_2} = R_{1,3}e^{\widetilde{\eta}\widetilde{L}_2} + R_{2,3}e^{-\widetilde{\eta}\widetilde{L}_2} = R_{1,4}e^{\widetilde{\eta}\widetilde{L}_4} + R_{2,4}e^{-\widetilde{\eta}\widetilde{L}_4}, \\ R_{1,2}e^{\widetilde{\eta}\widetilde{L}_2} + R_{1,3}e^{\widetilde{\eta}\widetilde{L}_2} + R_{1,4}e^{\widetilde{\eta}\widetilde{L}_4} = R_{2,2}e^{-\widetilde{\eta}\widetilde{L}_2} + R_{2,3}e^{-\widetilde{\eta}\widetilde{L}_3} + R_{2,4}e^{-\widetilde{\eta}\widetilde{L}_4}, \\ R_{1,4} + R_{2,4} = 0, \\ \widetilde{\eta}(R_{1,1}e^{\widetilde{\eta}\widetilde{L}_1} - R_{2,1}e^{-\widetilde{\eta}\widetilde{L}_1}) = 0. \end{cases}$$

(3.18)

If $\widetilde{\eta} = 0$, we have $\widetilde{\varphi}_k(x) \equiv R_{1,k} + R_{2,k}$. Using the first, third, and fifth equation in (3.18), $\widetilde{\varphi}_k(x) \equiv 0$. We cannot obtain an eigenvalue, so we suppose $\widetilde{\eta} \neq 0$.

Using the fifth equation in (3.18), we can take $R_{1,4} = 1, R_{2,4} = -1$.

We use the right part of the first and third equations and obtain:

$$R_{1,2} = R_{1,3}, \quad R_{2,2} = R_{2,3}. \tag{3.19}$$

Then, using the third and fourth equations in (3.18) and (3.19), we obtain:

$$\begin{cases} R_{1,2} = R_{1,3} = e^{-\widetilde{\eta}\widetilde{L}_2}(\frac{1}{4}e^{\widetilde{\eta}\widetilde{L}_4} - \frac{3}{4}e^{-\widetilde{\eta}\widetilde{L}_4}), \\ R_{2,2} = R_{2,3} = e^{\widetilde{\eta}\widetilde{L}_2}(\frac{3}{4}e^{\widetilde{\eta}\widetilde{L}_4} - \frac{1}{4}e^{-\widetilde{\eta}\widetilde{L}_4}). \end{cases}$$

Using the first and second equations in (3.18) and (3.19), we obtain:

$$\begin{cases} R_{1,1} = \frac{3}{2}R_{2,2} - \frac{1}{2}R_{1,2}, \\ R_{2,1} = -\frac{1}{2}R_{2,2} + \frac{3}{2}R_{1,2}. \end{cases}$$

The last equation in (3.18) yields:

$$R_{1,1} = R_{2,1}e^{-2\widetilde{\eta}\widetilde{L}_1}. \tag{3.20}$$

Then we have:

$$9\cosh\Big(\widetilde{\eta}\hat{L}_1\Big) + 3\cosh\Big(\widetilde{\eta}\hat{L}_2\Big) - 3\cosh\Big(\widetilde{\eta}\hat{L}_3\Big) - \cosh\Big(\widetilde{\eta}\hat{L}_4\Big) = 0. \tag{3.21}$$

with

$\hat{L}_1 = \widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_4,\ \hat{L}_2 = \widetilde{L}_1 - \widetilde{L}_2 - \widetilde{L}_4,\ \hat{L}_3 = \widetilde{L}_1 + \widetilde{L}_2 - \widetilde{L}_4,\ \hat{L}_4 = \widetilde{L}_1 - \widetilde{L}_2 + \widetilde{L}_4.$

We now introduce the following lemma:

LEMMA 3.1 *Let*

$$\begin{aligned} H(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4, \widetilde{\eta}) = {} & 9\cosh\Big(\widetilde{\eta}(\widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_4)\Big) + 3\cosh\Big(\widetilde{\eta}(\widetilde{L}_1 + \widetilde{L}_2 - \widetilde{L}_4)\Big) \\ & - 3\cosh\Big(\widetilde{\eta}(\widetilde{L}_1 + \widetilde{L}_2 - \widetilde{L}_4)\Big) - \cosh\Big(\widetilde{\eta}(\widetilde{L}_1 - \widetilde{L}_2 + \widetilde{L}_4)\Big). \\ F(\widetilde{\eta}, \widetilde{\lambda}) = {} & (\widetilde{\lambda} + \varepsilon)^2 - c^2 - \widetilde{\eta}^2. \end{aligned}$$

For each $\lambda^{\pm}_{j,l}$, there exists an open neighborhood $V^{k,1}_{j,l}, V^{k,2}_{j,l}, V^{k,3}_{j,l}$ $(k = 1, 2)$ and holomorphic maps $h_{j,l} : V^{k,1}_{j,l} \to V^{k,3}_{j,l}$ and $g^k_{j,l} : V^{k,1}_{j,l} \to V^{k,2}_{j,l}$ such that

$$H\Big(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4, h_{j,l}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)\Big) = 0,$$
$$F\Big(h_{j,l}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4), g^k_{j,l}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)\Big) = 0.$$

for all $(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4) \in V^{k,1}_{j,l}$.
Furthermore, $h_{j,l}(L, L, L) = \eta_{j,l},\ g^1_{j,l}(L, L, L) = \lambda^+_{j,l},\ g^2_{j,l}(L, L, L) = \lambda^-_{j,l}$.

PROOF The sums of holomorphic functions are holomorphic, so $H(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4, \widetilde{\eta})$ is a holomorphic function. From Lemma 2.2, if $\frac{\partial H}{\partial \widetilde{\eta}}(L, L, L, \eta_{j,l}) \neq 0$, there is an open neighborhood $U_{j,l} = U'_{j,l} \times U''_{j,l} \subset \mathbb{R}^3 \times \mathbb{C}$ and a holomorphic function $h_{j,l} : U'_{j,l} \to U''_{j,l}$ such that

$$\{(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4, \widetilde{\eta}) \in U' \times U'' : H(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4, \widetilde{\eta}) = 0\} =$$
$$\Big\{\Big(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4, h(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)\Big) : (\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4) \in U'\Big\},$$

and $h_{j,l}(L, L, L) = \eta$.

We have:

$$\frac{\partial H}{\partial \widetilde{\eta}}(L, L, L, \eta_{j,l}) =$$
$$\eta_{j,l}\Big(9\sinh(3\eta_{j,l}L) \times 3L - 3\sinh(-\eta_{j,l}L)L - 3\sinh(\eta_{j,l}L)L - \sinh(\eta_{j,l}L)L\Big)$$
$$= \eta_{j,l}L\Big(27\sinh(3\eta_{j,l}L) - \sinh(\eta_{j,l}L)\Big). \tag{3.22}$$

Recall that for $x \in \mathbb{C}$

$$\begin{cases} \sinh(3x) = \sinh(x)\big(3 + 4\sinh^2(x)\big). \\ 27\sinh(3x) - \sinh(x) = \sinh(x)(80 + 108\sinh^2 x). \end{cases} \tag{3.23}$$

While $l = 1$, due to $\sinh(\eta_{j,1}L) = \pm\frac{\sqrt{2}}{3}\mathrm{i}$, $\sinh^2(\eta_{j,1}L) = -\frac{2}{9}$ and (3.23),

$$\frac{\partial H}{\partial \widetilde{\eta}}(L, L, L, \eta_{j,1}) = \pm\frac{56\sqrt{2}}{3}\eta_{j,1}L\mathrm{i} \neq 0. \tag{3.24}$$

While $l = 2$, due to $\sinh(\eta_{j,2}L) = \pm\mathrm{i}$, $\sinh^2(\eta_{j,2}L) = -1$ and (3.23),

$$\frac{\partial H}{\partial \widetilde{\eta}}(L, L, L, \eta_{j,2}) = \pm 28\eta_{j,2}L\mathrm{i} \neq 0. \tag{3.25}$$

Then, from Lemma 2.2, we obtain the existence of the holomorphic function $h_{j,l}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)$ with $h_{j,l}(L, L, L) = \eta_{j,l}$.

$F(\widetilde{\eta}, \widetilde{\lambda})$ is a holomorphic function. From Lemma 2.2, if $\frac{\partial F}{\partial \widetilde{\lambda}}(\eta_{j,l}, \lambda^+_{j,l}) \neq 0$, then there is an open neighborhood $W^1_{j,l} = W^{1,1}_{j,l} \times W^{1,2}_{j,l} \subset \mathbb{C} \times \mathbb{C}$ and a holomorphic function $f : W^{1,1}_{j,l} \to W^{1,2}_{j,l}$ such that

$$\{(\widetilde{\eta}, \widetilde{\lambda}) \in W^{1,1}_{j,l} \times W^{1,2}_{j,l} : F(\widetilde{\eta}, \widetilde{\lambda}) = 0\} = \Big\{\Big(\widetilde{\eta}, f^1_{j,l}(\widetilde{\eta})\Big) : \widetilde{\eta} \in W^{1,1}_{j,l}\Big\}$$

and $f^1_{j,l}(\eta_{j,l}) = \lambda^+_{j,l}$.

We compute $\frac{\partial F}{\partial \widetilde{\lambda}}$:

$$\frac{\partial F}{\partial \widetilde{\lambda}} = 2(\lambda_{j,l}^{+} + \varepsilon) \neq 0.$$

We obtain function $f_{j,l}^{1}(\widetilde{\lambda}) : W_{j,l}^{1,1} \to W_{j,l}^{1,2}$ with $f(\eta_{j,l}) = \lambda_{j,l}^{+}$.

The compositions of holomorphic functions are holomorphic. Denote

$$g_{j,l}^{1} := h_{j,l} \ \circ \ f_{j,l}^{1} : (h_{j,l})^{-1}(U_{j,l}'' \cap W_{j,l}^{1,1}) \to f_{j,l}^{1}(U_{j,l}'' \cap W_{j,l}^{1,1}).$$

$(h_{j,l})^{-1}$ is the inverse function of $h_{j,l}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)$. $g_{j,l}^{1}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)$ is still a holomorphic function. Moreover,

$$g_{j,l}^{1}(L, L, L) = f_{j,l}^{1}(\eta_{j,l}) = \lambda_{j,l}^{+}.$$

Making the same analysis of $F(\widetilde{\eta}, \widetilde{\lambda})$ on $(\eta_{j,l}, \lambda_{j,l}^{-})$, we obtain open neighborhoods $W_{j,l}^{2,1}, W_{j,l}^{2,2}$ and holomorphic functions $f_{j,l}^{2}(\widetilde{\eta}), g_{j,l}^{2}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4)$.

Denote the open neighborhood

$$\begin{aligned} V_{j,l}^{s,1} &:= (h_{j,l})^{-1}(U_{j,l}'' \cap W_{j,l}^{s,1}), \\ V_{j,l}^{s,2} &:= f_{j,l}^{1}(U_{j,l}'' \cap W_{j,l}^{s,1}), \\ V_{j,l}^{s,3} &:= U_{j,l}'' \cap W_{j,l}^{s,1}. \end{aligned}$$

$j \in \mathbb{Z}$ for $l = 1$ and $j \in \mathbb{N}$ for $l = 2$, $k = 1, 2$. Then we have

$$V_{j,l}^{k,1} \subset U_{j,l}, \ V_{j,l}^{k,2} \subset W_{j,l}^{k,2}, \ V_{j,l}^{k,3} \subset U_{j,l}''.$$

Furthermore, $g_{j,l}^{k}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_4) : V_{j,l}^{k,1} \to V_{j,l}^{k,2}$ satisfy:

$$g_{j,l}^{1}(L, L, L) = \lambda_{j,l}^{+}, \ g_{j,l}^{2}(L, L, L) = \lambda_{j,l}^{-}.$$

■

With Lemma 3.1, we obtain the following holomorphic functions:

$$\begin{aligned} p_{j,l}(r) &= h_{j,l}(L + c_1 r, L + c_2 r, L + c_4 r), \\ q_{j,l,k}(r) &= f_{j,l}^{k}(L + c_1 r, L + c_2 r, L + c_4 r). \end{aligned}$$

The eigenvalues $\widetilde{\lambda}_{j,l}^{\pm}$ and corresponding $\eta_{j,l}$ have the following asymptotic expansion in the $V_{j,l}^{k}$:

$$\widetilde{\eta}_{j,l} = \eta_{j,l} + \sum_{s=1}^{\infty} p_{j,l}^{(s)}(0) r^s$$

$$\widetilde{\lambda}_{j,l}^{+} = \lambda_{j,l}^{+} + \sum_{s=1}^{\infty} q_{j,l,1}^{(s)}(0) r^s$$

$$\widetilde{\lambda}_{j,l}^{-} = \lambda_{j,l}^{-} + \sum_{s=1}^{\infty} q_{j,l,2}^{(s)}(0) r^s$$

We suppose

$$\begin{cases}\widetilde{\lambda}^{\pm}_{j,l} = \lambda^{\pm}_{j,l} + \lambda^{\pm}_{j,l}(1)r + \lambda^{\pm}_{j,l}(2)r^2 + ... \\ \widetilde{\eta}_{j,l} = \eta_{j,l} + \eta_{j,l}(1)r + \eta_{j,l}(2)r^2 + ... \end{cases} \tag{3.26}$$

Using (3.14) (3.17) (3.26) and taking the first-order mean approximation, we have:

$$\eta_{j,l}\eta_{j,l}(1) = (\lambda^{\pm}_{j,l} + \varepsilon)\lambda^{\pm}_{j,l}(1). \tag{3.27}$$

Notice that if $\widetilde{l} = l + sr$, $\widetilde{\eta} = \eta + \eta(1)r + O(r^2)$,

$$e^{\widetilde{\eta}\widetilde{l}} = e^{\eta l + (s\eta + \eta(1)l)r + O(r^2)} = e^{\eta l}(1 + (s\eta + \eta(1)l)r) + O(r^2).$$

Thus,

$$\cosh\Big(\widetilde{\eta}\widetilde{l}\Big) = \frac{1}{2}(e^{\widetilde{\eta}\widetilde{l}} + e^{-\widetilde{\eta}\widetilde{l}}) = \cosh(\eta l) + \sinh(\eta l)(s\eta + \eta(1)l)r + O(r^2).$$

Then, using (3.15), (3.21) and (3.26) and taking the first-order mean approximation, we obtain:

$$\begin{aligned} &9\sinh(3\eta_{j,l}L)\Big(3\eta_{j,l}(1)L + \eta_{j,l}(c_1 + c_2 + c_4)\Big) + \\ &+3\sinh(\eta_{j,l}L)\Big(\eta_{j,l}(1)L + \eta_{j,l}(-c_1 + c_2 + c_4)\Big) \\ &-3\sinh(\eta_{j,l}L)\Big(\eta_{j,l}(1)L + \eta_{j,l}(c_1 + c_2 - c_4)\Big) - \\ &-\sinh(\eta_{j,l}L)\Big(\eta_{j,l}(1)L + \eta_{j,l}(c_1 - c_2 + c_4)\Big) = 0. \end{aligned}$$

After computation, we have:

$$\begin{aligned} 0 = {} &\eta_{j,l}(1)\Big(27\sinh(3\eta_{j,l}L) - \sinh(\eta_{j,l}L)\Big)L + \\ &+\eta_{j,l}\Big(9\sinh(3\eta_{j,l}L)(c_1 + c_2 + c_4) + \sinh(\eta_{j,l}L)(-7c_1 + c_2 + 5c_4)\Big). \end{aligned} \tag{3.28}$$

While $l = 1$, the equation (3.2) yields:

$$0 = 128\eta_{j,l}(1)L + \eta_{j,1}(27c_1 + 45c_2 + 54c_4). \tag{3.29}$$

While $l = 2$, the equation (3.2) yields:

$$0 = -7\eta_{j,l}(1)L - \eta_{j,2}(4c_1 + 2c_2 + c_4). \tag{3.30}$$

Then, from (3.27), we have:

$$\begin{cases} \lambda^{\pm}_{j,1}(1) = \frac{\eta^2_{j,1}(27c_1+45c_2+54c_4)}{128L(\lambda^{\pm}_{j,1}+\varepsilon)}, \\ \lambda^{\pm}_{j,2}(1) = \frac{\eta^2_{j,2}(4c_1+2c_2+c_4)}{7L(\lambda^{\pm}_{j,2}+\varepsilon)}. \end{cases} \tag{3.31}$$

There exists m_1 such that while $|m| < m_1$,

$$c^2 + \eta^2_{m,1} > 0,$$

and while $|m| \geq m_1$

$$c^2 + \eta^2_{m,1} \leq 0.$$

A finite number of eigenvalues (with sufficiently small m) lie on the real axis and the real part of the other eigenvalues is equal to $-\varepsilon$ (large m).

- While $|m| < m_1$, by (3.14),
$$\lambda^{\pm}_{m,1} + \varepsilon \text{ are on the real axis.}$$
$\lambda^{\pm}_{m,1}(1)$ are on the real axis. Recall the proof of Proposition 3.1,
$$Re(\lambda^{\pm}_{m,1}) < \lambda_{min} = -\varepsilon + \sqrt{c^2 - \frac{1}{L^2}}.$$
Using the continuity of $q_{j,l,k}(r)$, there exists a sufficiently small r_1 such that while $|m| < m_1$, $|r| < r_1$
$$Re(\widetilde{\lambda}^{\pm}_{m,1}) < \frac{1}{2}\lambda_{min} < 0.$$
- While $|m| \geq m_1$, by (3.14),
$$\lambda^{\pm}_{m,1} + \varepsilon \text{ are on the imaginary axis.}$$
Due to (3.14) (3.17), for higher derivatives $\lambda_{m,1}(s)$, we obtain:
$$\begin{aligned} &(\lambda^{\pm}_{m,1} + \varepsilon)\lambda^{\pm}_{m,1}(s) + G^s\big(\lambda^{\pm}_{m,1}(1), \lambda^{\pm}_{m,1}(2), ..., \lambda^{\pm}_{m,1}(s-1)\big) = \\ &= H^s\big(\eta_{m,1}, \eta_{m,1}(1), \eta_{m,1}(2), ..., \eta_{m,1}(s)\big). \end{aligned} \tag{3.32}$$
It is an important fact that G^s and F^s have a quadratic form.

  We claim the following statement and prove it later in Remark 3.1
$$\text{for all } s \in \mathbb{N}^+,\ \eta_{m,1}(s) \text{ are on the imaginary axis.} \tag{3.33}$$
We now use mathematical induction to prove that for $s \in \mathbb{N}^+$, $\lambda^{\pm}_{m,1}(s)$ are on the imaginary axis. Let $P(n)$ be the statement
$$\lambda^{\pm}_{m,1}(s) \text{ are on the imaginary axis for } s \leq n.$$

Base case: From (3.31), $\lambda^{\pm}_{m,1}(1)$ is on the imaginary axis, so the statement $P(1)$ holds.

Induction step:
If $P(n)$ holds, $G^{n+1}\big(\lambda^{\pm}_{m,1}(1), \lambda^{\pm}_{m,1}(2), ..., \lambda^{\pm}_{m,1}(n)\big) \in \mathbb{R}$.
From (3.33), $H^{s}\big(\eta_{m,1}, \eta_{m,1}(1), \eta_{m,1}(2), ..., \eta_{m,1}(s)\big) \in \mathbb{R}$ for all $s \in \mathbb{N}^{+}$.
Recall that $\lambda^{\pm}_{m,1} + \varepsilon$ ($\neq 0$) is on the imaginary axis, and from (3.2),

$$(\lambda^{\pm}_{m,1} + \varepsilon)\lambda^{\pm}_{m,1}(n+1) = -G^{n+1}\big(\lambda^{\pm}_{m,1}(1), ..., \lambda^{\pm}_{m,1}(n)\big) + H^{n+1}\big(\eta_{m,1}, \eta_{m,1}(1), ..., \eta_{m,1}(n+1)\big) \in \mathbb{R}.$$

Thus,

$$\lambda^{\pm}_{m,1}(n+1) = \frac{-G^{n+1} + H^{n+1}}{\lambda^{\pm}_{m,1} + \varepsilon},$$

is on the imaginary axis.

That is, the statement $P(n+1)$ also holds, establishing the induction step.

Since both the base case and the induction step have been proved as true, by mathematical induction, the statement $P(n)$ holds for every natural number n.

Then we have:

$$Re(\widetilde{\lambda}^{\pm}_{m,1}) = Re(\lambda^{\pm}_{m,1}) = -\varepsilon \leq 0.$$

There also exists n_1 such that while $n < n_1$, we have $c^2 + \eta^2_{n,2} > 0$ and while $|n| \geq n_1$, we have $c^2 + \eta^2_{n,2} \leq 0$. Using similar arguments as in the analysis of $\widetilde{\lambda}^{\pm}_{m,1}$, we obtain the following statements:

- while $n < n_1$, there exists sufficiently small r_2 such that

$$Re(\widetilde{\lambda}^{\pm}_{n,2}) < \frac{1}{2}\lambda_{min} < 0.$$

- while $n > n_1$,

$$Re(\widetilde{\lambda}^{\pm}_{n,2}) = Re(\lambda^{\pm}_{n,2}) = -\varepsilon \leq 0.$$

Taking $r_0 = \min\{r_1, r_2\}$, we know that for all $\widetilde{\lambda}^{\pm}_{j,l}$ ($|j| \in \mathbb{Z}$ for $l = 1$ and $j \in \mathbb{N}$ for $l = 2$), if $|r| < r_0$,

$$Re(\widetilde{\lambda}^{\pm}_{j,l}) < \frac{1}{2}\lambda_{min} < 0.$$

The system (3.12) is L^2-exponentially stable if r is sufficiently small. ∎

Remark 3.1 *For all $s \in \mathbb{N}^{+}$, $\eta_{j,l}(s)$ are on the imaginary axis.*

Proof of Remark 3.1

We still use mathematical induction to prove the result. Observing (3.21) and separately writing the real part and the imaginary part of $\eta(s), \widetilde{\eta}$, and η:

$$\begin{aligned}\widetilde{\eta} &= \widetilde{a} + \widetilde{b}\mathrm{i},\\ \eta &= a + b\mathrm{i},\\ \eta(s) &= a(s) + b(s)\mathrm{i},\end{aligned}$$

with $\widetilde{a}, \widetilde{b}, a, b, a(k), b(k) \in \mathbb{R}$, the characteristic equation (3.21) yields:

$$\begin{aligned}&9\cosh\left(\widetilde{a}\hat{L}_1\right)\cos\left(\widetilde{b}\hat{L}_1\right) + 3\cosh\left(\widetilde{a}\hat{L}_2\right)\cos\left(\widetilde{b}\hat{L}_2\right) - 3\cosh\left(\widetilde{a}\hat{L}_3\right)\cos\left(\widetilde{b}\hat{L}_3\right) -\\ &\cosh\left(\widetilde{a}\hat{L}_4\right)\cos\left(\widetilde{b}\hat{L}_4\right) = -\mathrm{i}\Big(9\sinh\left(\widetilde{a}\hat{L}_1\right)\sin\left(\widetilde{b}\hat{L}_1\right) + 3\sinh\left(\widetilde{a}\hat{L}_2\right)\sin\left(\widetilde{b}\hat{L}_2\right) -\\ &3\sinh\left(\widetilde{a}\hat{L}_3\right)\sin\left(\widetilde{b}\hat{L}_3\right) - \sinh\left(\widetilde{a}\hat{L}_4\right)\sin\left(\widetilde{b}\hat{L}_4\right)\Big).\end{aligned} \tag{3.34}$$

Let $Q(n)$ be the statement:

$$a(s) = 0 \text{ for } s \le n.$$

Base case: From the first order approximation before in (3.29) and (3.30), the numbers $\eta_m(1)$ are purely imaginary and $a(1) = 0$. The statement $Q(1)$ holds.

Induction step:
If $Q(n)$ holds,

$$\widetilde{a} = a(n+1)r^{n+1} + O(r^{n+2}).$$

Thus,

$$\begin{aligned}\cos\left(\widetilde{b}l\right) &= \cos(bl)\cos\left(\sum_{s=1}^{\infty} b(s)r^s\right) - \sin(bl)\sin\left(\sum_{s=1}^{\infty} b(s)r^s\right)\\ &= \cos(bl) - \sin(bl)b(1)lr + O(r^2),\\ \sin\left(\widetilde{b}l\right) &= \sin(bl)\cos\left(\sum_{s=1}^{\infty} b(s)r^s\right) + \cos(bl)\sin\left(\sum_{s=1}^{\infty} b(s)r^s\right)\\ &= \sin(bl) + \cos(bl)b(1)lr + O(r^2),\end{aligned}$$

and

$$\begin{aligned}\cosh(\widetilde{a}l) &= 1 + O(r^{2n+2}),\\ \sinh(\widetilde{a}l) &= a(n+1)lr^{n+1} + O(r^{n+2}).\end{aligned}$$

Taking the $n+1$-order approximation of the imaginary part of (3.2), we obtain:

$$a(n+1)L\big(27\sin(3bL) + 3\sin(bL) - 3\sin(bL) - \sin(bL)\big)b(1) = 0.$$

Using $a(1) = 0$ and (3.23) we obtain:

$$\begin{cases} b(1) = -\mathrm{i}\eta(1) \neq 0, \\ 27\sin(3bL) - \sin(bL) = -\mathrm{i}\left(27\sinh(3\eta L) - \sinh(\eta L)\right) \neq 0. \end{cases}$$

Thus, $a(n+1) = 0$, which means that the statement $Q(n+1)$ holds, establishing the induction step.

Since both the base case and the induction step have been proven as true, by mathematical induction the statement $Q(n)$ holds for every natural number n. ■

3.3. Perturbation of control parameter K_1

In this section, we want to prove that the system can be stabilized even if $K_1 \neq 0$, but is sufficiently small.

THEOREM 3.2 *Assume that $c > 0$ and $\varepsilon \in (0, c)$. The following system is exponentially stable if $L < L_{min}$ and $|K_1|$ is sufficiently small:*

$$\begin{cases} u^k_{tt} = u^k_{xx} - 2\varepsilon u^k_t - (\varepsilon^2 - c^2)u^k, \quad t \in (0, +\infty),\ x \in (0, L),\ k \in \{1, 2, 3, 4\}, \\ u^1(t, 0) = u^2(t, 0) = u^3(t, 0), \\ u^2(t, L) = u^3(t, L) = u^4(t, L), \\ \Sigma_{k=1,2,3} u^k_x(t, 0) = 0, \\ \Sigma_{k=2,3,4} u^k_x(t, L) = 0, \\ u^4(t, 0) = 0, \\ u^1_x(t, L) = K_1 u^1_t(t, L). \end{cases} \tag{3.35}$$

PROOF From Remark 2.1, the spectral properties of the system (1.1) directly determine the growth of the solution.

Let $\widetilde{\lambda} \in \mathbb{C}$. We look for a nontrivial solution $\widetilde{U}(t, x) = (\widetilde{u}^1(t, x), ..., \widetilde{u}^4(t, x))$ of the system with the form $\widetilde{u}^k(t, x) = e^{\widetilde{\lambda}t}\widetilde{\varphi}_k(x)$, with the eigenvalue $\widetilde{\lambda}$ of the system. The corresponding eigenfunctions of the $\widetilde{u}^k(t, x)$ are $\widetilde{\varphi}_k(x)$.

Similarly as in the analysis before, such a $\widetilde{U}(t, x)$ can only be a solution of the system if $\widetilde{\varphi}_k(x) = R_{1,k}e^{\widetilde{\eta}x} + R_{2,k}e^{-\widetilde{\eta}x}$ and $\widetilde{\eta}$ satisfy the following characteristic equation:

$$\Big[\widetilde{\eta}\cosh(\widetilde{\eta}L)\big(-5 + 9\cosh(2\widetilde{\eta}L)\big)\Big] + \Big[\widetilde{\Lambda}^{\pm}(\eta)K_1\sinh(\widetilde{\eta}L)\big(1 + 9\cosh(2\widetilde{\eta}L)\big)\Big] = 0. \tag{3.36}$$

The function $\widetilde{\Lambda}^{\pm}(\eta)$ is the solution of $\widetilde{\lambda}$ in the following equation (3.37):

$$(\widetilde{\lambda}+\varepsilon)^2 = \widetilde{\eta}^2 + c^2. \tag{3.37}$$

While $K_1 = 0$, the sequence of η is

$$\{\eta_{j,1} | \eta_{j,1} = \frac{1}{L}(\arctan\sqrt{\frac{2}{7}} + j\pi)\mathrm{i}\}_{j\in\mathbb{Z}} \cup \{\eta_{j,2} | \eta_{j,2} = (\frac{j\pi + \frac{\pi}{2}}{L})\mathrm{i}\}_{j\in\mathbb{N}}. \tag{3.38}$$

The eigenvalues $\lambda^{\pm}_{j,l}$ ($j \in \mathbb{Z}$ for $l = 1$ and $j \in \mathbb{N}$ for $l = 2$) satisfy

$$(\lambda^{\pm}_{j,l} + \varepsilon)^2 = \eta^2_{j,l} + c^2. \tag{3.39}$$

More precisely,

$$\lambda^{+}_{j,l} = \begin{cases} -\varepsilon + \sqrt{\eta^2_{j,l} + c^2}, & \eta^2_{j,l} + c^2 > 0 \\ -\varepsilon + \sqrt{-(\eta^2_{j,l} + c^2)}\mathrm{i}, & \eta^2_{j,l} + c^2 < 0, \end{cases}$$

$$\lambda^{-}_{j,l} = \begin{cases} -\varepsilon - \sqrt{\eta^2_{j,l} + c^2}, & \eta^2_{j,l} + c^2 > 0 \\ -\varepsilon - \sqrt{-(\eta^2_{j,l} + c^2)}\mathrm{i}, & \eta^2_{j,l} + c^2 < 0. \end{cases}$$

Furthermore, the characteristic equation is:

$$\eta\cosh(\eta L)\big(-5 + 9\cosh(2\eta L)\big) = 0. \tag{3.40}$$

Observe that for any $\varepsilon_1 > 0$ (that will be determined later) there exists R_1, such that while $|\widetilde{\eta}| > R_1$,

$$|\widetilde{\lambda} - \widetilde{\eta}| < \varepsilon_1.$$

We try to divide the solution of (3.36) into two parts $V_1 := \{\widetilde{\lambda} || (\widetilde{\lambda}+\varepsilon)^2 - c^2| > R_0^2\}$ and $V_2 := \{\widetilde{\lambda} || (\widetilde{\lambda}+\varepsilon)^2 - c^2| \le R_0^2\}$ (R_0 is decided later).

We use Rouché's theorem to state that there are only finitely many elements in the set V_1.

PROPOSITION 3.2 *If $K_1 = r$ is sufficiently small, there is only a finite number of eigenvalues of the system* (3.35) *in the set V_1 that we have defined above.*

PROOF Define:

$$f(\eta) = \eta\cosh(\eta L)\big(-5 + 9\cosh(2\eta L)\big),$$

$$g(\eta) = \widetilde{\Lambda}^{\pm}(\eta) r \sinh(\eta L)\big(1 + 9\cosh(2\eta L)\big),$$

$$h(\eta) = f(\eta) + g(\eta).$$

We have defined $h(\eta)$ in such a way that the roots of $h(\eta)$ are equivalent to solutions to (3.36) with $K_1 = r$ and the roots of $f(\eta)$ are equivalent to solutions to (3.40), that is

$$\{0\} \cup \{\eta_{j,1} | \eta_{j,1} = \frac{1}{L}(\arctan\sqrt{\frac{2}{7}} + j\pi)\mathrm{i}\}_{j\in\mathbb{Z}} \cup \{\eta_{j,2} | \eta_{j,2} = (\frac{j\pi + \frac{\pi}{2}}{L})\mathrm{i}\}_{j\in\mathbb{N}}.$$

Let $T = S(0, R_2) = \{\eta | -R_2 \le Re(\eta) \le R_2, -R_2 \le Im(\eta) \le R_2\}$, with Fig. 2 showing the range of T.

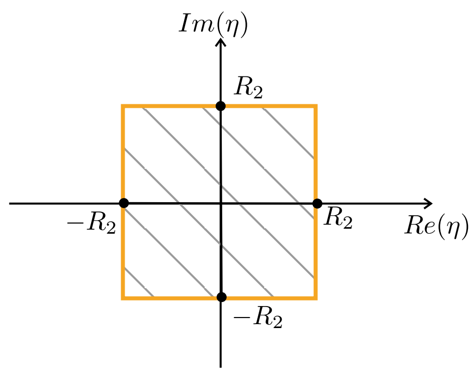


Figure 3.2: The range of $T = S(0, R_2)$

$$R_2 = \begin{cases} R_0 + q & if\ there\ exists\ j, l\ such\ that\ |\eta_{j,l}| = R_0, \\ R_0 & else. \end{cases}$$

We take $q > 0$ as an arbitrary small real number that makes sure $|\eta_{j,l}| \ne R_1 + q$ for any j, l. Define

$$\partial T_1 = \{\eta | Re(\eta) = \pm R_2, -R_2 \le Im(\eta) \le R_2\},$$

$$\partial T_2 = \{\eta | Im(\eta) = \pm R_2, -R_2 \le Re(\eta) \le R_2\}.$$

We have the following estimation on the boundary ∂T_1:

$$\begin{aligned} |f(\eta)| =& |\eta| \cdot |\cosh(\eta L)| \cdot |-5 + 9\cosh(2\eta L)| \\ \ge& \min_{\eta\in\partial T_1}\{|\eta|\} \cdot \min_{\eta\in\partial T_1}\{|\cosh(\eta L)|\} \cdot \min_{\eta\in\partial T_1}\{|-5 + 9\cosh(2\eta L)|\} \\ \ge& R_2 \sinh(R_2 L)\big(-5 + 9\cosh(2R_2 L)\big) > 0, \qquad \forall \eta \in \partial T_1. \end{aligned}$$

And

$$\begin{aligned}|g(\eta)| =& r \cdot |\widetilde{\Lambda}^{\pm}(\eta)| \cdot |\sinh(\eta L)| \cdot |1 + 9\cosh(2\eta L)| \\ \leq& |r| \max_{\eta \in \partial T_1} \{|\widetilde{\Lambda}^{\pm}(\eta)|\} \cdot \max_{\eta \in \partial T_1} \{|\sinh(\eta L)|\} \cdot \max_{\eta \in \partial T_1} \{|1 + 9\cosh(2\eta L)|\} \\ \leq& |r|(\sqrt{2R_2^2 + c^2} + \varepsilon)\cosh(2R_2 L)\big(1 + 9\cosh(2R_2 L)\big) \\ <& 10|r|(\sqrt{2R_2^2 + c^2} + \varepsilon)\cosh^2(2R_2 L), \qquad \forall \eta \in \partial T_1.\end{aligned}$$

Similarly, we have the following estimation on the boundary ∂T_2:

$$\begin{aligned}|f(\eta)| =& |\eta| \cdot |\cosh(\eta L)| \cdot |-5 + 9\cosh(2\eta L)| \\ \geq& \min_{\eta \in \partial T_2} \{|\eta|\} \cdot \min_{\eta \in \partial T_2} \{|\cosh(\eta L)|\} \cdot \min_{\eta \in \partial T_2} \{|-5 + 9\cosh(2\eta L)|\} \\ \geq& R_2|\cos(R_2 L)\big(-5 + 9\cos(2R_2 L)\big)| > 0, \qquad \forall \eta \in \partial T_2.\end{aligned}$$

And

$$\begin{aligned}&|g(\eta)| = |r| \cdot |\widetilde{\Lambda}^{\pm}(\eta)| \cdot |\sinh(\eta L)| \cdot |1 + 9\cosh(2\eta L)| \\ &\leq |r| \max_{\eta \in \partial T_2} \{|\widetilde{\Lambda}^{\pm}(\eta)|\} \cdot \max_{\eta \in \partial T_2} \{|\sinh(\eta L)|\} \cdot \max_{\eta \in \partial T_2} \{|1 + 9\cosh(2\eta L)|\} \\ &\leq |r|(\sqrt{2R_2^2 + c^2} + \varepsilon)\sqrt{\cosh^2(2R_2 L) + \sinh^2(2R_2 L)} \times \\ &\times \sqrt{\big(1 + 9\cosh(2R_2 L)\big)^2 + 81\sinh^2(2R_2 L)} \\ &< 20|r|(\sqrt{2R_2^2 + c^2} + \varepsilon)\cosh^2(2R_2 L), \qquad \forall \eta \in \partial T_2.\end{aligned}$$

Taking

$$q_1 = \min\{R_2 \sinh(R_2 L)\big(-5+9\cosh(2R_2 L)\big), R_2|\cos(R_2 L)\big(-5+9\cos(2R_2 L)\big)|\},$$

$$|f(\eta)| > q_1, \forall \eta \in \partial T.$$

Now, taking $r_1 < \frac{q_1}{20(\sqrt{2R_2^2+c^2}+\varepsilon)\cosh^2(2R_2 L)}$, if $|r| < r_1$:

$$|g(\eta)| < 20|r|(\sqrt{2R_2^2 + c^2} + \varepsilon)\cosh^2(2R_2 L) < q_1, \quad \forall \eta \in \partial T,$$

which means

$$|f(\eta)| > q_1 > |g(\eta)| = |h(\eta) - f(\eta)|, \forall \eta \in \partial T.$$

Notice that ∂T is a closed, simple curve (i.e. not self-intersecting). Using Lemma 2.3, we obtain that f and h have the same number of roots in T.

From the above analysis, there are finite numbers of roots of $f(\eta)$ in T, thus, there are finite numbers of roots of $h(\eta)$ in T. For fixed $\widetilde{\eta}$, there are no more than two numbers $\widetilde{\lambda}(\widetilde{\Lambda}_{j,l}^{\pm}(\eta_{j,l}))$ that satisfy (3.37).

Recall that $\{\widetilde{\eta} \mid |\widetilde{\eta}| < R_0, \widetilde{\eta}$ is the solution of (3.36)$\} \subset T$ and, by the definition of V_1, we obtain the result. ■

For every $\eta_{j,l} \in T$, we take a sufficiently small neighbourhood $V_{j,l}$ of $\eta_{j,l}$, and upon making the same analysis as in Proposition 3.2, we could obtain only one root $\widetilde{\eta}_{j,l}$ of $h(\eta)$ in $V_{j,l}$. With sufficiently small r ($|r| < r_{j,l}$), its corresponding $\widetilde{\lambda}^{\pm}_{j,l}(\widetilde{\Lambda}^{\pm}_{j,l}(\eta_{j,l}))$ still lie on the left half plane. Moreover, observe that f and h have the same number of roots in T, these $\widetilde{\eta}_{j,l}$ are all roots of $h(\eta)$ in T:

$$\cup_{\{j,l|\eta_{j,l}\in T\}}\{\eta \in V_{j,l}|h(\eta) = 0\} \cup \{0\} = \{\eta \in T|h(\eta) = 0\}.$$

From Proposition 3.2, there is a finite number of complex numbers $\eta_{j,l}$ in T, so we have finite numbers of $r_{j,l}$. Taking $|r| < \min\limits_{\{j,l|\eta_{j,l}\in T\}}\{r_1, r_{j,l}\}$, we have:

$$V_1 \subset \{\lambda|Re(\lambda) < 0\}.$$

We then make an analysis of the eigenvalues in V_2. Denote:

$$F(\widetilde{\eta}) = \frac{\widetilde{\Lambda}^{\pm}(\eta)K_1}{\widetilde{\eta}} - \frac{2 + 7\tanh^2(\widetilde{\eta}L)}{\tanh(\widetilde{\eta}L)\big(5 + 4\tanh^2(\widetilde{\eta}L)\big)}.$$

$$G(\widetilde{\eta}) = K_1 + \frac{2 + 7\tanh^2(\widetilde{\eta}L)}{\tanh(\widetilde{\eta}L)\big(5 + 4\tanh^2(\widetilde{\eta}L)\big)}.$$

Define $F(\eta)$ in such a way that the roots of $F(\widetilde{\eta})$ are equivalent to solutions to (3.36).

While $|\widetilde{\eta}| > R_0$ and $F(\widetilde{\eta}) = 0$:

$$|G(\widetilde{\eta})| \le |F(\widetilde{\eta})| + |F(\widetilde{\eta}) - G(\widetilde{\eta})| = |\frac{(\widetilde{\Lambda}^{\pm}(\eta) - \widetilde{\eta})K_1}{\eta}| < \frac{\varepsilon_1 K_1}{R_2}.$$

Without loss of generality, we suppose that $K_1 = r < 1, R_2 > 1$, then solve $|G(\widetilde{\eta})| < \varepsilon_1$.

- If $\lim\limits_{\eta\to\widetilde{\eta}} \tanh(\eta L) = \infty$, that is $\widetilde{\eta} = \frac{\frac{\pi}{2}+k\pi}{L}\mathrm{i}$ ($k \in \mathbb{Z}$), we have

$$G(\widetilde{\eta}) = r.$$

- If $\tanh(\widetilde{\eta}L) = \pm\sqrt{\frac{2}{7}}\mathrm{i}$, that is $\widetilde{\eta} = \frac{(\pm\arctan\sqrt{\frac{2}{7}}+2k\pi)}{L}\mathrm{i}$ ($k \in \mathbb{Z}$), we have

$$G(\widetilde{\eta}) = r.$$

Let $\tanh(\widetilde{\eta}L) = y$ and denote $g_1(y) := \frac{2+7y^2}{5y+4y^3}$. The inequality yields:

$$|g_1(y) + r| < \varepsilon_1. \tag{3.41}$$

Notice that the roots of the equation $g_1(y) = 0$ are $y_1 = \infty$, $y_2 = \sqrt{\frac{2}{7}}\mathrm{i}$, $y_3 = -\sqrt{\frac{2}{7}}\mathrm{i}$.

From the properties of the cubic polynomial, using the implicit function theorem, we obtain that for any $\varepsilon_2 > 0$, there exists δ_1 such that

$$\{y \in \mathbb{C} | |g_1(y)| \le \delta_1\} \subset B(y_1, \varepsilon_2) \cup B(y_2, \varepsilon_2) \cup B(y_3, \varepsilon_2).$$

We now analyze $y(s) = \tanh(s)$ near $y = y_1, y_2, y_3$:

- Let $\varepsilon_2 < \frac{\sqrt{\frac{2}{7}}}{2}$ be given. If $y(s) \in B(y_2, \varepsilon_2) \cup B(y_3, \varepsilon_2)$, define $s_2 = \arctan\Big(\sqrt{\frac{2}{7}}\Big)\mathrm{i}$, $s_3 = -\arctan\Big(\sqrt{\frac{2}{7}}\Big)\mathrm{i}$ that satisfy $ys_i) = y_i$.
  Then we have

$$\begin{aligned} |\tanh(s - s_i)| &= \frac{|\tanh(s) - \tanh(s_i)|}{|1 - \tanh(s)\tanh(s_i)|} \le \frac{\varepsilon_2}{1 - |\tanh(s)||\tanh(s_1)|} \\ &\le \frac{\varepsilon_2}{1 - \frac{1}{7}} \le \varepsilon_2. \end{aligned}$$

- If $y(s) \in B(y_1, \varepsilon_2)$, we define $s_1 = \pi\mathrm{i}$ that satisfies $y(s_i) = y_i$. So, we have

$$|\tanh(s - s_1)| = \frac{|\tanh(s) - \tanh(s_1)|}{|1 - \tanh(s)\tanh(s_1)|} \le \frac{\varepsilon_2}{|\tanh(s)||\tanh(s_1)| - 1} \le \varepsilon_2.$$

We obtain the following estimation:

Proposition 3.3 *If $\varepsilon_2 < 1$, $z \in \mathbb{C}$ satisfies $|\tanh(z)| \le \varepsilon_2$, then we have:*

$$\begin{cases} z_1 \in (-\tanh^{-1}(\sqrt{2}\varepsilon_2), \tanh^{-1}(\sqrt{2}\varepsilon_2)), \\ z_2 \in \big(-\arctan\big(\sqrt{2}\varepsilon_2\big) + k\pi, \arctan\big(\sqrt{2}\varepsilon_2\big) + k\pi\big) \ (k \in \mathbb{Z}), \end{cases}$$

where $\tanh^{-1}(x)$ *is the inverse function of* $\tanh(x)$.

Proof If $z = z_1 + z_2\mathrm{i} \in \mathbb{C}$ $(z_1, z_2 \in \mathbb{R})$ satisfies $|\tanh(z)| < \varepsilon_2$,

$$\tanh(z_1 + z_2\mathrm{i}) = \frac{\tanh(z_1) + \tan(z_2)\mathrm{i}}{1 + \tanh(z_1)\tan(z_2)\mathrm{i}},$$

$$|\tanh(z_1 + z_2\mathrm{i})|^2 = \frac{\tanh(z_1)^2 + \tan(z_2)^2}{1 + \big(\tanh(z_1)\tan(z_2)\big)^2} \le \varepsilon_2^2.$$

Suppose $\varepsilon_2 < 1$, if $|\tan(z_2)| > 1$,

$$|\tanh(z_1 + z_2\mathrm{i})|^2 \ge \frac{\tan(z_2))^2}{1 + \big(\tanh(z_1)\tan(z_2)\big)^2} \ge \tan(z_2))^2 > 1.$$

This contradicts $|\tanh(z)| < \varepsilon_2$, so we have $|\tan(z_2)| \leq 1$. Then we have

$$\tanh(z_1)^2 + \tan(z_2)^2 \leq \varepsilon_2^2\big(1 + (\tanh(z_1)\tan(z_2))^2\big) \leq 2\varepsilon_2^2,$$

which means that

$$z_1 \in \big(-\tanh^{-1}(\sqrt{2}\varepsilon_2), \tanh^{-1}(\sqrt{2}\varepsilon_2)\big),$$
$$z_2 \in \Big(-\arctan\Big(\sqrt{2}\varepsilon_2\Big) + k\pi, \arctan\Big(\sqrt{2}\varepsilon_2\Big) + k\pi\Big)\ (k \in \mathbb{Z}).$$

∎

For brevity, denote $A_1 = \tanh^{-1}(\sqrt{2}\varepsilon_2)$, $A_2 = \arctan\Big(\sqrt{2}\varepsilon_2\Big)$, then we have:

$$\begin{aligned}&\{s \in \mathbb{C}||g_1(\tanh(s))| \leq \delta_1\}\\ \subset&\{s \in \mathbb{C}|y(s) \in B(y_1, \varepsilon_2) \cup B(y_2, \varepsilon_2) \cup B(y_3, \varepsilon_2)\} \subset \{s|| \tanh(s - s_i)| \leq \varepsilon_2\}\\ \subset&\{s_i + z_1 + z_2\mathrm{i}|z_1 \in (-A_1, A_1), z_2 \in (-A_2 + k\pi, A_2 + k\pi)\ , k \in \mathbb{Z}, i = 1, 2, 3\}.\end{aligned}$$

Substituting $s = \widetilde{\eta}L$, and supposing that $|r| \leq \frac{\delta_1}{4}$, we have:

$$\begin{aligned}&\{\widetilde{\eta}||G(\widetilde{\eta})| < \frac{\delta_1}{2}\} \subset \{\widetilde{\eta}||g_1(\tanh(\widetilde{\eta}L)) \leq \delta_1\} \subset\\ &\quad\{\widetilde{\eta}|y(\widetilde{\eta}L) \in B(y_1, \varepsilon_2) \cup B(y_2, \varepsilon_2) \cup B(y_3, \varepsilon_2)\}\\ &\subset \{\frac{s_i}{L} + z_1 + z_2\mathrm{i}|z_1 \in (-\frac{A_1}{L}, \frac{A_1}{L}), z_2 \in (-\frac{A_2}{L} + k\frac{\pi}{L}, \frac{A_2}{L} + k\frac{\pi}{L}),\\ &k \in \mathbb{Z}, i = 1, 2, 3\}.\end{aligned} \tag{3.42}$$

Recall (3.37), if $\widetilde{\eta} = \eta_1 + \eta_2\mathrm{i}$:

$$Re(\widetilde{\lambda}) = -\varepsilon \pm \sqrt{\frac{\eta_1^2 - \eta_2^2 + c^2 + \sqrt{(\eta_1^2 - \eta_2^2 + c^2)^2 + 4\eta_1^2\eta_2^2}}{2}}.$$

$Re(\widetilde{\lambda}) < 0$ is equivalent to

$$\frac{\eta_1^2 - \eta_2^2 + c^2 + \sqrt{(\eta_1^2 - \eta_2^2 + c^2)^2 + 4\eta_1^2\eta_2^2}}{2} < \varepsilon^2,$$

that is

$$\eta_1^2\eta_2^2 + (\eta_1^2 - \eta_2^2 + c^2)\varepsilon^2 < \varepsilon^4. \tag{3.43}$$

Recall (3.3), we know that $\eta_1 \in (-\frac{A_1}{L}, \frac{A_1}{L})$, and so, if we take sufficiently small ε_2 such that $\frac{A_1}{L} < \frac{\varepsilon}{2}$, then (3.43) can be written as follows:

$$\eta_2^2 \geq \frac{(c^2 - \varepsilon^2)\varepsilon^2 + \eta_1^2\varepsilon^2}{\varepsilon^2 - \eta_1^2}.$$

Observing that:

$$\frac{(c^2-\varepsilon^2)\varepsilon^2+\eta_1^2\varepsilon^2}{\varepsilon^2-\eta_1^2}=\frac{c^2\varepsilon^2}{\varepsilon^2-\eta_1^2}-\varepsilon^2\leq c^2-\varepsilon^2$$

if $|\widetilde{\eta}|^2>c^2-\frac{3}{4}\varepsilon^2$, we obtain

$$\eta_2^2=|\widetilde{\eta}|^2-\eta_1^2>c^2-\frac{3}{4}\varepsilon^2-\frac{1}{4}\varepsilon^2=c^2-\varepsilon^2.$$

The corresponding $Re(\widetilde{\lambda})<0$.

We take sufficiently small $\varepsilon_2<1$ such that $\frac{A_1}{L}\leq\frac{\varepsilon}{2}$, then we have the corresponding δ_1.

Taking $\varepsilon_1=\frac{\delta_1}{2}$, $|r|<\frac{\varepsilon_1}{2}$, $R_0=\max\{\frac{4}{3}c^2-\frac{5}{12}\varepsilon^2,R_1\}$. We obtain that if $|\widetilde{\eta}|>R_0$ and $F(\widetilde{\eta})=0$, the real part

$$Re(\widetilde{\lambda})<0.$$

In conclusion, while $|r|<\min\{\frac{\delta_1}{4},r_1,r_{j,l}\}$, the system (1.1) is exponentially stable. ∎

4. Results on instability

In this section, we prove that for sufficiently large lengths of the edges and for any $K_1\in\mathbb{R}$, there exists an eigenvalue that lies in the right half plane, and thus system (1.1) with $c_k=c_1=c$, $\varepsilon_k=\varepsilon_1=\varepsilon$ and $L_k=L_1=L$ cannot be L^2- exponentially stable.

Proposition 4.1 *While $L\geq L_{max}=\frac{\pi}{2\sqrt{c^2-\varepsilon^2}}$, $c_k=c_1=c>0$, $\varepsilon_k=\varepsilon_1=\varepsilon\in[0,c)$ and $L_k=L_1=L$, for any $K_1\in\mathbb{R}$, the system* (1.1) *is not L^2-exponentially stable.*

Proof Let $\lambda\in\mathbb{C}$. We look for a nontrivial solution $U(t,x)=(u^1(t,x),...,u^4(t,x))$ of the system having the form $u^k(t,x)=e^{\lambda t}\varphi_k(x)$, with the eigenvalue λ of the system. The corresponding eigenfunctions of the $u^k(t,x)$ are $\varphi_k(x)$.

Such a $U(t,x)$ can only be a solution of the system if

$$(\lambda+\varepsilon)^2\,\varphi_k=\varphi_k''+c^2\varphi_k. \tag{4.1}$$

From (4.1), we have $\varphi_k(x)=R_{1,k}e^{\eta x}+R_{2,k}e^{-\eta x}$ and

$$\eta^2=(\lambda+\varepsilon)^2-c^2. \tag{4.2}$$

Similarly as in the analysis before, using the boundary conditions, we get the following characteristic equation:

$$\Big[\eta\cosh(\eta L)\big(-5+9\cosh(2\eta L)\big)\Big]+\Big[\lambda K_1\sinh(\eta L)\big(1+9\cosh(2\eta L)\big)\Big]=0. \quad (4.3)$$

We want to discuss the solution $\eta = \omega\mathrm{i}$ ($\omega \in \mathbb{R}$) on the imaginary axis. The real part of the corresponding λ is greater than 0 if and only if $\omega \in (-\sqrt{c^2-\varepsilon^2}, \sqrt{c^2-\varepsilon^2})$. Moreover,

$$\lambda = -\varepsilon + \sqrt{c^2-\omega^2}.$$

We can rewrite the characteristic equation:

$$\begin{aligned}-K_1 = \frac{\omega}{-\varepsilon+\sqrt{c^2-\omega^2}}\big(\cot(\omega L)\frac{-5+9\cos(2\omega L)}{1+9\cos(2\omega L)}\big) = \\ \frac{\omega}{-\varepsilon+\sqrt{c^2-\omega^2}} \times \frac{2-7\tan^2(\omega L)}{5-4\tan^2(\omega L)}\cot(\omega L).\end{aligned} \quad (4.4)$$

For $\omega \in (-\sqrt{c^2-\varepsilon^2}, \sqrt{c^2-\varepsilon^2})$, we define:

$$\begin{aligned}F(\omega) &= \frac{\omega}{-\varepsilon+\sqrt{c^2-\omega^2}} \times \frac{2-7\tan^2(\omega L)}{5-4\tan^2(\omega L)}\cot(\omega L),\\ f(\omega) &= \frac{\omega}{\sqrt{c^2-\omega^2}-\varepsilon},\\ g(\omega) &= \frac{2-7\tan^2(\omega L)}{5-4\tan^2(\omega L)}\cot(\omega L).\end{aligned}$$

Figure 3 shows the graphs of f and g.

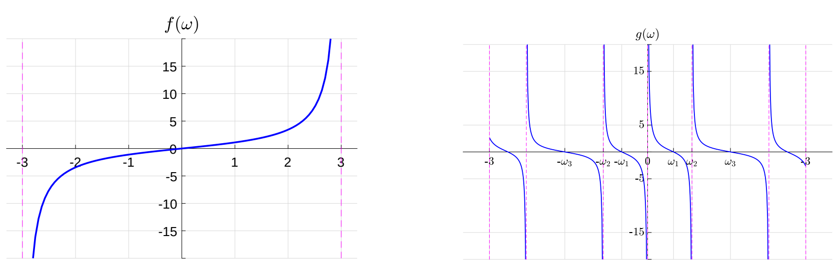


Figure 3: $c = 5,\ \varepsilon = 4,\ L = 1,\ \sqrt{c^2-\varepsilon^2} = 3$

If we can prove that the range of $F(\omega)$ covers $\mathbb{R}$, the system has an eigenvalue $\lambda > 0$ for any $K_1 \in \mathbb{R}$. Thus, the system cannot be exponentially stable for any $K_1 \in \mathbb{R}$.

We have $f'(\omega) = \frac{\frac{c^2}{\sqrt{c^2-\omega^2}}-\varepsilon}{(\sqrt{c^2-\omega^2}-\varepsilon)^2} > 0,\ g'(\omega) = -\frac{28\tan^4(\omega L)+11\tan^2(\omega L)+10}{\sin^2(\omega L)(4\tan^2(\omega L)-5)^2}L < 0.$

We now discuss the range of $F(\omega)$, firstly,

$$\lim_{\omega\to 0^+} F(\omega) = \lim_{\omega\to 0^+} f(\omega)g(\omega) = \frac{f'(\omega)}{(1/g(\omega))'} = \frac{\frac{1}{c-\varepsilon}}{5/2} = \frac{2}{5(c-\varepsilon)},$$

$$\lim_{\omega\to\sqrt{c^2-\varepsilon^2}-} F(\omega) = \lim_{\omega\to\sqrt{c^2-\varepsilon^2}-} f(\omega)g(\omega) = \infty\times g(\sqrt{c^2-\varepsilon^2}) = \infty.$$

As long as $\sqrt{c^2-\varepsilon^2} \geq \frac{\pi}{2L}$, we define the set of discontinuity points and roots of $F(\omega)$ in $[0,\sqrt{c^2-\varepsilon^2})$ as χ. We obtain

$$\{\omega_1,\omega_2,\omega_3\} \subset \chi,$$

that is, the set χ contains at least three elements: $\omega_1 = \frac{\arctan\sqrt{\frac{2}{7}}}{L}$, $\omega_2 = \frac{\arctan\sqrt{\frac{5}{4}}}{L}$, $\omega_3 = \frac{\pi}{2L}$ in χ . Besides,

$$\lim_{\omega\to\omega_1} F(\omega) = \lim_{\omega\to\omega_1} f(\omega)g(\omega) = f(\omega_1)\times 0 = 0,$$

$$\lim_{\omega\to\omega_2^-} F(\omega) = \lim_{\omega\to\omega_2^-} f(\omega)g(\omega) = f(\frac{\arctan\sqrt{\frac{5}{4}}}{L})\times(-\infty) = -\infty,$$

$$\lim_{\omega\to\omega_2^+} F(\omega) = \lim_{\omega\to\omega_2^+} f(\omega)g(\omega) = f(\frac{\arctan\sqrt{\frac{5}{4}}}{L})\times\infty = \infty,$$

$$\lim_{\omega\to\omega_3^-} F(\omega) = \lim_{\omega\to\omega_3^-} f(\omega)g(\omega) = f(\frac{\pi}{2L})\times 0 = 0.$$

From the continuity of $F(\omega)$, the range of $F(\omega)$ covers $\mathbb{R}$ in the interval $[\omega_1,\omega_2]\cup[\omega_2,\omega_3]$.

In conclusion, while $L \geq L_{max} = \frac{\pi}{2\sqrt{c^2-\varepsilon^2}}$, for any $K_1\in\mathbb{R}$, the system has a real eigenvalue that is greater than 0, so it cannot be stabilized. ■

Remark 4.1 *We have the following estimate for $L < L_{max} = \frac{\pi}{2\sqrt{c^2-\varepsilon^2}}$:*

- *If $\sqrt{c^2-\varepsilon^2} < \frac{\arctan\sqrt{\frac{2}{7}}}{L}$, i.e. $L < \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}}$, from Proposition 3.1, $K_1 = 0$ can stabilize the system.*
- *If $\frac{\arctan\sqrt{\frac{2}{7}}}{L} < \sqrt{c^2-\varepsilon^2} < \frac{\arctan\sqrt{\frac{5}{4}}}{L}$, the set of discontinuity points and roots is equal to $\{\omega_1\}$. From the continuity of $F(\omega)$, the range of $F(\omega)$ is $(-\infty, C_1)$,*

$$C_1 := \inf_{\omega\in(0,\omega_1)} F(\omega) < \infty.$$

Hence, the range of $F(\omega)$ does not cover $\mathbb{R}$.

- *If* $\frac{\arctan\sqrt{\frac{5}{4}}}{L} < \sqrt{c^2-\varepsilon^2} < \frac{\pi}{2L}$*, the set of discontinuity points and roots is equal to* $\{\omega_1, \omega_2\}$*. From the continuity of* $F(\omega)$*, the range of* $F(\omega)$ *is* $(-\infty, C_1) \cup (C_2, \infty)$,

$$C_1 := \inf_{\omega\in(0,\omega_1)} F(\omega) < \infty, \quad C_2 := \sup_{\omega\in(\omega_2,\sqrt{c^2-\varepsilon^2})} F(\omega) > 0.$$

We try to make a simulation of $F(\omega)$ for a definite case.

Taking $c = 5,\ \varepsilon = 4,\ L = 0.5$, we obtain the following Fig. 4 for the graph of $F(\omega)$. It shows that the range of $F(\omega)$ does not cover $\mathbb{R}$ in this case ($C_1 < C_2$).

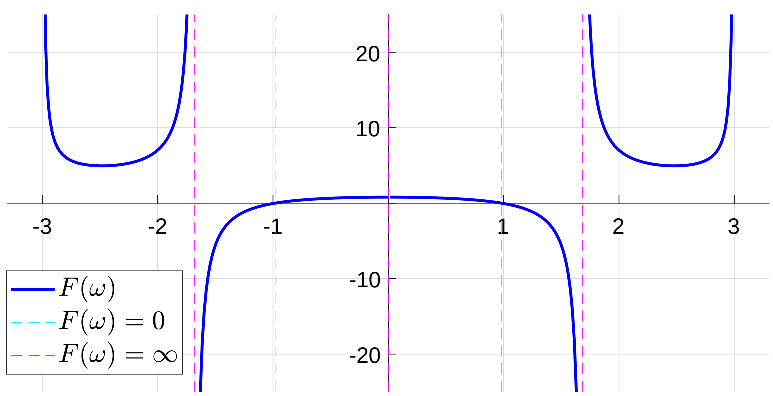


Figure 4: $F(\omega)$ with $c = 5,\ \varepsilon = 4,\ L = 0.5$

Thus, while $L_{min} < L < L_{max}$, we cannot prove that for any control K_1, the corresponding system has an eigenvalue bigger than 0. However, the numerical results indicate that the system can probably not be stabilized by some K_1 (see Section 6, Example 2).

5. Examples

From the stability result in Section 3, we can obtain an explicit expression of the solutions if $K_1 = 0$ and all arcs have the same length. So, in this section, we give some results of the explicit solution.

5.1. Illustration of the eigenfunctions

In this subsection, we present the figures of some of the eigenfunctions in system (3.1) generated with MATLAB. We take $\varepsilon = \pi, c = \sqrt{1.01}\varepsilon, L = 1$, from (3.8) (3.9) in Proposition 3.1 we can obtain the eigenvalues and the corresponding eigenfuctions. The following figures (Fig. 5) show the eigenfunctions corresponding to the eigenvalues

$$\begin{aligned}
\lambda_{0,1}^{-} &= -\pi + \sqrt{1.01\pi^2 - \arctan^2\sqrt{\frac{2}{7}}}, & \lambda_{0,2}^{-} &= (-1+\sqrt{0.76})\pi,\\
\lambda_{1,1}^{-} &= -\pi + \sqrt{-1.01\pi^2 + (\arctan\sqrt{\frac{2}{7}}+\pi)^2\mathrm{i}}, & \lambda_{1,2}^{-} &= -\pi + \sqrt{1.24}\pi\mathrm{i},\\
\lambda_{2,1}^{-} &= -\pi + \sqrt{-1.01\pi^2 + (\arctan\sqrt{\frac{2}{7}}+2\pi)^2\mathrm{i}}, & \lambda_{2,2}^{-} &= -\pi + \sqrt{5.24}\pi\mathrm{i}.
\end{aligned}$$

5.2. The time–wise evolution of the state

In this subsection, we present solutions of system (3.1) generated with MATLAB for some special initial values. We also give some figures which show the evolution of the L^2-norm energy over time.

We take $\varepsilon = \pi$, $c = \sqrt{1.01}\varepsilon$, and from Theorem 1.1 we obtain the critical length

$$L_0 = \frac{\arctan\sqrt{\frac{2}{7}}}{\sqrt{c^2-\varepsilon^2}} \approx 1.5625.$$

While $L < L_0$, the system is exponentially stable, and while $L > L_0$, the system is not exponentially stable. We take $L_1 = 1 < L_0$ and $L_2 = 2 > L_0$ with the same initial value (IV1),

$$\begin{cases} u^k(0,x) = \mathrm{Im}\big(\sum\limits_{0\le j\le 49} \varphi_k^{j,1}(x) + \varphi_k^{j,2}(x)\big), \\ u_t^k(0,x) = \mathrm{Im}\big(\sum\limits_{0\le j\le 49} \varphi_k^{j,1}(x) + \varphi_k^{j,2}(x)\big), \quad k \in \{1,2,3,4\}, \end{cases} \tag{IV1}$$

i.e. $c_{m,j,l} = \begin{cases} 1, & 0 \le j \le 49,\ m = 1,2,\ l = 1,2 \\ 0, & else \end{cases}$, $\varphi_k^{j,l}(x)$ is defined before in (3.9).

From (3.10) in Section 3, we obtain that the explicit solution of the system (3.1) is

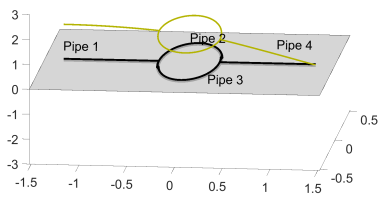


Figure 5.1.: The eigenfunction $\varphi^-_{0,1}(x)$

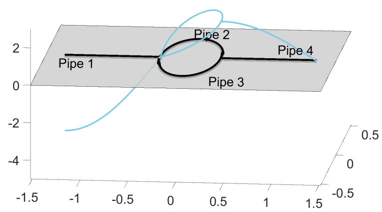


Figure 5.2: The eigenfunction $\varphi^-_{0,2}(x)$

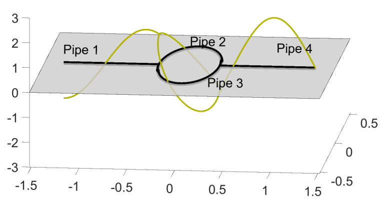


Figure 5.3: The eigenfunction $\varphi^-_{1,1}(x)$

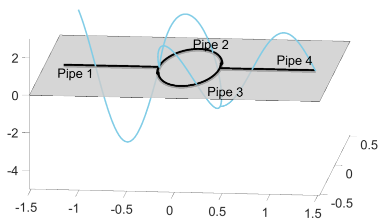


Figure 5.4: The eigenfunction $\varphi^-_{1,2}(x)$

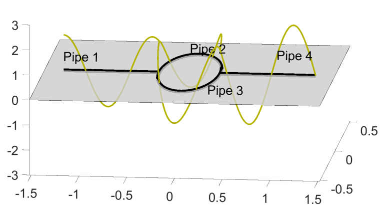


Figure 5.5: The eigenfunction $\varphi^-_{2,1}(x)$

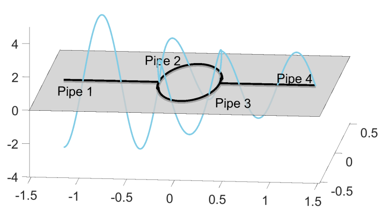


Figure 5.6: The eigenfunction $\varphi^-_{2,2}(x)$

$$u^k(t,x) = \sum_{l=1,2} \mathrm{Im}\Big[\sum_{j=0}^{49} \Big(\frac{-\lambda^+_{m,1}}{\lambda^+_{m,1}-\lambda^-_{m,1}} e^{\lambda^+_{m,1}t} + \frac{\lambda^-_{m,1}}{\lambda^+_{m,1}-\lambda^-_{m,1}} e^{\lambda^-_{m,1}t}\Big)\varphi^{j,l}_k(x)\Big],$$
$$k \in \{1,2,3,4\}.$$

The time evolution of the network can be shown in Figs. 6 and 7 generated by MATLAB. The initial data used for Figs. 6 and 7 contain highly oscillatory parts that vanish rather quickly with time.

We also present the variation of the L^2-energy for both two values in Figure 8 ($L=1$) and Figure 9 ($L=2$):

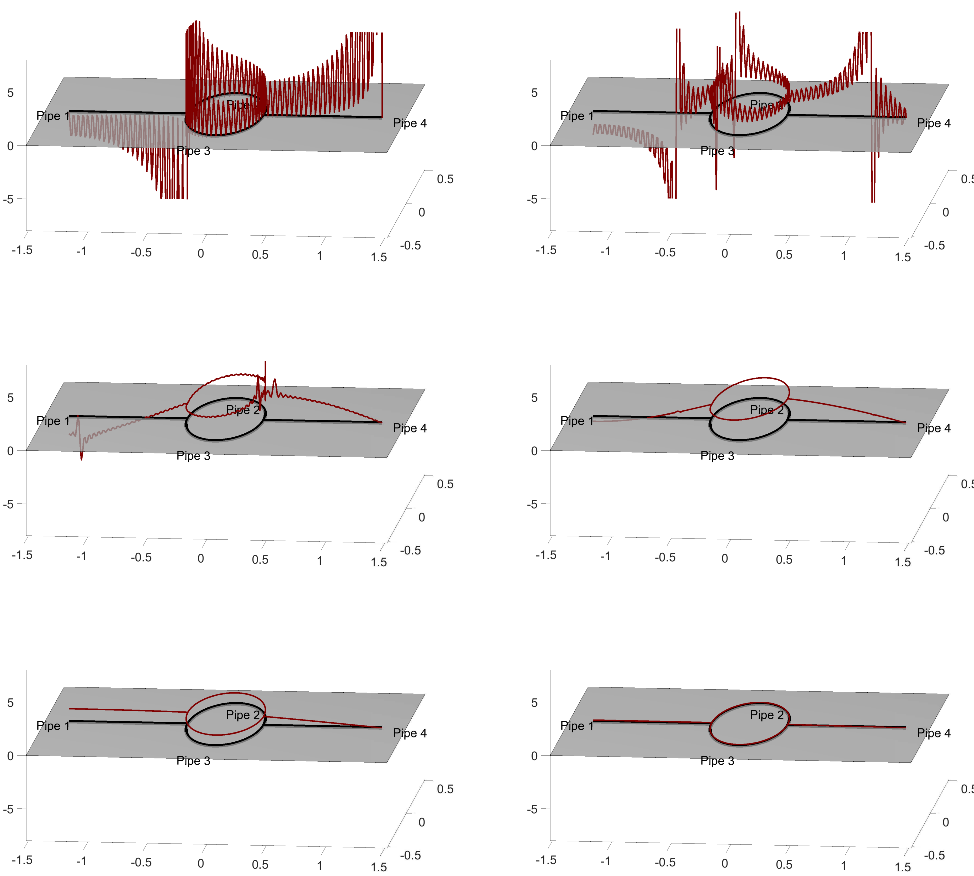


Figure 6: The time evolution (from left to right and then from up to down) of the network with the initial value (IV1) and $\varepsilon = \pi, c = \sqrt{1.01}\pi, L = 1$

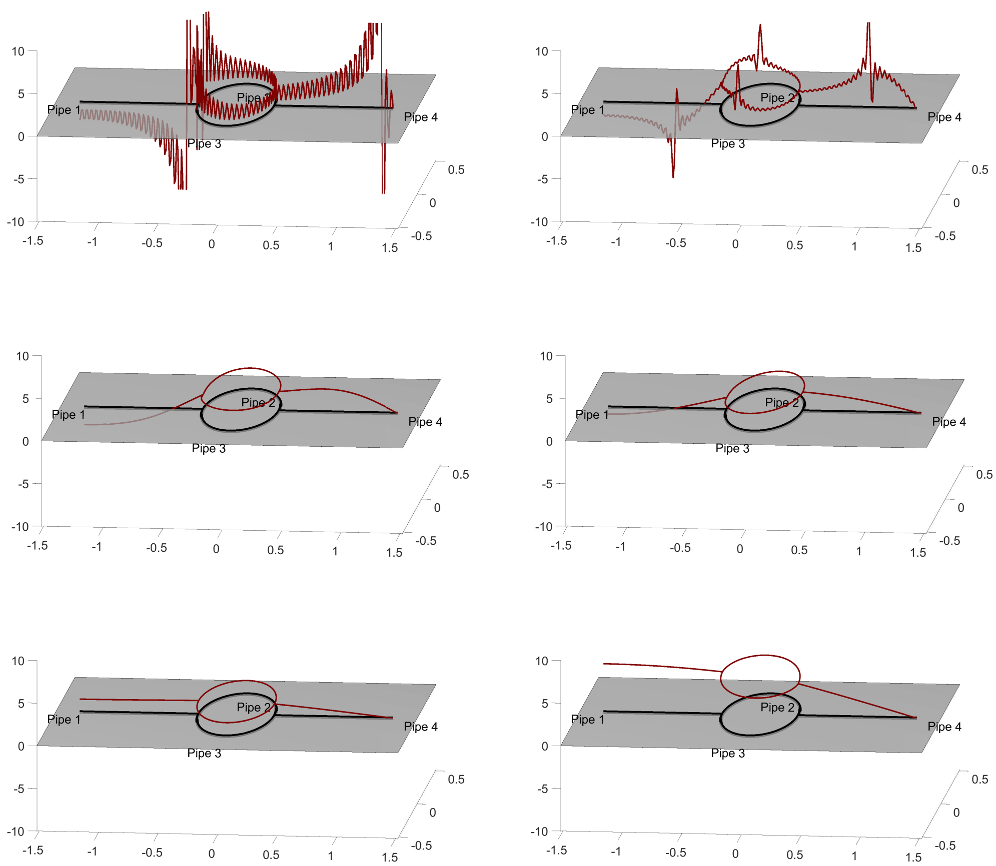


Figure 7: The time evolution (from left to right and then from up to down) of the network with the initial value (IV1) and $\varepsilon = \pi, c = \sqrt{1.01}\pi, L = 2$

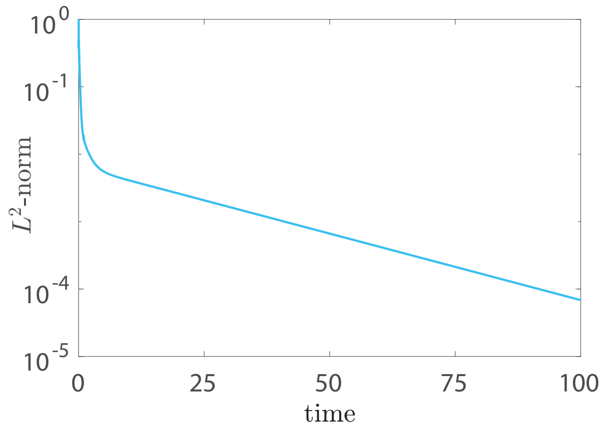


Figure 8: The L^2-energy of the network under the time evolution with the initial value (IV1) $K_1 = 0,\ L = 1,\ T = 100$

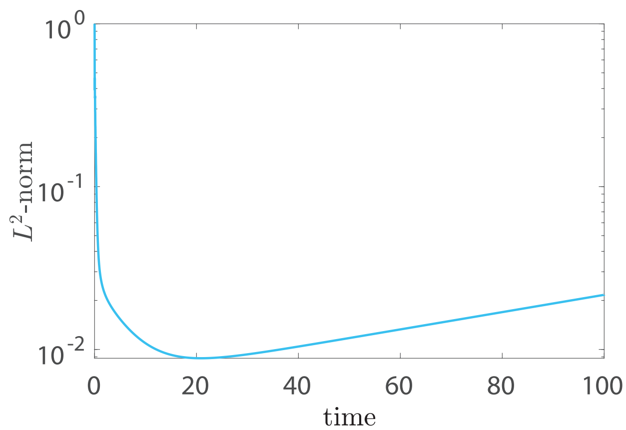


Figure 9: The L^2-energy of the network under the time evolution with the initial value (IV1) $K_1 = 0,\ L = 2,\ T = 100$

6. Simulations

In this section, we present some numerical results generated with MATLAB of the upwind implicit scheme for the system (1.1). Gugat and Gerster in (2019) also use the upwind scheme for simulations for the star-shaped system.

We first use the variable substitution $v^i = -\frac{1}{c_i}(u_x^i + u_t^i + \varepsilon_i u^i)$ to rewrite the system as a 2×2 system:

$$\begin{cases} U_t^i + AU_x^i + B^i U^i = 0, \quad t \in (0, +\infty),\ x \in [0, L_i], i \in \{1,2,3,4\}, \\ u^1(t,0) = u^2(t,0) = u^3(t,0), \\ u^2(t,L_2) = u^3(t,L_3) = u^4(t,L_4), \\ \Sigma_{k=1,2,3} u_x^k(t,0) = 0, \\ \Sigma_{k=2,3,4} u_x^k(t,L_k) = 0, \\ u^4(t,0) = 0, \\ u_x^1(t,L_1) = -K_1 v^1(t,L_1), \end{cases} \tag{6.1}$$

with $U^i = (u^i, v^i)^T,\ A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ B^i = \begin{pmatrix} \varepsilon_i & c_i \\ c_i & \varepsilon_i \end{pmatrix}$.

For numerical illustrations, each arc $[0, L_i]$ is divided into J_i cells by a space discretization $\Delta x > 0$ such that $\Delta x J_i = L_i$ with cell centers $x_j := (j - \frac{1}{2})\Delta x$ for $j = 1, 2, ..J_i$. Ghost cells with centers x_0 and x_{J_i+1} are added outside the domain. The discrete time steps are denoted as $t_k := k\Delta t$ for $k \in \mathbb{N}$ and $\Delta t > 0$ such that the CFL-condition holds. Cell averages at t_k are approximated by

$$u_j^{k,i} \approx \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^i(t_k, x)dx, \quad v_j^{k,i} \approx \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v^i(t_k, x)dx.$$

The advection part can be approximated by the left and right-sided upwind scheme and the reaction part by an implicit Euler step that takes the characteristic speeds into account, i.e.

$$\begin{aligned} u_j^{k+1,i} &= u_j^{k,i} - \frac{\Delta t}{\Delta x}(u_j^{k+1,i} - u_{j-1}^{k+1,i}) - \Delta t(\varepsilon u_j^{k+1,i} + c v_j^{k+1,i}), \\ v_j^{k+1,i} &= v_j^{k,i} + \frac{\Delta t}{\Delta x}(v_{j+1}^{k+1,i} - v_j^{k+1,i}) - \Delta t(c u_j^{k+1,i} + \varepsilon v_j^{k+1,i}). \end{aligned}$$

In order to successfully use the upwind scheme, we use ghost grid point $u_0^{k,i}, v_0^{k,i}, u_{J+1}^{k,i}, v_{J+1}^{k,i}$ to apply the boundary condition:

$$\begin{aligned} & u^i(t_k, 0) = \frac{u_0^{k,i} + u_1^{k,i}}{2},\ \partial_x u^i(t_k, 0) = \frac{u_1^{k,i} - u_0^{k,i}}{\Delta x}, \\ & \partial_t u^i(t_k, 0) = \frac{u_0^{k+1,i} + u_1^{k+1,i} - u_0^{k+1,i} - u_1^{k+1,i}}{2\Delta t}, \\ & u^i(t_k, L) = \frac{u_J^{k,i} + u_{J+1}^{k,i}}{2},\ \partial_x u^i(t_k, L) = \frac{u_{J+1}^{k,i} - u_J^{k,i}}{\Delta x}, \\ & \partial_t u^i(t_k, L) = \frac{u_{J+1}^{k+1,i} + u_J^{k+1,i} - u_{J+1}^{k+1,i} - u_J^{k+1,i}}{2\Delta t}. \end{aligned}$$

All simulations are done in MATLAB. The space discretization $\Delta x = 200$ and the CFL-condition 0.99 are used. For our cases, Theorem 1.1 shows that while $K_1 = 0, \varepsilon = \pi, c = \sqrt{1.01}\pi$, as stated in Section 5, the system is exponentially stable with $L_i = L$ $(i = 1, 2, 3, 4)$.

The time evolution of the network is shown in Fig. 10, generated by MATLAB with the initial value (IV2),

$$\begin{cases} u^1(0,x) = -4\sin\left(\frac{\pi}{2}x\right), \\ u^i(0,x) = 2\sin\left(\frac{\pi}{2}x\right), \quad i \in \{2,3,4\}, \\ u_t^k(0,x) = 0, \quad k \in \{1,2,3,4\}. \end{cases} \tag{IV2}$$

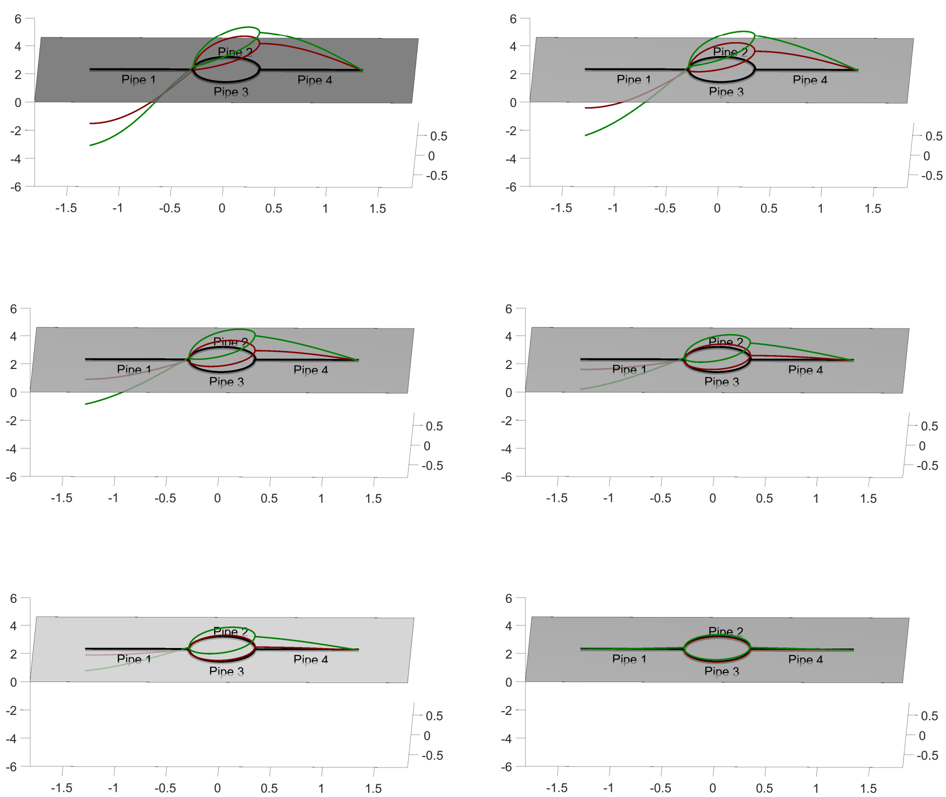


Figure 10: The time evolution of the network with the initial value (IV2)
$\varepsilon = \pi, \quad c = \sqrt{1.01}\pi, \quad L = 1$
Green line: Numerical simulation result Red line: Exact solution

From the figure we can observe that the simulation result of the scheme is quite good. We then present the time evolution of the network concerning the L^2-norm for both stability and instability cases in the following figure. Stability is measured in the L^2-norm

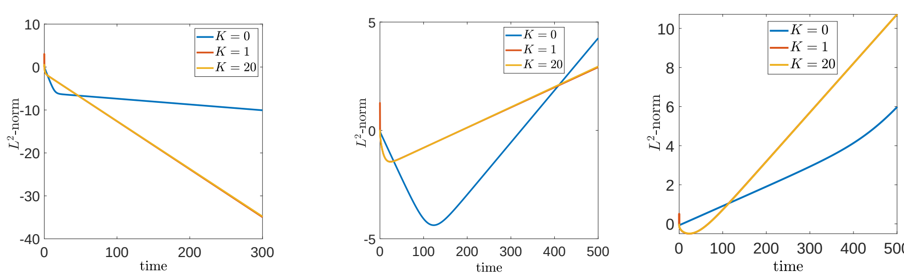
$$L_2(t_k) := \sum_{i=1}^{4} \sum_{j=0}^{J_i+1} [(u_j^{k,i})^2 + (v_j^{k,i})]^2.$$

Example 1

We normalize the initial L^2 energy as 1. We take $\varepsilon = \pi, c = \sqrt{1.01}\pi$, Theorem 1.1 gives us the $L_{min} = 1.5625,\ L_{max} = 5$. The time evolution of the log of L^2-energy of the networks with different length of the arcs can be shown in Figure 6.2 for $K_1 = 0, 1, 20$. We take the initial value:

$$\begin{cases} u^1(0,x) = \sin\left(\frac{\pi x}{L}\right) + \frac{\pi}{L}x, \\ u^2(0,x) = u^3(0,x) = -\sin\left(\frac{\pi x}{L}\right), \\ u^4(0,x) = -\frac{2\pi}{L}x, \\ u_t^k(0,x) = 0, \quad k \in \{1,2,3,4\}. \end{cases} \tag{6.2}$$

The numerical results indicate that if there exists a critical length L_c that



(a) $L_1 = 1 < L_{min}$ (b) $L_2 = 3 \in (L_{min}, L_{max})$ (c) $L_3 = 6 > L_{max}$

Figure 11 The time evolution of the log of L^2-energy with different lengths of the arcs

determines the stabilizability of the system, it is likely that it is equal L_{min}.

Example 2

We take $\varepsilon = 4, c = 5$. Fig. 4 in Remark 4.1 shows that if $L = \frac{1}{2}$, we cannot prove the existence of an eigenvalue λ with positive real part if $K_1 \in (0.8, 5.0)$.

We have tried to make the simulation for $K_1 \in \{0.9, 1, 2, 3, 4, 4.5\}$ with the initial data (6.2).

However, since the five lines, representing the time evolution of the L^2-energy on the network are too close and all increasing, we only present the logarithm of the energy for $K_1 = 3$ in Fig. 12.

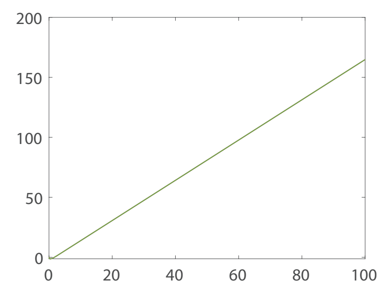


Figure 12: $c = 5,\ \varepsilon = 4,\ L = \frac{1}{2},\ K_1 = 3$

The numerical results show that the system is not exponentially stable even if we cannot theoretically prove there exists an eigenvalue in the right part of the plane.

7. Conclusion

We have discussed the limits of stabilization of a networked hyperbolic system with a circle that is governed by a wave equation with nondissipative source terms depending on the position and the velocity. If the lengths of the arcs are small enough, the system is exponentially stable with the control parameter $|K_1|$ sufficiently small if the arcs in cycle have the same length. Similar to the example presented by Bastin and Coron (2016), the system cannot be exponentially stable for any feedback parameters if the length of arcs is sufficiently large. For stability, we have proven that the lengths of arcs could be slightly different from each other. If we have no restriction on the length of arcs, the complexity of the characteristic equation leads to additional difficulties in spectral analysis.

For our future research we are interested in the existence of the critical length to precisely separate the domains of stability and instability. Moreover, it is interesting to consider more general graphs, for example a cycle made from three edges and three attached single links. This topic is linked to the analysis in Leugering and Sokolowski (2008), where the elliptic case has been considered, see also Gugat, Qian and Sokolowski (2023) for the topological derivative method for control of wave equation on networks. A disadvantage of the spectral approach is that for more complex graphs also the spectral equations become more complicated. Therefore, it would be useful to have a simpler method for the general case, even if it would provide only less precise results. We clearly expect that also for more general graphs there exists a limit of stabilizability.

Another approach to extend the analysis is to allow for additional feedback control applied to the Kirchhoff conditions to improve the stability, similarly as in Avdonin, Edward and Leugering (2023). Since additional feedback control action would improve the stability of the system, we expect that in this case also systems with graphs with several intertwined cycles could be stabilized with suitable feedback parameters at all interior nodes.

Also an analysis of the exponential stability with respect to the H^2-norm would be of interest (see, e.g., Hayat, 2019; Hayat and Shang, 2021): Does this change the critical length where stabilization is impossible compared with the L^2-case?

Acknowledgements

This work was supported by China Scholarship Council (CSC) under the Grant CSC No.20210610098 and Deutsche Forschungsgemeinschaft (DFG) in the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Projects C03 and C05, Projektnummer 239904186 during the visit of Xu Huang to FAU.

References

Avdonin, S., Edward, J. and Leugering, G. (2023) Controllability for the wave equation on graph with cycle and delta-prime vertex conditions. *Evolution Equations and Control Theory*, **12**, 6.

Bastin, G. and Coron, J. M. (2016) Stability and boundary stabilization of 1-D hyperbolic systems. *Progress in Nonlinear Differential Equations and their Applications* **88**. Birkhäuser/Springer, Cham. *Subseries in Control*.

Coron, J. M. (2007) *Control and Nonlinearity. Mathematical Surveys and Monographs* **136**. American Mathematical Society, Providence, RI, 2007.

Evans, L. C. (2010) *Partial Differential Equations. Graduate Studies in Mathematics*, **19** American Mathematical Society, Providence, RI, second edition.

Fritzsche, K. and Grauert, H. (2002) *From Holomorphic Functions to Complex Manifolds. Graduate Texts in Mathematics*, **213**. Springer-Verlag, New York.

Gu, Q. and Li, T. (2009) Exact boundary controllability for quasilinear wave equations in a planar tree-like network of strings. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**, 6, 2373–2384.

Gugat, M. and Gerster, S. (2019) On the limits of stabilizabi-lity for networks of strings. *Systems Control Lett.*, 131:104494, 10.

Gugat, M. and Giesselmann, J. (2021) Boundary feedback stabilization of a semilinear model for the flow in star-shaped gas networks. ESAIM: *Control, Optimisation and Calculus of Variations*, 27:67.

Gugat, M. and Herty, M. (2022) Limits of stabilizability for a semilinear model for gas pipeline flow. *Optimization and Control for Partial Differential Equations - Uncertainty Quantification, Open and Closed-Loop Control, and Shape Optimization. Radon Ser. Comput. Appl. Math.* De Gruyter, Berlin, **29**, 59–71.

Gugat, M., Leugering, G., Martin A., Schmidt, M., Sirvent, M. and Wintergerst, D. (2018) MIP-based instantaneous control of mixed-integer PDE-constrained gas transport problems. *Comput. Optim. Appl.*, **70**(1):267–294.

Gugat, M., Leugering, G. and Wang, K. (2017) Neumann boundary feedback stabilization for a nonlinear wave equation: a strict H2-Lyapunov function. *Math. Control Relat. Fields*, **7**(3):419–448.

Gugat, M., Qian, M. and Sokolowski, J. (2023) Topological derivative method for control of wave equation on networks. In: *2023 27th International Conference on Methods and Models in Automation and Robotics (MMAR)*, 320–325.

Gugat, M. and Weiland, S. (2021) Nodal stabilization of the flow in a network with a cycle. *Журнал з оптимізації, диференціальних рівнянь та їх застосувань*, **29**(2):1–23.

Hayat, A. (2019) On boundary stability of inhomogeneous 2×2 1-d hyperbolic systems for the c1 norm. *ESAIM: Control, Optimisation and Calculus of Variations*, 25:82.

Hayat, A. and Shang, P. (2021) Exponential stability of density-velocity systems with boundary conditions and source term for the h2 norm. *Journal de mathématiques pures et appliquées*, 153:187–212.

Huang, X., Wang, Z. and Zhou, S. (2023) The Dichotomy Property in Stabilizability of 2 x 2 Linear Hyperbolic Systems. arXiv e-prints, arXiv:2308. 09235, August 2023.

Krug, R., Leugering, G., Martin, A., Schmidt, M. and Weninger, D. (2021) Time-domain decomposition for optimal control problems governed by semilinear hyperbolic systems. *SIAM J. Control Optim.*, **59**(6): 4339-–4372.

Leugering, G. and Schmidt, E. J. P. G. (2002) On the modelling and stabilization of flows in networks of open canals. *SIAM J. Control Optim.*, **41**(1):164–180.

Leugering, G. and Sokolowski, J. (2008) Topological sensitivity analysis for elliptic problems on graphs. *Control and Cybernetics*, **37**(4): 971–997.

Li, T. (2010) Controllability and Observability for Quasilinear Hyperbolic Systems. *AIMS Series on Applied Mathematics*, **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing, 2010.

Li, T. and Rao, B. (2004) Exact boundary controllability of unsteady flows in a tree-like network of open canals. *Methods Appl. Anal.*, **11**(3):353–365.

Nakić, I. and Veselić, K. (2020) Perturbation of eigenvalues of the Klein-Gordon operators. *Rev. Mat. Complut.*, **33**(2):557–581.

Pazy, A. (1983) Semigroups of linear operators and applications to partial differential equations. *Applied Mathematical Sciences*, **44**. Springer-Verlag, New York.

Schmidt, M., Assmann, D., Burlacu, R., Humpola, J., Joormann, I., Kanelakis, N., Koch, T., Oucherif, D., Pfetsch, M., Schewe, L., Schwarz, R. and Sirvent, M. (2017) Gaslib–a library of gas network instances. *Data*, **2**:40, 12.

von Below, J. (1988) Sturm-Liouville eigenvalue problems on networks. *Math. Methods Appl. Sci.*, **10**(4):383–395.

von Below, J. and François, G. (2005) Spectral asymptotics for the Laplacian under an eigenvalue dependent boundary condition. *Bull. Belg. Math. Soc. Simon Stevin*, **12**(4):505–519.

Young, R. M. (1980) An Introduction to Nonharmonic Fourier Series. *Pure and Applied Mathematics*, **93** Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London.