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The Application of Weights in the Weighted Arithmetic Mean to Obtain the Optimal Solution of Degenerate Transportation Problem

Abstract

The transportation problem is a special type of linear programming problems, which involve linear cost functions and constraints. In this research, two aspects of the transportation problem are discussed. Firstly, this paper proposes the weighted arithmetic mean algorithm to find an initial basic feasible solution. Secondly, it explicates the application of weights to achieve optimality. The weights form additional parameters that appear in the matrix defining the cost. Furthermore, after studying and analyzing the algorithms, a special type of degenerate transportation problem is considered and a solution that is optimal or near to optimal is found.

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1. Introduction

Transportation Problem is a special kind of Linear Programming Problem (LPP) (cf. Hadley (1962)). The main objective is to transport goods from many sources to various destinations. The goal is to find an appropriate transportation arrangement for which the cost of transportation is minimized. The transportation problem may be solved by the simplex method (cf. Dantzig (1951)), but this method is not suitable for large-scale problems. The stepping stone method was developed for efficiency reasons due to its special model by Charnes and Cooper in 1954.

The original models of transportation were established in 1941, when the study by F.L. Hitchcock was published (v. [4]). This presentation is considered as the first major contribution to solving transportation problems. In 1947, the study by Koopmans (1949)[1] was presented. The development of the transportation models involving many production sources and several destinations mainly relies on these two contributions.

[1] *Optimum use of the transport system*

To solve the transportation problems, the stepping stone method is very frequently used (cf. Quddoos et al. (2012), Sudhakar et al. (2012)). It is a method by which all non-basic cells that are empty and are evaluated move from an initial feasible solution to an optimal solution. To evaluate an empty cell, the method adopts a path tracing approach. The modification distribution method (MODI) is another way to evaluate empty cells. The stepping stone method is similar to the MODI method. Degeneracy, which appears when there are few basic cells in a feasible solution, is a serious problem of the stepping stone method. To deal with the problem of degeneracy one may follow the work of Shafaat and Goyal (1988).

Although degeneracy in the simplex method does not cause major difficulties, it may lead to computational problems in transportation problem (cf. Sharma (2009), Taha (1992)). In such a case it is not possible to make closed paths (loops) for every vacant cell in the stepping-stone method, so it is not possible to take into account all vacant cells. If the MODI method is used, it is not possible to find all the needed variables, since the number of assigned cells is not sufficient. Therefore, a degenerate transportation problem must be identified and appropriate steps must be taken to help avoid computing problems. Degeneracy can occur in the initial solution or during some subsequent iteration.

In this paper we develop a method to obtain the optimal solution with weights. We find an initial basic feasible solution with the Weighted Arithmetic Mean and optimal solution or near to the optimal solution to the transportation problem. Optimality is tested by using the stepping stone concept via the sum of the weights assigned in the transportation cost matrix and determining the net difference in the weights of empty cells. This method utilizes the newly developed weighted arithmetic mean employing the weights other than used in previously established methods like Arithmetic Mean, Geometric Mean and the Harmonic Mean (cf. Sathyavathy and Shalini (2019)). The new algorithm discussed here gives the idea of optimality. We also provide a numerical example that illustrates the new algorithm.

2. Definition of Transportation Problem

Let us consider the transportation problem with m sources S_i (with supplies a_i, $i = 1, 2, 3, \dots, m$) and n destinations D_j (with demands b_j, $j = 1, 2, 3, \dots, n$).

Let c_{ij} denotes the cost of transporting the unit load from the source S_i to the destination D_j.

Let x_{ij} denotes the number of load units moved from S_i to D_j.

Mathematically the problem can be stated as:

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to constraints:

$\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1, 2, 3, \dots, m$ (supply constraints)

$\sum_{i=1}^{m} x_{ij} = b_j$ for $j = 1, 2, 3, \dots, n$ (demand constrains) and $x_{ij} \geqslant 0$ for all i and j.

The above details can be given in the form of a matrix shown in Table 1: Balanced Transportation Problem

Table 1: Balanced transportation problem

Source	**Destination**				**Supply**
	D_1	D_2	....	D_n	
S_1	(x_{11}) c_{11}	(x_{12}) c_{12}	....	(x_{1n}) c_{1n}	a_1
S_2	(x_{21}) c_{21}	(x_{22}) c_{22}	....	(x_{2n}) c_{2n}	a_2
....	....	....	....	....	....
S_m	(x_{m1}) c_{m1}	(x_{m2}) c_{m2}	....	(x_{mn}) c_{mn}	a_m
Demand	b_1	b_2	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

A problem that satisfies the additional condition stated in Table 1 i.e., $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is called a balanced (standard) transportation problem. Most techniques developed for solving transportation problems work only for balanced problems. It is required that every non-standard transportation problem, in which supplies and demands are not balanced ($\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$), be converted to a standard transportation problem before it can be addressed. The non-standard transportation problem can be converted into a standard transportation problem by using a dummy source or destination.

2.1. Basic terms of the *Transportation Problem*. A few terms related to the transportation problem are described below.

Feasible Solution: A set of non-negative individual allocations ($x_{ij} \geq 0$) that satisfies the demand and supply constraints (rim conditions).

Basic Feasible Solution: A feasible solution that consists of no more than $m+n-1$ non-negative allocations (where m is the number of rows and n is the number of columns).

Initial Basic Feasible Solution (IBFS): An initial basic solution that satisfies the conditions:

1) The solution is feasible, i.e. all the supply and demand constraints (also known as rim conditions) are satisfied.
2) The number of non-negative allocations is equal to $m+n-1$ (where m is the number of rows and n is the number of columns).

Optimal Solution: An optimal solution is a feasible solution that optimizes (minimizes) the total transportation cost.

Balanced Transportation Problem: A transportation problem for which the total supply from all sources equals the total demand in the destinations. i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Unbalanced Transportation Problem: A transportation problem for which the total supply from all sources does not equal the total demand in the destinations. *i.e.* $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$.

Non-degenerate Basic Feasible Solution: A basic feasible solution for which the number of non-negative allocations is strictly $m+n-1$ (*i.e.* equal to the number of independent constraint equations), and these allocations are in independent positions.

Degenerate Basic Feasible Solution: A basic feasible solution in which the total number of non-negative allocations is less than $m+n-1$.

This type of solution is not easy to change because at this point it is difficult to draw a closed loop for each occupied cell. Degeneracy must also be removed for the solution achieved to be optimal. Thus, in two different stages, degeneracy occurs.

1) At the stage of Initial Basic Feasible Solution, where the number of occupied cells is less than $m+n-1$.
2) When moving towards an optimal solution at any point; two or more occupied cells can simultaneously become unoccupied.

Optimality Test: The optimality test will be performed if the number of allocation cells in an initial basic feasible solution is equal to $m+n-1$. Otherwise, the optimality test cannot be performed.

In the optimality test we check whether the total transportation cost is decreased if we put an allocation in a vacant cell.

3. Weights in Transportation Problem. Now let w_{ij} be weights assigned to the costs c_{ij} for each $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. We assign the weights to the costs c_{ij} from the highest to lowest transportation cost for each row and each column. That is, we assign the weight 1 to the highest (maximum) cost, subsequent weights to the subsequent costs, and n to the lowest cost for each row. Similarly, we assign the weight 1 to the highest (maximum) cost, subsequent weights to the subsequent costs, and m to the lowest cost for each column. We note that weights can be possibly assigned *chronologically or randomly* in case of tied costs.

We denote the weights assigned to the cost c_{ij} by w_{ij}^r considering the cell in a row and w_{ij}^c considering it in a column.

We define the sum of weights $S_{ij}=w_{ij}^r+w_{ij}^c$ for each cell in the transportation cost matrix. We will consider it for the optimality test. We will define later the net difference in the weights $d_{w_{ij}}$ for each unoccupied cell using the sum of weights S_{ij}.

3.1. Weighted Arithmetic Mean (WAM). The Weighted Arithmetic Mean is derived from the formula

$$\overline{x}_w = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots + w_nx_n}{w_1 + w_2 + w_3 + \ldots + w_n} = \frac{\sum_{i=1}^{n} w_ix_i}{\sum_{i=1}^{n} w_i}$$

The Weighted Arithmetic Mean is an average determined by giving different weights to the set of observations in the data. The Weighted Arithmetic Mean plays a significant role in the systems of data analysis, weighted differential and integral calculus. We apply this arithmetic average in the transportation problem in this paper.

4. Algorithm. This approach provides a procedure for finding an initial basic feasible solution using the Weighted Arithmetic Mean

Step 1: Determine whether the transportation problem is balanced or not. If it is not balanced, then add a dummy row or column to make the problem balanced. Then proceed to the next step.

Step 2: Using weights in the transportation matrix, find the weighted arithmetic mean for each row and column.

Step 3: Select the highest values for each row and each column from Step 2 and allot the minimum supply or demand of the lowest cost value for the corresponding row or column.

Step 4: Repeat Steps 2 and 3 until the demands are met and all the supplies are exhausted.

Step 5: Total cost is calculated as the sum of the products of the cost and the corresponding allocated value of supply or demand.

Step 6: If the degeneracy is found, i.e. the number of allocations is less than $m + n - 1$, to resolve degeneracy at the initial basic feasible solution we proceed by allocating a small quantity close to zero to one or more (if needed) unoccupied cells to get $m + n - 1$ occupied cells. The cell containing this extremely small quantity is considered to be an occupied cell and is denoted by Δ. This small quantity will not affect the total cost and supply and demand values. It is better to allocate Δ to unoccupied cells that have the lowest transportation cost in a minimization problem. On the other hand, in a maximization problem, the cell with the maximum transportation cost should be allocated. The quantity of Δ is considered so small that when transferred to an occupied cell it does not change the quantity of allocation.

4.1. The Optimality Test. To verify the optimality, we follow the following procedure:

Step 1:

1. Start with an initial basic feasible solution by the Weighted Arithmetic Mean containing $m+n-1$ allocations in independent positions.
2. If the initial basic feasible solution is found by any method other than WAM, then we must assign the weights to the transportation cost matrix as discussed earlier in this paper.

Step 2:

1. Trace a closed path (or loop) starting from an unoccupied cell through at least three occupied cells, and then back to the selected unoccupied cell.
2. Assign plus (+) and minus (-) signs alternatively to the sums S_{ij} for each corner cell of the closed path just being traced, starting with plus (+) sign for the unoccupied cell.
3. Compute the net difference $d_{w_{ij}}$ of the weights for each unoccupied cell by adding the sums of weights S_{ij} from the cells having plus sign and subtracting the sums of weights S_{ij} from the cells having minus sign along with the closed path traced in Step 2.
4. Repeat this process for all other unoccupied cells in the matrix.

Step 3:

1. The optimal solution is achieved when the net difference in the weights $d_{w_{ij}} \leq 0$ for the unoccupied cells.
2. Otherwise, select the unoccupied cell with the highest positive net difference ($d_{w_{ij}} > 0$).

Step: 4 Repeat Step 2 and Step 3 until the optimal solution is obtained.

Step: 5 The total transportation cost is calculated as the sum of the product of value and corresponding allotted cost of supply or demand, *i.e.* total cost=$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$.

This optimality test, using the sum of weights given for each cell, produces the optimal solution or a solution close to the optimal solution. In our optimality test, we use the net difference in the weights $d_{w_{ij}}$. The net difference in the weights $d_{w_{ij}}$ is nothing but the resultant weight for each unoccupied cell. If the resultant weight is positive for an unoccupied cell, it will contribute to the optimal cost of the transportation matrix.

5. Numerical example. A steel company has three open hearth furnaces and five rolling mills. The transportation costs (rupees per quintal) for shipping steel from furnaces to rolling mills are given in Table 2. What is the optimal shipping schedule?

Table 2: Formulation of the problem.

	M_1	M_2	M_3	M_4	M_5	Supply
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	7	14
Demand	4	4	6	8	8	

5.1. Solution. The given problem is an unbalanced transportation problem. Accordingly, the initial basic feasible solution from WAM is established after making the balanced transportation problem by adding dummy column considered as M_6.

Table 3: Stage 1 IBFS

	M_1	M_2	M_3	M_4	M_5	M_6	Supply
F_1	$\boxed{4}$ 4 *2* *3*	2 *4* *3*	$\boxed{4}$ 3 *3* *3*	2 *5* *2*	6 *1* *2*	0 *6* *1*	**8**
F_2	5 *1* *2*	$\boxed{4}$ 4 *3* *2*	5 *2* *1*	2 *4* *3*	$\boxed{8}$ 1 *5* *3*	0 *6* *2*	**12**
F_3	6 *3* *1*	5 *4* *1*	$\boxed{2}$ 4 *5* *2*	$\boxed{8}$ 7 *1* *1*	7 *2* *1*	$\boxed{4}$ 0 *6* *3*	**14**
Demand	**4**	**4**	**6**	**8**	**8**	**4**	**34**

Number of occupied cells: $7 \neq m+n-1 = 3+6-1 = 8$.
Total transportation cost of WAM: $4{\cdot}4+3{\cdot}4+4{\cdot}4+1{\cdot}8+4{\cdot}2+7{\cdot}8+0{\cdot}4 = 116$.

We obtained the IBFS that has the number of occupied cells less than $m+n-1 = 3+6-1 = 8$. Consequently, the IBFS is Degenerate. To remove the degeneracy, allocate Δ at the unoccupied cells that have minimum transportation cost among the unoccupied cells. So, here we choose (F_2,M_4) cell of the cost matrix because we consider a minimization problem (see Table 4).

Table 4: Stage 2 IBFS

	M_1	M_2	M_3	M_4	M_5	M_6	Supply
F_1	$\boxed{4}$ 4 *2* *3*	2 *4* *3*	$\boxed{4}$ 3 *3* *3*	2 *5* *2*	6 *1* *2*	0 *6* *1*	**8**
F_2	5 *1* *2*	$\boxed{4}$ 4 *3* *2*	5 *2* *1*	$\boxed{\Delta}$ 2 *4* *3*	$\boxed{8}$ 1 *5* *3*	0 *6* *2*	**12**
F_3	6 *3* *1*	5 *4* *1*	$\boxed{2}$ 4 *5* *2*	$\boxed{8}$ 7 *1* *1*	7 *2* *1*	$\boxed{4}$ 0 *6* *3*	**14**
Demand	**4**	**4**	**6**	**8**	**8**	**4**	**34**

Now, with the optimality test, we have the optimal solution as shown in Table 5:

Table 5: Stage 3 Optimality

	M_1	M_2	M_3	M_4	M_5	M_6	Supply
F_1	4 2 3	$\boxed{4}$ 2 4 3	3 3 3	$\boxed{4}$ 2 5 2	6 1 2	0 6 1	8
F_2	5 1 2	4 3 2	5 2 1	$\boxed{4}$ 2 4 3	$\boxed{8}$ 1 5 3	0 6 2	12
F_3	$\boxed{4}$ 6 3 1	5 4 1	$\boxed{6}$ 4 5 2	7 1 1	7 2 1	$\boxed{4}$ 0 6 3	14
Demand	**4**	**4**	**6**	**8**	**8**	**4**	**34**

$d_{w_{13}} = -2$, $d_{w_{15}} = -5$, $d_{w_{16}} = -3$, $d_{w_{21}} = -2$, $d_{w_{22}} = -3$, $d_{w_{23}} = -5$, $d_{w_{26}} = -2$, $d_{w_{32}} = -1$, $d_{w_{34}} = -4$, $d_{w_{35}} = -4$

So, all $d_{w_{ij}} \leq 0$ and optimality is achieved.

Total Minimum Cost $= 2 \cdot 4 + 2 \cdot 4 + 2 \cdot 4 + 1 \cdot 8 + 6 \cdot 4 + 4 \cdot 6 + 0 \cdot 4 = 80$.

6. Conclusion. It was shown in the paper that there are two important methods for solving transportation problems: finding an initial basic feasible solution (IBFS) and checking optimality. It is possible to solve standard and non-standard, non-degenerate and degenerate transportation problems with the proposed procedure. The aim of this paper was to address degeneracy while solving a well-defined problem. We strived to derive the optimal solution to the transportation problem through the optimality test. The main advantage of this procedure is that the external parameters of the weights used for the costs in each iteration provide the optimal solution or near-optimal solution for a Degenerate Transportation Problem in fewer steps and shorter time to effectively solve the problem. The future extent of this algorithm is that the decision-maker correlates this algorithm with certain credible changes with fewer steps. Additionally, this algorithm achieves the most effective result for real-world problems.

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Ważone średnie arytmetyczne wag w optymalnych rozwiązaniach zdegenerowanych problemów transportowych.

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Streszczenie Zagadnienie transportowe to specjalne zadanie programowania liniowego, dla którego zostały opracowane dedykowane algorytmy. W tej pracy zaproponowano dwa podejścia do zagadnienia transportowego. W pierwszym podejściu podajemy algorytm średniej ważonej. Jest on przeznaczony dla początkowego podstawowego rozwiązania dopuszczalnego. W drugim podejściu wyjaśniamy zastosowanie wag do osiągnięcia optymalności. Wagi, to dodatkowe parametry, które są ujęte w macierzy kosztu. Po przestudiowaniu i przeanalizowaniu algorytmu analizujemy specjalny przypadek zdegenerowany, dla którego uzyskujemy rozwiązanie optymalne lub bliskie optymalnemu.

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