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An Application of Mathematical Theory of Evidence in Navigation

W. Filipowicz
Gdynia Maritime University, Gdynia, Poland

ABSTRACT: Plenty of various quality data are available to the officer of watch. The data of various qualities comes from different navigational aids. This kind of data creates new challenge regarding information association. The challenge is met by Mathematical Theory of Evidence. The theory delivers methods enabling combination of various sources of data. Results of association have informative context increased. Associated data enable the navigator to refine his position and his status regarding dangerous places. The procedure involves uncertainty, ambiguity and vague evidence. Imprecise and incomplete evidence can be combined using extended Dempster-Shafer reasoning scheme.

1 INTRODUCTION

There are stochastic and epistemic uncertainties distinguished. Stochastic also called aleatory uncertainty reflects unknown, usually unpredictable behaviour of a system. The system behaves in stochastic way when its states are random ones. They can be identified based on traditional probability theory. In maritime traffic engineering attempt to find deviation from intended track is related to the aleatory uncertainty.

Shortage of knowledge or incomplete evidence creates another kind of uncertainty. Epistemic or subjective uncertainty results from insufficient or vague evidence. Question of identity of new spotted object refers to this sort of uncertainty. It is quite often when observer at monitoring station spots new radar mark and tries to find out what vessel this could be. Usually there is some evidence available, for example radar echo signature and estimate of speed can be helpful. Modern AIS technology transfers data useful in identification. Problem was discussed by the author in his previous papers (Filipowicz 2007 & Filipowicz 2008).). Navigational aids deliver plenty of data used for position fixing. The quality of data is different and depends on many factors. Such imprecise and sometimes incomplete data are further combined for position refinement. Quantifying navigational status regarding an obstacle is crucial from safety standards point of view.

In classical probability theory the knowledge of probability of an event can be used to calculate likelihood of the contrary statement. In this approach if one navigational aid indicates position within certain area with probability of 0.6, that mean that navigator believes that he is outside the area with the probability of 0.4. The theory also requires that data regarding probability of all considered events is at disposal. The theory is limited in its ability when dealing with epistemic uncertainty.

Mathematical Theory of Evidence (MTE for short) is more flexible in this respect. MTE is a theory (initiated by Dempster & Shafer) based on belief and plausibility functions and scheme of reasoning in order to combine separate pieces of evidence to calculate the probability of an event. Contrary to probability theory it enables modelling knowledge and ignorance. Evidence can be combined therefore even partial knowledge associated with less meaningful facts may end up in valuable conclusions. Combining evidence leads to data enrichment and improved probability judgments can be obtained for each considered hypothesis. Fundamental for MTE is Dempster-Shafer scheme of reasoning initially intended for crisp values. New extensions to cope with imprecision are also available since it is often that to obtain precise figures is infeasible. Imprecision is expressed as interval values or fuzzy figures. In the paper and elsewhere fuzzy values are considered as a set of intervals given for selected possibility levels.

Problem of position refinement that involves epistemic uncertainty could be defined as below.

Given:

– navigation aids indicating different positions, different distances from an obstacle

– each aid has reliability and accuracy characteristic assigned to it
– linguistic terms referring to close, sufficient and safe distances are available as membership functions

Question:

– what is credibility that the real distance to the obstacle is safe one?

First part of the paper is devoted to basic probability assignment. Then necessity to deal with imprecision is depicted. Further on interval values are introduced and belief structure defined. Short description of Dempster-Shafer method is also included. Last part of the paper deals with identification of navigational status referring to an obstacle. Two navigational aids are considered. Their indications are combined in order to quantify distance from certain shallow water area.

2 PROBABILITY ASSIGNMENT

Frame of discernment in Mathematical Theory of Evidence consists of possible events. Events are understood very widely. Examples of events that are of interest in navigation could be: route taken by a spotted vessel, position fixing based on an electronic aid, attempt to refine unidentified object etc. Events are considered as atomic or structured ones. Considering limited set of objects as a single entity means dealing with molecular or structured event. For example new spotted object must be large container carrier or medium bulk vessel because no other traffic is expected within the area. It is assumed that in case of structured event all constituents are equally possible.

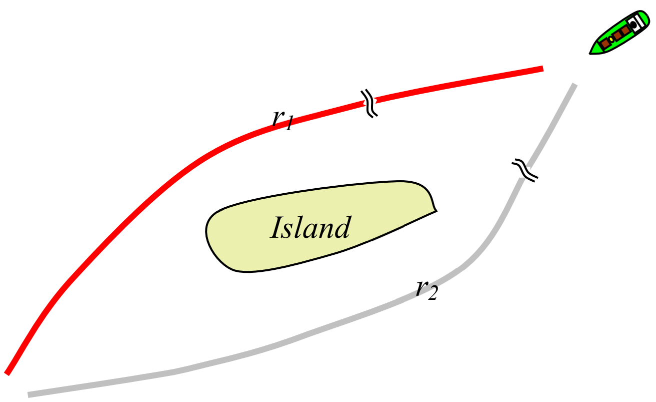

Figure 1. Intended route forecast problem involves three events: taking route r_1, taking r_2 and joint r_1 or r_2

Let us consider example on reasoning which of possible and treated as equivalent routes r1 or r2 will be taken by the vessel shown at figure 1. The frame of discernment embraces three events $\Omega = (\{r_1\}, \{r_2\}, \{r_1, r_2\})$. First event is related to route r1 as possibly taken by the vessel, selecting route r2 means occurrence of the second event. Third molecular event expresses uncertainty, it constituents r_1 or r_2 are assumed to be equally possible.

Some evidence supporting reasoning on intended ship's itinerary is assumed. Recorded cases with southwest bound vessels of similar tonnage were examined. For all *n* stored cases *x* out of the all have chosen route r_1. Appropriate masses related to each of the events can be calculated according to formula (1). Assuming that data stored in traffic related database gives x=24 and n=39 masses of likelihood that this time particular route is taken should be assigned as shown in formula (1)

$$\begin{aligned} m(\{r_1\}) &= \frac{x}{n+1} = 0.600 \\ m(\{r_2\}) &= \frac{n-x}{n+1} = 0.375 \\ m(\{r_1, r_2\}) &= \frac{1}{n+1} = 0.025 \end{aligned} \qquad (1)$$

It is easy to find out that all masses sum up to one so probability requirement is satisfied. The theory also requires that: $m(\{r_1, r_2\}) = 1 - m(\{r_1\}) - m(\{r_2\})$. Note that set $\{r_1, r_2\}$ expresses some sort of uncertainty since it reflects that both available routes can be taken with the same credibility level.

Let us again consider example on guessing which of routes r_1 or r_2 will be taken by the vessel. This time we assume different evidence supporting reasoning on intended ship's itinerary. We assume that various samples of recorded cases are available. Registered routes for similar ships referred to different weather conditions. Number of records in the samples varied within range of [20, 50]. Data analyses discovered that number of southwest bound ships that have chosen route r_1 was around 70% of all stored cases. The percentage never fell below 60% and did not exceed 80% of the total number. Under these assumptions one is not able to calculate masses of evidence using before presented way of reasoning. The task is seemingly unsolved due to limitation imposed by crisp values. Interval values are to be used instead. Counting all pros and cons and numbers of records in the samples interval-valued masses presents formula (2).

$$\begin{aligned} m(\{r_1\}) &= [m_1^-, m_1^+] = \left[\frac{12}{21}, \frac{40}{51}\right] = [0.571, 0.784] \\ m(\{r_2\}) &= [m_2^-, m_2^+] = \left[\frac{10}{51}, \frac{8}{21}\right] = [0.196, 0.381] \\ m(\{r_1, r_2\}) &= [m_3^-, m_3^+] = \left[\frac{1}{51}, \frac{1}{21}\right] = [0.020, 0.048] \end{aligned} \qquad (2)$$

In this case all masses cannot sum up to one so basic probability requirement cannot be satisfied. The approach stipulates that exists a set of sub ranges within defined intervals within which summation to one is observed. More formally conditions 1 and 2 in definition (1) are to be true. Definition (1) refers to interval-valued probability assignment that is also called as interval-valued belief structure.

Definition (1):

Interval-valued masses attributed to respective elements of the frame of discernment, namely: $[m^-_1, m^+_1]$, $[m^-_2, m^+_2]$,… , $[m^-_n, m^+_n]$ define adequate probability assignment if there is a set **m** such that for m∈**m** following are satisfied:

- within each interval there is a value: $m^-_i \le m_i \le m^+_i$, for each i∈{1, .., n}
- for all such values: $\sum_{i=1}^{n} m_i = 1$

For example three interval-valued masses the set of legal probability assignment is shown as two dimensional shape in figure 2. Procedure of establishing such shape can also lead to tightening interval bounds since some values may appear as unreachable.

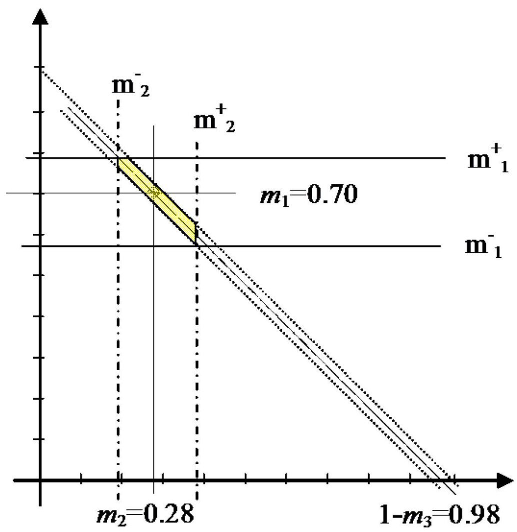

Figure 2. Graphical presentation of the set of valid probability assignment in interval-valued belief structure

We again consider above example on guessing which of routes r_1 or r_2 will be taken by the vessel using fuzzy approach. We assume that it is experienced radar observer who reasons on intended ship's itinerary. His subjective way of thinking is like this: the vessel is a medium one, visibility is rather good, wind moderate so to his best knowledge it is "likely" the vessel will take route r_1. He also observed quite many similar vessels have taken route r_2 so it is "fairly likely" that this route will be chosen this time. Judging from his experience uncertainty of his opinion is very low. Formal expression of the above statement requires introduction of meaning terms: "likely", "fairly likely" and "very low". All of them are linguistic terms referring to fuzzy reasoning. Such terms are characterized by membership functions.

2.1 *Theoretical membership functions*

Set with elements like: "very unlikely", "unlikely", "fairly likely", "likely", "very likely" and "certain" consists of linguistic terms which human beings use for estimated reasoning. To evaluate uncertainty one can use "very low", "low", "medium", "high", "very high" and "totally uncertain" as the highest term. Both sets contain six elements and membership functions can be used interchangeably depending on the context.

Counting elements from 0 up to n_c-1 one can use formula (3) to calculate normalized and regular fuzzy membership functions. Trapezoid shapes obtained for w_T=0.8 are presented in figure 3 and triangular ones for w_T=0 in figure 4.

$$F_k = \begin{cases} (0, 0, w_T * w, w) & if\ k = 0 \\ ((k-1) * w, k * w - w_T * w, \\ k * w + w_T * w, (k+1) * w) & if\ 0 < k < n_c - 1 \\ (1 - w, 1 - w_T * w, 1, 1) & if\ k = n_c - 1 \end{cases} \quad (3)$$

where:

- $w = \frac{1}{n_c - 1}$
- n_c - is a number of selected terms
- $w_T \in [0, 1]$– is the shape parameter, $w_T = 0$ means that membership function is a triangular one and $w_T = 1$ means rectangular shape.

Formula (4) defines trapezoid fuzzy-valued masses assignment for the third discussed case of probability assignment on expected route taken by the spotted craft. The formula contains membership functions for terms respectively "likely" (k=3), "fairly likely" (k=2) and "very low" (k=0). Functions are quads calculated with formula (1) for listed above *k* value. Membership functions are also presented as intervals for three selected possibility levels α = 0, 0.5 and 1. Possibility equal to zero denotes support of a fuzzy value. Possibility equal to one refers to the core of imprecise value.

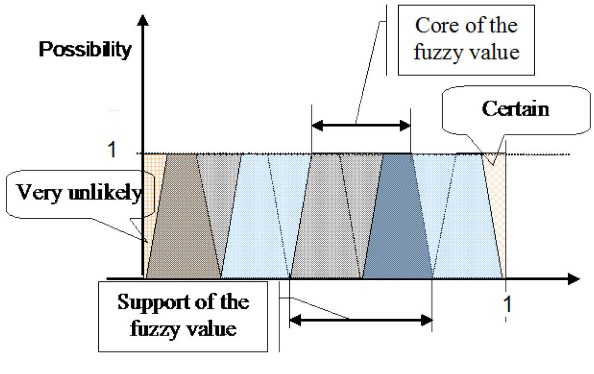

Figure 3. Trapezoid membership functions (w_T=0.8)

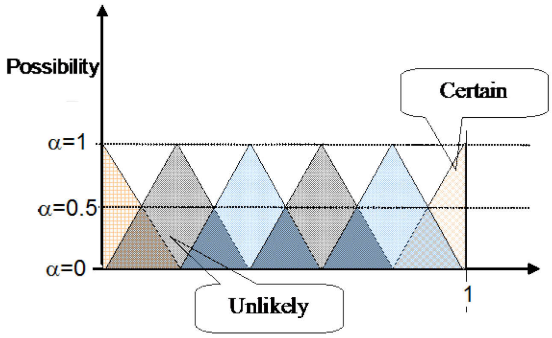

Figure 4. Triangular membership functions expressing six linguistic terms (w_T=0)

$$
\begin{aligned}
m(\{r_1\}) &= (0.4, 0.44, 0.76, 0.8) \approx \begin{bmatrix} \alpha=0 & [0.4, 0.8] \\ \alpha=0.5 & [0.42, 0.78] \\ \alpha=1 & [0.44, 0.76] \end{bmatrix} \\
m(\{r_2\}) &= (0.2, 0.24, 0.56, 0.6) \approx \begin{bmatrix} \alpha=0 & [0.2, 0.6] \\ \alpha=0.5 & [0.22, 0.58] \\ \alpha=1 & [0.24, 0.56] \end{bmatrix} \\
m(\{r_1, r_2\}) &= (0, 0, 0.16, 0.2) \approx \begin{bmatrix} \alpha=0 & [0, 0.2] \\ \alpha=0.5 & [0, 0.18] \\ \alpha=1 & [0, 0.16] \end{bmatrix}
\end{aligned} \tag{4}
$$

3 COMBINATION OF TWO BELIEF STRUCTURES

Probability assignments that examples are showed above can be combined in order to increase result information context. Probability assignment to events from frame of discernment at hand is called as belief structure. Belief structures are supposed to verify certain constraints (see for example definition (1)). Depending on type of assigned masses basic, interval-valued and fuzzy-valued structures are distinguished. It is said that combination of belief structures creates new assignment characterized by enrichment of engaged data. To take benefit of this enrichment other sources of data are to be available. In the above interval-valued example on guessing which of routes r_1 or r_2 will be taken by the vessel single source of data was assumed. Let us consider yet another archive that contains different sets of recorded cases. Registered routes for similar ships referred to similar weather conditions were analyzed. Number of records in the samples varied within range of [40, 60]. Data analyses revealed that number of southwest bound ships that have chosen route r_1 never fell below 50% and did not exceed 65% of the total number. Masses attributed to each event are shown in formula (5).

$$
\begin{aligned}
m_2(\{r_1\}) &= [m^-_{21}, m^+_{21}] = [0.488, 0.639] \\
m_2(\{r_2\}) &= [m^-_{22}, m^+_{22}] = [0.341, 0.492] \\
m_2(\{r_1, r_2\}) &= [m^-_{23}, m^+_{23}] = [0.016, 0.024]
\end{aligned} \tag{5}
$$

Combination procedure for ranges and fuzzy values extends original Dempster-Shafer method initially proposed for crisp masses in basic belief structures. Comprehensive way of two sources combinations is summarized below. The scheme was further used for example combination of the two discussed sources results are shown in table 1.

Dempster-Shafer rules of combination:

1 Create table with rows that refer to events embraced in second source.
  Columns refer to the events of first source.
  Each event has mass of evidence (fuzzy or interval-valued) that is assigned to it
2 For each intersection of a row and a column product of masses involved is calculate and attributed to a common, for the two sets, event.
  In case of crisp events inconsistency occurs if the two sets have empty intersection. Therefore, for particular cell, the product of masses of evidence is assigned to an empty set
  In case of fuzzy events conjunctive operator is applied and search for minimum values on membership functions involved carried out
3 Calculate masses for each resulting set of events
4 Calculate belief functions (and if required plausibility) values

Definition (2):

There are two sets of interval-valued masses attributed to elements of the same frame of discernment, namely: $\mathbf{m_1}$, $\mathbf{m_2}$. Each of them embraces certain set of events referred to as: $\mathcal{F}(\mathbf{m_1})$ and $\mathcal{F}(\mathbf{m_2})$. Their combination defines probability assignment as a set $\mathbf{m}$ such that for $m \in \mathbf{m}$ appropriate limits (Denoeux 1999) are given by formula (6).

$$
\begin{aligned}
m^-(A) &= \min_{(m_1, m_2) \in (\mathbf{m_1} \times \mathbf{m_2})} \sum_{B \cap C = A} m_1(B) m_2(C) \\
m^+(A) &= \max_{(m_1, m_2) \in (\mathbf{m_1} \times \mathbf{m_2})} \sum_{B \cap C = A} m_1(B) m_2(C)
\end{aligned} \tag{6}
$$

Table 1. Combination of two sources of crisp data

	Source I		
	$m_1(\{s_1\})$ [0.571, 0.784]	$m_1(\{s_2\})$ [0.196, 0.381]	$m_1(\{s_1, s_2\})$ [0.020, 0.048]
$m_2(\{s_1\})$ [0.488, 0.639]	$m_{1\text{-}2}(\{s_1\})$ [0.488, 0.639]	$m_{1\text{-}2}(\{\varnothing\})$ [0.096, 0.243]	$m_{1\text{-}2}(\{s_1\})$ [0.010, 0.031]
$m_2(\{s_2\})$ [0.341, 0.492]	$m_{1\text{-}2}(\{\varnothing\})$ [0.195, 0.386]	$m_{1\text{-}2}(\{s_2\})$ [0.067, 0.187]	$m_{1\text{-}2}(\{s_2\})$ [0.007, 0.024]
$m_2(\{s_1, s_2\})$ [0.016, 0.024]	$m_{1\text{-}2}(\{s_1\})$ [0.009, 0.019]	$m_{1\text{-}2}(\{s_2\})$ [0.003, 0.009]	$m_{1\text{-}2}(\{s_1, s_2\})$ [0.000, 0.001]

Using formula (6) one can obtain limits of joint masses that are as follows:

- $m^-_{1\text{-}2}(\{s_1\}) = 0.279 + 0.009 + 0.01 = 0.298$
- $m^+_{1\text{-}2}(\{s_1\}) = 0.501 + 0.019 + 0.031 = 0.551$
- $m^-_{1\text{-}2}(\{s_2\}) = 0.067 + 0.003 + 0.007 = 0.077$
- $m^+_{1\text{-}2}(\{s_2\}) = 0.187 + 0.009 + 0.024 = 0.220$
- $m^-_{1\text{-}2}(\{s_1, s_2\}) = 0.0003$
- $m^+_{1\text{-}2}(\{s_1, s_2\}) = 0.0012$

Since in two cases there were empty intersections therefore inconsistency occurred. Limits of the empty set are as below:

- $m^-(\varnothing) = 0.096 + 0.195 = 0.291$
- $m^+(\varnothing) = 0.386 + 0.243 = 0.529$

Result belief structure with its interval-valued probability assignment enables determination of evidential functions. Lower and upper limits of belief function can be calculated with formula (7).

$$bel^-(A) = \max\Big(\sum_{B\subseteq A; B\neq\varnothing} m^-(B),\ 1-\sum_{B\not\subset A; B\neq A} m^+(B) - m^+(\varnothing)\Big)$$
$$bel^+(A) = \min\Big(\sum_{B\subseteq A; B\neq\varnothing} m^+(B),\ 1-\sum_{B\not\subset A; B\neq A} m^-(B) - m^-(\varnothing)\Big) \quad (7)$$

Taking into account limits of empty sets obtained during combination ranges of believes for each of the events are as shown in table 2.

Table 2. Joint masses, belief function values and tighten bounds

Event	Joint masses	Interval-valued beliefs	Tighten intervals
$\{s_1\}$	[0.298, 0.551]	[0.298, 0.551]	[0.313, 0.529]
$\{s_2\}$	[0.077, 0.220]	[0.077, 0.220]	[0.078, 0.219]
$\{s_1, s_2\}$	[0.0003, 0.0012]	[0.471, 0.709]	[0.0003, 0.0012]

MTE defines belief function in terms of the mass of evidence assigned to each event and its constituents, if available. Thus in order to obtain total belief committed to the set, masses of evidence associated with all the sets that are subsets of the given set must be added. Consequently beliefs of atomic event remain unchanged and equal to combined values. Joint events increase their belief values according to constituents masses (see last row in table 2).

3.1 *Evidence combination as optimization problem*

Presented procedure is an extension of initial Dempster proposal intended for structures with crisp events as well as crisp masses assigned to the events. Extension of the approach substitute crisp values with interval-valued probabilities. Subsequently principles of adequate mathematics are to be applied.

Unfortunately such direct modification can lead to results that are too broad. The new approach toward data association is to be considered since its results are to be tightened. Problem of combination of interval-valued structures can be introduced as following optimization task (Denoeux 1999).

Search for lower and upper limits of combined structure:

$$m^-_A(m_1, m_2) = \min \sum_{B\cap C=A} m_1(B) * m_2(C)$$
$$m^+_A(m_1, m_2) = \max \sum_{B\cap C=A} m_1(B) * m_2(C) \quad (8)$$

Under constraints:

$$\sum_{B\in\mathcal{F}(m_1)} m_1(B) = 1$$
$$\sum_{C\in\mathcal{F}(m_2)} m_1(C) = 1$$
$$m_1^-(B) \le m_1(B) \le m_1^+(B) \quad \forall B\in\mathcal{F}(m_1)$$
$$m_1^-(C) \le m_1(C) \le m_1^+(C) \quad \forall C\in\mathcal{F}(m_2) \quad (9)$$

Adequate optimization problem was solved using available software and results are shown in the rightmost part of table 2. It is seen that optimization leads to results falling within limits established with previous method. All further results of combination presented in the paper were obtained using software available at website:

http://www.hds.utc.fr/~tdenoeux/.

The software implements procedures solving above defined optimization problem.

4 BELIEF STRUCTURES IN MARITIME NAVIGATION

Previously presented case of guessing which route will be taken by unknown vessel, although interesting, is not very much representative for maritime navigation. Its typical problems are related to position fixing. The aim of the position interpretation is to find out what the distance from nearest obstacle could be. The distance given as crisp value is not of primary importance instead it subjective assessment really matters. Subjectivity should embrace local condition. Confined water distance of 4Nm must be differently perceived than the same distance in the open sea. Nevertheless safe or sufficient distance value is to be maintained everywhere and all the time. Example of the set of fuzzy-valued subjective distances is shown in figure 5.

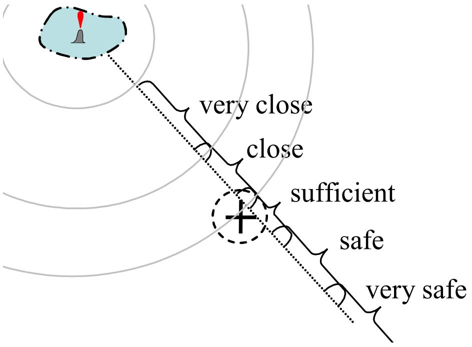

Figure 5. Distances from an obstacle expressed as fuzzy values

Fixing can be directly transferred into appropriate state referring to the obstacle. Being within the state can be treated as an event in MTE terminology. From figure 5 it is also clear that limits of a state are imprecise values therefore event is not crisp any longer. In figure 5 circle around position cross reflects error, standard deviation attributed to particular system. Marked spot is somewhere in between "close" and "sufficient" distance if proposed limits are assumed. Instead of establishing borders one can ask experts what they think about, for example, 4Nm off the buoy. They are to use scale that covers five terms from "very close" to "very safe". Table 3 contains results of the inquiry with 16 unity intervals scale. Each linguistic term covers four adjacent unity intervals. Extreme interval is assumed to be shared with neighbour term.

Table 3. Meaning of 4Nm off safe water buoy in the given area

	very close							sufficient						very safe		
					close						safe					
Expert	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1									+	+	+					
2								+	+	+	+					
3						+	+	+	+							
4								+	+	+						
5								+	+							
frequency						0.2		0.8		0.6						
							0.2		1.0		0.4					

Last two rows of the table 3 embrace relative frequencies of answers for non-zero unity intervals. Set of these figures creates irregular membership function that will be written as:

$$\mu_{d1}(x_i) = \{0.2/6, 0.2/7, 0.8/8, 1/9, 0.6/10, 0.4/11\}$$

Unlike theoretical membership functions these similar to presented in table 3 are called empirical membership functions.

Accuracy of distance measured by a navigational aid depends on method and appliance involved. Different credibility is attributed to various aids. To conclude reasoning regarding measured distance one has to attribute mass of credibility to engaged system. Let us assume that example system's credibility is high. In this case using suggested 6-grade scale and formula 3 factor k will be assumed as equal to 4 (trapezoid regular membership function with w_T=0.8 are further used). Doubtfulness regarding proper functionality of the aid and outcome of expert opinions is rather low (k=1).

Above statements define following belief structure.

Measured distance to the obstacle expressed subjectively:

$$\mu_{d1}(x_i) = (0.2/6, 0.2/7, 0.8/8, 1/9, 0.6/10, 0.4/11)$$

Mass of credibility attributed to navigational aid and quality of expert opinions:

$$m_1(d_1) = (0.8, 0.84, 0.96, 1)$$

The last can be approximately expressed as:

$$m_1(d_1) \approx \begin{cases} \alpha = 1 & [0.84,\ 0.96] \\ \alpha = 0.5 & [0.82,\ 0.98] \\ \alpha = 0.0 & [0.80,\ 1] \end{cases}$$

Mass of uncertainty attributed to navigational aid and to quality of expert opinions:

$$m_1(any) = (0, 0.04, 0.36, 0.4)$$

This can be equivalent to:

$$m_1(any) \approx \begin{cases} \alpha = 1 & [0.04,\ 0.36] \\ \alpha = 0.5 & [0.02,\ 0.38] \\ \alpha = 0.0 & [0,\ \ 0.40] \end{cases}$$

The latest reflects statement that contradicts membership function shown in table 5. It expresses conclusion that engaged system might not work properly and indicates wrong data. Consequently every distance is equally possible. Membership function attached to such uncertainty consists of all one:

$$\mu_{any}(x_i) = (1/1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11, 1/12, 1/13, 1/14, 1/15, 1/16).$$

Assuming approximation of fuzzy values by interval values at selected possibility levels conditions of definition 1 are observed for all levels thus the above assignment is appropriate belief structure.

In order to enrich knowledge and reduce uncertainty regarding distance from the obstacle we assume that there is another navigational aid that indicates different distance and the aid is also reputed in different way. Another belief structure is as follows.

Measured distance to the obstacle expressed in subjective way:

$$\mu_{d2}(x_i) = (0.2/5, 0.4/6, 0.6/7, 1/8, 0.6/9, 0.2/10)$$

Mass of credibility attributed to another navigational aid and quality of new expert opinions assumed as trapezoid fuzzy value (k = 3 and $w_T = 0.8$):

$$m_2(d_2) = (0.4, 0.44, 0.76, 0.8)$$

$$m_2(d_2) \approx \begin{cases} \alpha = 1 & [0.44,\ 0.76] \\ \alpha = 0.5 & [0.42,\ 0.78] \\ \alpha = 0.0 & [0.40,\ 0.80] \end{cases}$$

Mass of uncertainty attributed to this positioning system and quality of other expert opinions expressed as trapezoid fuzzy value with k = 3 and w_T = 0.8):

$$m_2(any) = (0.2, 0.24, 0.56, 0.6)$$

$$m_2(any) \approx \begin{cases} \alpha = 1 & [0.24,\ 0.56] \\ \alpha = 0.5 & [0.22,\ 0.58] \\ \alpha = 0.0 & [0.20,\ 0.60] \end{cases}$$

Same as before fuzzy values were approximated by interval values at three selected possibility levels. Conditions of definition 1 are observed for each of the levels thus the second assignment is also correct belief structure.

Indications coming from two sources were associated using extended Dempster-Shafer scheme and optimization approach. Obtained results are shown in table 4.

Table 4. Combination of two navigational aids

			$m_1(\mu_{d1})$	$m_1(any)$
	α = 1		[0.84, 0.96]	[0.04, 0.36]
	α =0.5		[0.82, 0.98]	[0.02, 0.38]
	α =0.0		[0.80, 1]	[0, 0.40]
			$m_{1\text{-}2}(\mu_{d1}\wedge\mu_{d2})$	$m_{1\text{-}2}(\mu_{d2})$
	α = 1	[0.44, 0.76]	[0.37, 0.73]	[0.018, 0.27]
$m_2(\mu_{d2})$	α =0.5	[0.42, 0.78]	[0.34, 0.76]	[0.008 0.30]
	α =0.0	[0.40, 0.80]	[0.32, 0.80]	[0.0, 0.32]
			$m_{1\text{-}2}(\mu_{d1})$	$m_{1\text{-}2}(any)$
	α = 1	[0.24, 0.56]	[0.20, 0.54]	[0.01, 0.20]
$m_2(any)$	α =0.5	[0.22, 0.58]	[0.18, 0.57]	[0.004 0.22]
	α =0.0	[0.20, 0.60]	[0.16, 0.60]	[0.0 0.24]

In table 4 there is expression $m_{1\text{-}2}(\mu_{d1}\wedge\mu_{d2})$ that remains to be explained. It is at the intersection of $m_2(\mu_{d2})$ row and $m_1(\mu_{d1})$ column and mean joint confidence regarding distances to the same obstacle measured by different navigational aid. In case of crisp events the mass would be assigned to empty set (∅). In case when events are fuzzy the expression should be written as $m_{1\text{-}2}(\mu_{d1}(x_i)\wedge\mu_{d2}(x_i))$ and interpreted as a mass of confidence attributed to conjunction of two fuzzy values respectively $\mu_{d1}(x_i)$ and $\mu_{d2}(x_i)$. In this case $\mu_{d1}(x_i)\wedge\mu_{d2}(x_i)$ =(0/5, 0.2/6, 0.2/7, 0.8/8, 1/9, 0.6/10, 0.4/11)∧(0.2/5, 0.4/6, 0.6/7, 1/8, 0.6/9, 0.2/10, 0/11) = (0/5, 0.2/6, 0.2/7, 0.8/8, 0.6/9, 0.2/10, 0/11). Note that conjunction ∧ means minimum operation in the two sets. As a result of combination of fuzzy events apart from initial sets appear yet another membership functions. The more sources are combined the more numerous count of such extra events. Note that such events bring some support for certain classes of fuzzy events.

Seemingly this phenomenon makes the approach vague. To some extent the statement is true. At the other hand result of combination could be treated as an encoded knowledge base. Having such database one is supposed to ask questions and get answers. As a matter of fact this is main advantage of the approach.

Kind of questions that can be submitted to the knowledge base depend on the problem at hand. In discussed case it could be interesting to know support for a statement that the distance from the obstacle is safe or sufficient one. Table 5 contains interval values of belief functions for different regular fuzzy functions related to considered scale of distances.

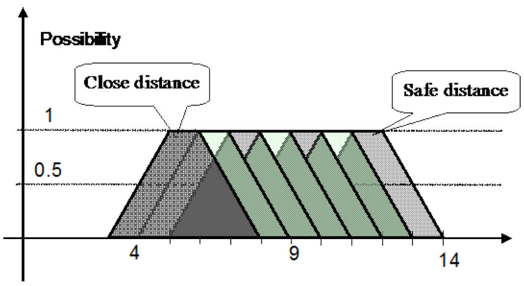

Figure 6. Bundle of benchmark membership functions

Benchmark membership functions used in table 5 are regular trapezoid ones presented in figure 6. They are based on sixteen unity interval scale as presented in table 3. First of the functions reflects term "safe", second one is shifted left (closer to the obstacle) by 1 unit and so on. In this way fourth function is related to sufficient distance and seventh to close condition.

Fuzzy belief functions values are given as α-cuts for α=1, 0.5 and 0 in top to bottom order.

Figure 7 shows diagrams of three belief values marked with asterisk in table 5. They represent interval-valued beliefs that the distance is close, sufficient and safe, for the highest possibility level. The highest credibility with upper limit approaching 0.74 receives sufficient distance.

Table 5. Fuzzy beliefs for obtained combination results and selected fuzzy distances

	Pattern fuzzy value		Belief function	
			α = 1	[0.074, 0.146]*
1	(0.5/10, 1/11, 1/12, 0.5/13)	safe	α =0.5	[0.069, 0.153]

			α =0.0	[0.064, 0.160]
			α = 1	[0.114, 0.195]
2	(0.5/9, 1/10, 1/11, 0.5/12)		α =0.5	[0.105, 0.197]
			α =0.0	[0.096, 0.200]
			α = 1	[0.376, 0.493]
3	(0.5/8, 1/9, 1/10, 0.5/11)		α =0.5	[0.364, 0.497]
			α =0.0	[0.352, 0.500]
			α = 1	[0.510, 0.714]*
4	(0.5/7, 1/8, 1/9, 0.5/10)	sufficient	α =0.5	[0.493, 0.727]
			α =0.0	[0.476, 0.740]
			α = 1	[0.358, 0.475]
5	(0.5/6, 1/7, 1/8, 0.5/9)		α =0.5	[0.347, 0.480]
			α =0.0	[0.336, 0.484]
			α = 1	[0.155, 0.315]
6	(0.5/5, 1/6, 1/7, 0.5/8)		α =0.5	[0.141, 0.326]
			α =0.0	[0.128, 0.336]
			α = 1	[0.074, 0.146]*
7	(0.5/4, 1/5, 1/6, 0.5/7)	close	α =0.5	[0.069, 0.153]
			α =0.0	[0.064, 0.160]

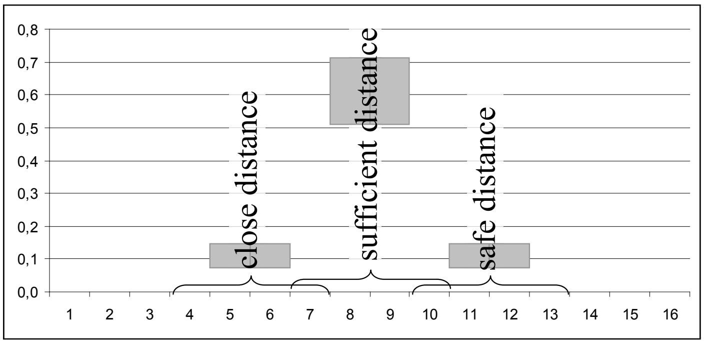

Figure 7. Belief intervals for close, sufficient and safe distances

5 CONCLUSIONS

Bridge officer has to use different navigational aids in order to refine position of the vessel. To combine various sources he uses his common sense or relies on traditional way of data association. So far Kalman filter proved to be most famous method of data integration. Mathematical Theory of Evidence delivers new ability. It can be used for data combination that results in their enrichment. Dempster-Shafer scheme initially designed for crisp data association now is widely used to cope with imprecision, which is expressed by intervals or fuzzy values. Assignment of masses of evidence to each of events at hand creates belief structure. Crisp, interval-valued and fuzzy-valued belief structures are distinguished.

In the paper interval-valued belief structure is defined. It is also shown that transition from interval to fuzzy-valued structure is straightforward. Example of such structures for position fixing was presented. The structures were then combined and results discussed. The most important conclusion that can be drawn from included example is that with help of MTE quantification of imprecise statement is possible. With at least two navigational aids engaged credibility that the distance from an obstacle is safe receives its unique, although interval or fuzzy-valued belief.

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