Opuscula Math. 44, no. 5 (2024), 727–748
https://doi.org/10.7494/OpMath.2024.44.5.727 Opuscula Mathematica

WINTNER-TYPE NONOSCILLATION THEOREMS FOR CONFORMABLE LINEAR STURM–LIOUVILLE DIFFERENTIAL EQUATIONS

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Communicated by Josef Diblík

Abstract. In this study, we addressed the nonoscillation of the Sturm–Liouville differential equation with a differential operator, which corresponds to a proportional-derivative controller. The equation is a conformable linear differential equation. A Wintner-type nonoscillation theorem was established to be applied to such equations. Using this theorem, we provided a sharp nonoscillation condition that guarantees that all nontrivial solutions to Euler-type conformable linear equations do not oscillate. The main nonoscillation theorems can be proven by introducing a Riccati inequality, which corresponds to the conformable linear equation of the Sturm–Liouville type.

Keywords: nonoscillation, conformable differential equation, proportional-derivative controller, Riccati technique, Euler equation.

Mathematics Subject Classification: 34C10, 26A24.

1. INTRODUCTION

Typical examples of fractional differential equations include using the Riemann–Liouville and Caputo definitions, which are used to define phenomena that occur in fields such as engineering, physics, economics, and science (for example, see [21]). However, the Riemann–Liouville and Caputo definitions do not satisfy results applicable to ordinary differentiation (see [12, 19, 23] for details). The development of a novel differential that can express the properties of ordinary differentials with neither excess nor deficiency has attracted considerable research attention (for example, see [1, 2, 4, 5, 16, 28]). This derivative is called the *conformable fractional derivative*. For example, a common derived conformable fractional derivative is given by the following differential operator:

$$T_\alpha f(t) := \lim_{s\to t} \frac{f(t)-f(s)}{t-s} t^{1-\alpha}, \quad t > 0,$$

as defined by Khalil *et al.* [19] to cover the Riemann–Liouville and Caputo definitions. However, the definition by Khalil *et al.* [19] denote that the zeroth derivative of a function does not necessarily return to the function itself. Thus, this definition does not satisfy the identity. The strict-sense criteria for the fractional derivative are (i) linearity, (ii) identity, (iii) backward compatibility, (iv) index law, and (v) generalized Leibniz rule, as considered by Ortigueira *et al.* [23]. According to the strict-sense criteria, calling the definition by Khalil *et al.* [19] a fractional derivative could be incorrect. As an improvement to the definition by Khalil *et al.* [19], Anderson and Ulness [10] provided the following definition:

Definition 1.1. Let α be a constant defined on interval $[0,1]$. Two continuous functions, $\kappa_0 : [0,1] \times \mathbb{R} \to [0,\infty)$ and $\kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$, satisfy

$$\begin{cases} \lim\limits_{\alpha \to 0+} \kappa_0(\alpha,t) = 0, & \lim\limits_{\alpha \to 0+} \kappa_1(\alpha,t) = 1, \\ \lim\limits_{\alpha \to 1-} \kappa_0(\alpha,t) = 1, & \lim\limits_{\alpha \to 1-} \kappa_1(\alpha,t) = 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \kappa_0(\alpha,t) \neq 0, & \alpha \in (0,1], \\ \kappa_1(\alpha,t) \neq 0, & \alpha \in [0,1). \end{cases} \tag{1.2}$$

Next, the differential operator D^α is defined as follows:

$$D^\alpha f(t) = \kappa_1(\alpha,t) f(t) + \kappa_0(\alpha,t) \frac{\mathrm{d}}{\mathrm{d}t} f(t), \tag{1.3}$$

where κ_0 and κ_1 are the functions that satisfy (1.1) and (1.2), respectively.

Remark 1.2. From condition (1.1), we have

$$\lim_{\alpha \to 0+} D^\alpha f(t) = D^0 f(t) = f(t) \quad \text{and} \quad \lim_{\alpha \to 1-} D^\alpha f(t) = D^1 f(t) = f'(t).$$

Furthermore, for arbitrary $\alpha \in [0,1]$ and $\beta \in [0,1]$, $D^\alpha D^\beta \neq D^\beta D^\alpha$ in general. However, if the two continuous functions, κ_0 and κ_1, are constant, then $D^\alpha D^\beta = D^\beta D^\alpha$.

Remark 1.3. For $\alpha \in (0,1)$, the nonzero condition of κ_1 in (1.2) ($\kappa_1(\alpha,t) \equiv 0$), $\kappa_0(\alpha,t) = \alpha t^{1-\alpha}$, is relaxed. Next, we have a special case of the definition by Khalil *et al.* [19]. For that definition [19], if function f is differentiable, then $T_\alpha f(t) = t^{1-\alpha} f'(t)$.

Definition 1.1 is independent of fractional differentiation, and it is intended for proportional–derivative control expressed as

$$x(t) = \kappa_p E(t) + \kappa_d \frac{\mathrm{d}}{\mathrm{d}t} E(t)$$

where the controller provides x at time t, κ_p is the proportional gain, κ_d is the derivative gain, and E is the input deviation. As an application, this controller is used in robotics (see, for example, [6, 13]). Therefore, Definition 1.1 is called a *proportional–derivative controller.* Furthermore, because Definition 1.1 is considered independent of the fractional derivative, it is also called a *conformable derivative.* However, Definition 1.1 provides certain fractional derivative properties. According to Gao and Chi [16], if we compare the solutions of the equation using defining formula (1.3) for $\kappa_1(\alpha, t) = 1 - \alpha$ and $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$ with the equation with the Riemann–Liouville and Caputo definitions obtained by numerical simulation, the behavior of both solutions is similar (see [16, Section 3]).

We consider the nonoscillation of conformable linear Sturm–Liouville differential equations with the following form:

$$D^\alpha[r(t)D^\alpha x] + c(t)x = 0 \tag{1.4}$$

using (1.3) in Definition 1.1, where $\alpha \in (0, 1]$ and $r, c : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $t_0 \geq 0$ and $r(t) > 0$ for $t \geq t_0$. The uniqueness of the solution to the initial value problem of (1.4) is guaranteed ([7, Theorem 3.2]). We define the oscillation of the nontrivial solution of (1.4). A nontrivial solution x of (1.4) is said to be *nonoscillatory* on $[t_0, \infty)$ if it is eventually either positive or negative. Otherwise, the nontrivial solution x of (1.4) is said to be *oscillatory.* Furthermore, Sturm's comparison and separation theorems for (1.4) have been established in [7, Theorem 7.2] and [9, Theorem 8.3 .6], respectively. Therefore, because the oscillatory and nonoscillatory solutions of (1.4) are separate, if a nontrivial solution of (1.4) is oscillatory (or nonoscillatory), then all nontrivial solutions of (1.4) are also oscillatory (or nonoscillatory).

As a special case of (1.4), when $\alpha = 1$, (1.4) becomes an ordinary second-order linear differential equation:

$$(r(t)x')' + c(t)x = 0. \tag{1.5}$$

Several results have been provided for the oscillation theory regarding (1.5) over a long time (see [3, 24] for details). The classification of global solutions of (1.5) into oscillatory and nonoscillatory types is based on the magnitudes of coefficients r and c in (1.5). In particular, for (1.5), we introduce a preliminary result that extends Wintner's nonoscillation theorem [25], which is conventional in oscillation theory. This result has been addressed in various studies (see, for example, [14, 15, 17, 27]).

Theorem 1.4 ([25]). *Let*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \mathrm{d}s = \infty \tag{1.6}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} c(s)\mathrm{d}s \textit{ is convergent,} \tag{1.7}$$

that is, $\int_{t_0}^{\infty} c(s)\mathrm{d}s < \infty$. *If*

$$-\frac{3}{4} < \liminf_{t\to\infty} A_1(t) \leq \limsup_{t\to\infty} A_1(t) < \frac{1}{4} \tag{1.8}$$

then all nontrivial solutions of (1.5) *are nonoscillatory, with*

$$A_1(t) = \int_{t_0}^{t} \frac{1}{r(s)} \mathrm{d}s \int_{t}^{\infty} c(s) \mathrm{d}s. \tag{1.9}$$

A theorem corresponding to Theorem 1.4 is well-known and described as follows:

Theorem 1.5. *Let*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \mathrm{d}s < \infty. \tag{1.10}$$

If

$$-\frac{3}{4} < \liminf_{t\to\infty} A_2(t) \le \limsup_{t\to\infty} A_2(t) < \frac{1}{4} \tag{1.11}$$

then all nontrivial solutions of (1.5) *are nonoscillatory, where*

$$A_2(t) = \int_{t}^{\infty} \frac{1}{r(s)} \mathrm{d}s \int_{t_0}^{t} c(s) \mathrm{d}s. \tag{1.12}$$

Upper limit 1/4 of conditions (1.8) and (1.11) in Theorems 1.4 and 1.5, respectively, is a common reference value for determining whether the global solution of Euler's equation oscillates. This limit is known as the *oscillation constant*. The global solution of Euler's equation can be classified as an oscillatory or nonoscillatory solution based on the oscillation constant. Therefore, the solution is an important test equation. Additionally, 1/4 appears as a threshold in the Kneser-type oscillation and nonoscillation criteria [20]. Wu and Sugie [27] discussed the case expressed as

$$\liminf_{t\to\infty} A_1(t) < -\frac{3}{4} \quad \text{or} \quad \liminf_{t\to\infty} A_2(t) < -\frac{3}{4}.$$

Çetinkaya and Cuchta [11] and Ishibashi [18] provided oscillation and nonoscillation theorems for equations including (1.4). However, limited progress has been achieved on the long standing oscillation and nonoscillation theorems corresponding to (1.4). Therefore, this study focused on establishing a Wintner-type nonoscillation theorem corresponding to Theorems 1.4 and 1.5 and that can be applied to (1.4). We provide a nonoscillation theorem for (1.4) corresponding to Theorems 1.4 and 1.5. The proof of the main theorems uses the generalized Riccati inequality corresponding to (1.4).

This paper is organized as follows. Background on differential and integral calculus corresponding to the conformable derivative defined by (1.3) is provided in the Appendix. Section 2 introduces the Riccati technique used to prove the main theorem. In particular, we demonstrated that all the nontrivial solutions of (1.4) are nonoscillatory and that the existence of a global solution to the Riccati inequality corresponding to (1.4) is equivalent. In Section 3, we introduce the nonoscillation theorems for (1.4), which correspond to Theorems 1.4 and 1.5. In Section 4, we use the main theorem to provide the nonoscillatory conditions for the Euler-type conformable differential equation. For example, we provide a nonoscillatory condition for the Euler-type conformable differential equation with κ_1 and κ_0, which is a relaxation of the nonzero condition (1.2) in Remark 1.2. Numerical simulations demonstrated that the constant that appears in the nonoscillatory condition of the Euler-type conformable differential equation is close to the oscillation constant. Furthermore, when $0 < \alpha < 1$, we provide a nonoscillatory condition for the Sturm–Liouville-type conformable differential equation with $\kappa_1(\alpha, t) = 1 - \alpha$ and $\kappa_0(\alpha, t) = \alpha$ satisfying (1.1) and (1.2). Section 5 presents our conclusions and directions of future work.

2. RICCATI CONFORMABLE DIFFERENTIAL INEQUALITY RELATED TO STURM–LIOUVILLE TYPE

Equation (1.4) and the first-order conformable nonlinear differential inequality introduced below have the following relationship.

Lemma 2.1. *The following are equivalent:*

(i) *All nontrivial solutions to* (1.4) *are nonoscillatory.*

(ii) *A differentiable function v exists that satisfies*

$$\begin{aligned} D^\alpha v &\leq -c(t) - \frac{v^2}{r(t)} + \kappa_1(\alpha, t)v \\ &= -c(t) - \frac{1}{r(t)}\Big(v - \frac{1}{2}r(t)\kappa_1(\alpha, t)\Big)^2 + \frac{1}{4}r(t)\kappa_1^2(\alpha, t) \end{aligned} \tag{2.1}$$

for large t.

(iii) *A differentiable function w exists that satisfies*

$$\begin{aligned} D^\alpha w &\geq c(t) + \frac{w^2}{r(t)} + \kappa_1(\alpha, t)w \\ &= c(t) + \frac{1}{r(t)}\Big(w + \frac{1}{2}r(t)\kappa_1(\alpha, t)\Big)^2 - \frac{1}{4}r(t)\kappa_1^2(\alpha, t) \end{aligned} \tag{2.2}$$

for large t.

Proof. (i)$\Rightarrow$(ii) Assume that for any $t \geq t_1$, there exists $t_1 \geq t_0$ that satisfies $x(t) > 0$. We define differentiable function v as

$$v(t) = \frac{r(t)D^\alpha x(t)}{x(t)}.$$

Then, from

$$\begin{aligned} 0 &= D^\alpha(r(t)D^\alpha x(t)) + c(t)x(t) \\ &= D^\alpha(v(t)x(t)) + c(t)x(t) \\ &= D^\alpha v(t)x(t) + v(t)D^\alpha x(t) - v(t)x(t)\kappa_1(\alpha, t) + c(t)x(t) \\ &= D^\alpha v(t)x(t) + \frac{v^2(t)x(t)}{r(t)} - v(t)x(t)\kappa_1(\alpha, t) + c(t)x(t) \\ &= \Big(\frac{v^2(t)}{r(t)} + D^\alpha v(t) - \kappa_1(\alpha, t)v(t) + c(t)\Big)x(t). \end{aligned}$$

Here, v satisfies inequality (2.1).

(ii)⇒(i) Suppose that for $t \in \mathbb{R}$ and any $t \geq t_0$, $v(t)$ exists such that it satisfies inequality (2.1). Consider

$$D^\alpha(r(t)D^\alpha y) + (c(t) + C(t))y = 0, \tag{2.3}$$

where

$$C(t) = -D^\alpha v(t) - c(t) - \frac{v^2(t)}{r(t)} + \kappa_1(\alpha, t)v(t) \geq 0.$$

Equation (2.3) has a nonoscillatory solution of

$$\begin{aligned} y(t) &= \exp\left(\int_{t_0}^t \frac{\frac{v(s)}{r(s)} - \kappa_1(\alpha, s)}{\kappa_0(\alpha, s)} \mathrm{d}s\right) \\ &= \exp\left(\int_{t_0}^t \left(\frac{v(s)}{r(s)} - \kappa_1(\alpha, s)\right) \mathrm{d}_\alpha s\right) = e_{\frac{v}{r}}(t, t_0), \end{aligned}$$

where $\mathrm{d}_\alpha s = \mathrm{d}s/\kappa_0(\alpha, s)$ (see Appendix for the conformable exponential function, $e_\phi(t, t_0)$). From Sturm's separation theorem, we demonstrate that y is a solution to (2.3) because all its nontrivial solutions (1.4) are nonoscillatory. From

$$D^\alpha y(t) = \frac{v(t)}{r(t)} e_{\frac{v}{r}}(t, t_0)$$

and

$$\begin{aligned} D^\alpha(r(t)D^\alpha y(t)) &= D^\alpha\left(r(t)\frac{v(t)}{r(t)} e_{\frac{v}{r}}(t, t_0)\right) \\ &= D^\alpha v(t) e_{\frac{v}{r}}(t, t_0) + v(t) D^\alpha e_{\frac{v}{r}}(t, t_0) - v(t) e_{\frac{v}{r}}(t, t_0)\kappa_1(\alpha, t) \\ &= D^\alpha v(t) e_{\frac{v}{r}}(t, t_0) + \frac{v^2(t)}{r(t)} e_{\frac{v}{r}}(t, t_0) - v(t) e_{\frac{v}{r}}(t, t_0)\kappa_1(\alpha, t), \end{aligned}$$

we see that

$$\begin{aligned} &D^\alpha(r(t)D^\alpha y(t)) + (c(t) + C(t))y(t) \\ &= D^\alpha v(t) e_{\frac{v}{r}}(t, t_0) + \frac{v^2(t)}{r(t)} e_{\frac{v}{r}}(t, t_0) - v(t) e_{\frac{v}{r}}(t, t_0)\kappa_1(\alpha, t) \\ &\quad + \left(c(t) - D^\alpha v(t) - c(t) - \frac{v^2(t)}{r(t)} + v(t)\kappa_1(\alpha, t)\right) e_{\frac{v}{r}}(t, t_0) \\ &= 0. \end{aligned}$$

Therefore, y is a solution to (2.3). By comparing the coefficients of (1.4) and (2.3), we find that

$$c(t) + C(t) \geq c(t),$$

Thus, according to Sturm's comparison theorem, all nontrivial solutions of (1.4) are nonoscillatory.

(ii)⇒(iii) Suppose that a differentiable function v satisfies inequality (2.1). In this case, if differentiable function w is $w(t) = -v(t)$, w satisfies inequality (2.2).

(iii)⇒(ii) Suppose that a differentiable function w satisfies inequality (2.2). In this case, if differentiable function v is $v(t) = -w(t)$, v satisfies inequality (2.1). □

Remark 2.2. When $\alpha = 1$, inequality (2.1) becomes the typical Riccati inequality ([3, Lemma 2.2.1]). Therefore, (2.1) and (2.2) are Riccati inequalities that correspond to the conformable derivative.

3. WINTNER-TYPE NONOSCILLATION THEOREMS

The Wintner-type nonoscillation theorem for (1.4) is described as follows.

Theorem 3.1. *Suppose that*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s = \infty \tag{3.1}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} c(s) e_0(t_0, s) \mathrm{d}_\alpha s < \infty \tag{3.2}$$

Let

$$B_1(t) = \int_{t_0}^{t} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s \int_{t}^{\infty} c(s) e_0(t,s) \mathrm{d}_\alpha s. \tag{3.3}$$

Next, we obtain the following:

(i) *If* $r(t)\kappa_1(\alpha,t) \int_{t_0}^{t} r^{-1}(s) e_0(t,s) \mathrm{d}_\alpha s < 1$ *and*

$$-\frac{3}{4} + r(t)\kappa_1(\alpha,t) \int_{t_0}^{t} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s \le B_1(t) \le \frac{1}{4} \tag{3.4}$$

then all nontrivial solutions of (1.4) *are nonoscillatory.*

(ii) *If* $r(t)\kappa_1(\alpha,t) \int_{t_0}^{t} r^{-1}(s) e_0(t,s) \mathrm{d}_\alpha s > 1$ *and*

$$\frac{1}{4} \le B_1(t) \le -\frac{3}{4} + r(t)\kappa_1(\alpha,t) \int_{t_0}^{t} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s \tag{3.5}$$

then all nontrivial solutions of (1.4) *are nonoscillatory.*

Remark 3.2. For condition (i) of Theorem 3.1, if $\alpha = 1$, then $e_0(t,s) = 0$ and $\mathrm{d}_\alpha s = \mathrm{d}s$. Thus, conditions (3.1), (3.2), and (3.3) correspond to conditions (1.6) and (1.7) and match (1.9). Furthermore, condition (3.4) corresponds to condition (1.8) in Theorem 1.4. Theorem 3.1(ii) is a unique result of the conformable linear differential equation given by (1.4).

The counterpart nonoscillation theorem for condition (3.1) in Theorem 3.1 is described as follows.

Theorem 3.3. *Suppose that*

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{e_0(t_0,s)}{r(s)}\mathrm{d}_\alpha s<\infty, \tag{3.6}$$

and let

$$B_2(t)=\int_t^\infty\frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\int_{t_0}^{t}c(s)e_0(t,s)\mathrm{d}_\alpha s. \tag{3.7}$$

If

$$-\frac{3}{4}-r(t)\kappa_1(\alpha,t)\int_t^\infty\frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\le B_2(t)\le\frac{1}{4} \tag{3.8}$$

then all nontrivial solutions of (1.4) *are nonoscillatory.*

Remark 3.4. For the conditions of Theorem 3.3, if $\alpha=1$, then $e_0(t,s)=0$ and $\mathrm{d}_\alpha s=\mathrm{d}s$, and conditions (3.6) and (3.7) match conditions (1.10) and (1.12) of Theorem 1.5, respectively. Additionally, condition (3.8) corresponds to condition (1.11) in Theorem 1.5.

We prove Theorems 3.1 and 3.3 using Lemma 2.1.

Proof of Theorem 3.1. First, we prove (i). If we can find the global solution v of Riccati inequality (2.1) corresponding to (1.4), we can prove Theorem 3.1 using Lemma 2.1. To determine global solution v, consider the following fitness differential inequality:

$$D^\alpha\rho(t)\le-\frac{1}{r(t)}\left(\rho(t)+C(t)-\frac{1}{2}r(t)\kappa_1(\alpha,t)\right)^2+\frac{1}{4}r(t)\kappa_1^2(\alpha,t) \tag{3.9}$$

related to the inequality (2.1), where

$$C(t)=\int_t^\infty c(s)e_0(t,s)\mathrm{d}_\alpha s.$$

Finding global solution v to Riccati inequality (2.1) is equivalent to finding global solution ρ to fitness differential inequality (3.9). Let $v(t)$ be a differentiable global solution that satisfies Riccati inequality (2.1) and

$$\rho(t)=v(t)-C(t).$$

Next, for inequality (2.1) and differentiable ρ,

$$\begin{aligned} D^\alpha \rho(t) &= D^\alpha v(t) - D^\alpha C(t) \\ &\leq -c(t) - \frac{1}{r(t)}\left(\rho(t) + C(t) - \frac{1}{2}\kappa_1(\alpha,t)r(t)\right)^2 + \frac{1}{4}r(t)\kappa_1^2(\alpha,t) \\ &\quad - D^\alpha\left(\int_{t_1}^{\infty} c(s)e_0(t,s)\mathrm{d}_\alpha s - \int_{t_1}^{t} c(s)e_0(t,s)\mathrm{d}_\alpha s\right) \\ &= -c(t) - \frac{1}{r(t)}\left(\rho(t) + C(t) - \frac{1}{2}\kappa_1(\alpha,t)r(t)\right)^2 + \frac{1}{4}r(t)\kappa_1^2(\alpha,t) \\ &\quad - D^\alpha\left(\int_{t_1}^{\infty} c(s)e_0(t,s)\mathrm{d}_\alpha s\right) + D^\alpha\left(\int_{t_1}^{t} c(s)e_0(t,s)\mathrm{d}_\alpha s\right) \\ &= -c(t) - \frac{1}{r(t)}\left(\rho(t) + C(t) - \frac{1}{2}\kappa_1(\alpha,t)r(t)\right)^2 + \frac{1}{4}r(t)\kappa_1^2(\alpha,t) \\ &\quad - D^\alpha\left(\int_{t_1}^{\infty} c(s)e_0(t,t_1)e_0(t_1,s)\mathrm{d}_\alpha s\right) + c(t) \\ &= -\frac{1}{r(t)}\left(\rho(t) + C(t) - \frac{1}{2}\kappa_1(\alpha,t)r(t)\right)^2 + \frac{1}{4}r(t)\kappa_1^2(\alpha,t) \\ &\quad - D^\alpha\left(e_0(t,t_1)\int_{t_1}^{\infty} c(s)e_0(t_1,s)\mathrm{d}_\alpha s\right). \end{aligned}$$

From condition (3.2) and $D^\alpha e_0(t,t_1) = 0$, we have

$$D^\alpha\left(e_0(t,t_1)\int_{t_1}^{\infty} c(s)e_0(t_1,s)\mathrm{d}_\alpha s\right) = \int_{t_1}^{\infty} c(s)e_0(t_1,s)\mathrm{d}_\alpha s\,(D^\alpha e_0(t,t_1)) = 0,$$

obtaining inequality (3.9). By contrast, suppose that $\rho(t)$ exists satisfying inequality (3.9). In this case, if we set $v(t) = \rho(t) + C(t)$, we obtain inequality (2.1). Therefore, we demonstrated the existence of a global solution to inequality (3.9). For any $t \geq t_0$, we define function ρ as

$$\rho(t) = \frac{1}{4\int_{t_0}^{t} \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s}.$$

In this case, the left-hand side of inequality (3.9) is expressed as follows:

$$
\begin{aligned}
D^{\alpha}\rho(t) &= \frac{1}{4}D^{\alpha}\left(\frac{1}{\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s}\right)\\
&= \frac{1}{4}\left\{\frac{D^{\alpha}(1)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s - D^{\alpha}\left(\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)}{\left(\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2} + \frac{\kappa_1(\alpha,t)}{\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s}\right\}\\
&= -\frac{1}{4r(t)}\left\{\frac{1}{\left(\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2} - \frac{2r(t)\kappa_1(\alpha,t)}{\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s}\right\}.
\end{aligned}
$$

Furthermore, if we complete the square, the left-hand side of inequality (3.9) becomes

$$
D^{\alpha}\rho(t) = -\frac{\left(1 - r(t)\kappa_1(\alpha,t)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2}{4r(t)\left(\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2} + \frac{1}{4}r(t)\kappa_1^2(\alpha,t). \tag{3.10}
$$

By contrast, the right-hand side of inequality (3.9) is expressed as follows:

$$
\begin{aligned}
&-\frac{1}{r(t)}\left(\rho(t) + C(t) - \frac{1}{2}\kappa_1(\alpha,t)r(t)\right)^2 + \frac{1}{4}r(t)\kappa_1^2(\alpha,t)\\
&= -\frac{\left(\frac{1}{4} + \int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\, C(t) - \frac{1}{2}r(t)\kappa_1(\alpha,t)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2}{r(t)\left(\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2}\\
&\quad + \frac{1}{4}r(t)\kappa_1^2(\alpha,t)
\end{aligned}
$$

and the left-hand side of the inequality (3.9) is given by (3.10). Thus, we obtain the following expression:

$$
\begin{aligned}
&\frac{1}{4}\left(1 - r(t)\kappa_1(\alpha,t)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2\\
&\geq \left(\frac{1}{4} + \int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\, C(t) - \frac{1}{2}r(t)\kappa_1(\alpha,t)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_{\alpha}s\right)^2.
\end{aligned} \tag{3.11}
$$

Function ρ being a global solution to inequality (3.9) is equivalent to inequality (3.11) holding. We can confirm that inequality (3.11) holds by assuming

$$
r(t)\kappa_1(\alpha,t)\int_{t_0}^{t} r^{-1}(s)e_0(t,s)\mathrm{d}_{\alpha}s < 1
$$

and condition (3.4). Moreover, by finding global solution ρ that satisfies inequality (3.9), global solution v satisfying inequality (2.1) is expressed as follows:

$$v(t) = \frac{1}{4\int_{t_0}^{t} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s} + \int_{t}^{\infty} c(s) e_0(t,s) \mathrm{d}_\alpha s. \tag{3.12}$$

Therefore, from Lemma 2.1, all nontrivial solutions of equation (1.4) are nonoscillatory.

Next, we prove (ii) of Theorem 3.1 in the same manner as we proved (i). Function ρ is a global solution of inequality (3.9) and inequality (3.11) holds. From condition (ii),

$$r(t)\kappa_1(\alpha,t) \int_{t_0}^{t} r^{-1}(s) e_0(t,s) \mathrm{d}_\alpha s > 1$$

and (3.5), and inequality (3.11) holds. Therefore, because global solution v that satisfies inequality (2.1) is (3.12), by Lemma 2.1, all nontrivial solutions of (1.4) are nonoscillatory. $\square$

Proof of Theorem 3.3. Consider the following fitness differential inequality:

$$D^\alpha \tilde{\rho}(t) \le -\frac{1}{r(t)} \left(\tilde{\rho}(t) - \tilde{C}(t) - \frac{1}{2} r(t) \kappa_1(\alpha,t) \right)^2 + \frac{1}{4} r(t) \kappa_1^2(\alpha,t) \tag{3.13}$$

related to inequality (2.1), where

$$\tilde{C}(t) = \int_{t_0}^{t} c(s) e_0(t,s) \mathrm{d}_\alpha s.$$

Finding the global solution v of Riccati inequality (2.1) is equivalent to finding global solution $\tilde{\rho}$ of fitness differential inequality (3.13). Let $v(t)$ be a differentiable global solution that satisfies Riccati inequality (2.1) and let it be

$$\tilde{\rho}(t) = v(t) + \tilde{C}(t).$$

Because inequality (2.1) and differentiable $\tilde{\rho}$ become

$$\begin{aligned} D^\alpha \tilde{\rho}(t) &= D^\alpha v(t) + D^\alpha \tilde{C}(t) \\ &\le -c(t) - \frac{1}{r(t)} \left(\tilde{\rho}(t) - \tilde{C}(t) - \frac{1}{2}\kappa_1(\alpha,t) r(t) \right)^2 + \frac{1}{4} r(t) \kappa_1^2(\alpha,t) \\ &\quad + D^\alpha \left(\int_{t_0}^{t} c(s) e_0(t,s) \mathrm{d}_\alpha s \right) \\ &= -c(t) - \frac{1}{r(t)} \left(\tilde{\rho}(t) - \tilde{C}(t) - \frac{1}{2}\kappa_1(\alpha,t) r(t) \right)^2 + \frac{1}{4} r(t) \kappa_1^2(\alpha,t) + c(t) \\ &= -\frac{1}{r(t)} \left(\tilde{\rho}(t) - \tilde{C}(t) - \frac{1}{2}\kappa_1(\alpha,t) r(t) \right)^2 + \frac{1}{4} r(t) \kappa_1^2(\alpha,t), \end{aligned}$$

we obtain inequality (3.13). By contrast, suppose that $\tilde{\rho}(t)$ satisfies inequality (3.13). In this case, by letting $v(t) = \tilde{\rho}(t) - \tilde{C}(t)$, we obtain inequality (2.1). Therefore, we demonstrated the existence of a global solution to inequality (3.13). For any $t \geq t_0$, let function $\tilde{\rho}$ be

$$\tilde{\rho}(t) = -\frac{1}{4\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s}.$$

Considering condition (3.6), we can calculate the left-hand side of inequality (3.13) from

$$\begin{aligned}
D^\alpha\left(\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right) &= D^\alpha\left(\int_{t_1}^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s - \int_{t_1}^t \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)\\
&= D^\alpha\left(\int_{t_1}^\infty \frac{e_0(t,t_1)e_0(t_1,s)}{r(s)}\mathrm{d}_\alpha s\right) - D^\alpha\left(\int_{t_1}^t \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)\\
&= \int_{t_1}^\infty \frac{e_0(t_1,s)}{r(s)}\mathrm{d}_\alpha s(D^\alpha e_0(t,t_1)) - \frac{1}{r(t)}\\
&= -\frac{1}{r(t)}
\end{aligned}$$

as follows:

$$\begin{aligned}
D^\alpha\tilde{\rho}(t) &= -\frac{1}{4}D^\alpha\left(\frac{1}{\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s}\right)\\
&= -\frac{1}{4}\left\{\frac{D^\alpha(1)\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s - D^\alpha\left(\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)}{\left(\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2} + \frac{\kappa_1(\alpha,t)}{\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s}\right\}\\
&= -\frac{1}{4r(t)}\left\{\frac{1}{\left(\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2} + \frac{2r(t)\kappa_1(\alpha,t)}{\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s}\right\}.
\end{aligned}$$

Furthermore, if we complete the square, the left-hand side of inequality (3.13) becomes

$$D^\alpha\tilde{\rho}(t) = -\frac{\left(1 + r(t)\kappa_1(\alpha,t)\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2}{4r(t)\left(\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2} + \frac{1}{4}r(t)\kappa_1^2(\alpha,t). \tag{3.14}$$

By contrast, the right-hand side of the inequality (3.13) is expressed as follows:

$$\begin{aligned}
&-\frac{1}{r(t)}\left(\tilde{\rho}(t)-\tilde{C}(t)-\frac{1}{2}\kappa_1(\alpha,t)r(t)\right)^2+\frac{1}{4}r(t)\kappa_1^2(\alpha,t)\\
&=-\frac{\left(\frac{1}{4}+\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\,\tilde{C}(t)+\frac{1}{2}r(t)\kappa_1(\alpha,t)\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2}{r(t)\left(\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2}\\
&\quad+\frac{1}{4}r(t)\kappa_1^2(\alpha,t)
\end{aligned}$$

and the left-hand side of inequality (3.13) becomes (3.14). Thus, we obtain the following:

$$\begin{aligned}
&\frac{1}{4}\left(1+r(t)\kappa_1(\alpha,t)\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2\\
&\geq\left(\frac{1}{4}+\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\,\tilde{C}(t)+\frac{1}{2}r(t)\kappa_1(\alpha,t)\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\right)^2.
\end{aligned}\tag{3.15}$$

Therefore, function $\tilde{\rho}$ being a global solution to inequality (3.13) is equivalent inequality (3.15) holding. We can confirm that inequality (3.15) holds because of the following:

$$1+r(t)\kappa_1(\alpha,t)\int_t^\infty r^{-1}(s)e_0(t,s)\mathrm{d}_\alpha s>0$$

and assuming condition (3.8). Furthermore, by finding global solution $\tilde{\rho}$ that satisfies inequality (3.13), we know that global solution v satisfying inequality (2.1) is expressed as follows:

$$v(t)=-\frac{1}{4\int_t^\infty \frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s}-\int_{t_0}^t c(s)e_0(t,s)\mathrm{d}_\alpha s.$$

Therefore, from Lemma 2.1, all nontrivial solutions of (1.4) are nonoscillatory. □

4. EULER-TYPE CONFORMABLE DIFFERENTIAL EQUATION

In this section, we present the application examples of the main theorem. First, consider an example of (1.4), which relaxes nonzero condition (1.2).

Example 4.1. Consider the following conformable linear Sturm–Liouville differential equation:

$$D^\alpha D^\alpha x+\frac{\lambda}{t^{2\alpha}}x=0,\quad t\geq t_0=1,\tag{4.1}$$

where $\alpha\in(0,1)$, $\lambda\in\mathbb{R}$, $\kappa_1(\alpha,t)\equiv 0$, and $\kappa_0(\alpha,t)=\alpha t^{1-\alpha}$. If $\lambda_*\geq\lambda$, then all nontrivial solutions of (4.1) are nonoscillatory, where $\lambda_*=\alpha^4/4$.

Proof. To show that all nontrivial solutions of (4.1) for $\lambda_* \geq \lambda$ do not oscillate, we should show that all nontrivial solutions of equation

$$D^\alpha D^\alpha x + \frac{\lambda_*}{t^{2\alpha}} x = 0, \quad t \geq t_0 = 1 \tag{4.2}$$

are nonoscillatory because if we use Sturm's comparison theorem for (4.2) and (4.1), all the nontrivial solutions of (4.1) for $\lambda_* \geq \lambda$ are nonoscillatory. Therefore, we reveal that all the nontrivial solutions of (4.2) are nonoscillatory. Comparing (4.2) and (1.4) yields the following:

$$r(t) \equiv 1 \quad \text{and} \quad c(t) = \frac{\lambda_*}{t^{2\alpha}}.$$

We apply Theorem 3.1 to coefficient functions r and c in (4.2). Considering

$$\kappa_1(\alpha, t) \equiv 0 \quad \text{and} \quad \kappa_0(\alpha, t) = \alpha t^{1-\alpha},$$

because $e_0(t, s) = 1$ and $\mathrm{d}_\alpha s = \mathrm{d}s / \alpha s^{1-\alpha}$, we have the following expression:

$$r(t)\kappa_1(\alpha, t) \int_{t_0}^{t} \frac{e_0(t, s)}{r(s)} \mathrm{d}_\alpha s = 0 < 1$$

Furthermore, we have the following:

$$\lim_{t\to\infty} \int_1^t \frac{e_0(t, s)}{r(s)} \mathrm{d}_\alpha s = \frac{1}{\alpha} \lim_{t\to\infty} \int_1^t \frac{1}{s^{1-\alpha}} \mathrm{d}s = \frac{1}{\alpha^2} \lim_{t\to\infty} (t^\alpha - 1) = \infty$$

and

$$\begin{aligned} \lim_{t\to\infty} \int_1^t c(s) e_0(t_0, s) \mathrm{d}_\alpha s &= \frac{\lambda_*}{\alpha} \lim_{t\to\infty} \int_1^t \frac{1}{s^{\alpha+1}} \mathrm{d}s \\ &= -\frac{\lambda_*}{\alpha^2} \lim_{t\to\infty} \left(\frac{1}{t^\alpha} - 1 \right) = \frac{\lambda_*}{\alpha^2}. \end{aligned}$$

Thus, conditions (3.1) and (3.2) of Theorem 3.1 are satisfied. Function B_1 is calculated as follows:

$$\begin{aligned} B_1(t) &= \int_1^t \frac{e_0(t, s)}{r(s)} \mathrm{d}_\alpha s \int_t^\infty c(s) e_0(t, s) \mathrm{d}_\alpha s \\ &= \frac{\lambda_*}{\alpha^4 t^\alpha} (t^\alpha - 1) = \frac{1}{4} - \frac{1}{4t^\alpha} < \frac{1}{4}, \end{aligned}$$

and because $1/t^\alpha$ is a monotonically decreasing function, we obtain the following:

$$B_1(t) > -\frac{1}{4t^\alpha} \geq -\frac{1}{4} > -\frac{3}{4}$$

from $-1/t^\alpha \geq -1$ for any $t \geq 1$. Hence, B_1 satisfies condition (3.4). Therefore, all the nontrivial solutions of (4.2) are nonoscillatory. Furthermore, assuming $\lambda_* \geq \lambda$, we obtain relation

$$\frac{\lambda_*}{t^{2\alpha}} \geq \frac{\lambda}{t^{2\alpha}}$$

for the coefficients of (4.2) and (4.1). According to Sturm's comparison theorem, all the nontrivial solutions of (4.1) are nonoscillatory. $\square$

Remark 4.2. If $\alpha = 1$, (4.1) becomes a typical Euler equation, and $\lambda_* = 1/4$ is its oscillation constant.

We numerically simulated the global solution of (4.2). Figure 1 displays the six solution curves for (4.2) from initial value $(x(1), D^{\alpha}x(1)) = (0, 1)$, where the green, red, orange, black, magenta, and blue curves are the solutions for $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.8$, $\alpha = 0.7$, $\alpha = 0.6$, and $\alpha = 0.5$. Because no solution curve has a zero crossing, we can confirm that all nontrivial solution curves are nonoscillatory.

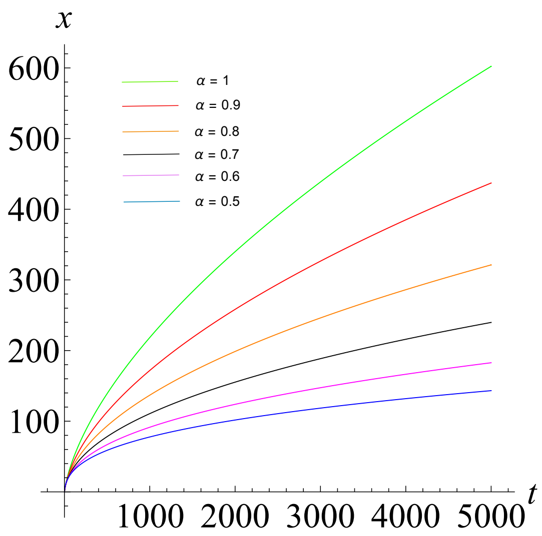

Fig. 1. Solution curves of (4.2) for $\alpha = 1, 0.9, 0.8, 0.7, 0.6, 0.5$

We evaluated the additional values of α. Figure 2 displays the six solution curves for (4.2) from initial value $(x(1), D^{\alpha}x(1)) = (0, 1)$, where the green, red, orange, black, magenta, and blue curves are the solutions for $\alpha = 0.4$, $\alpha = 0.3$, $\alpha = 0.2$, $\alpha = 0.1$, $\alpha = 0.05$, and $\alpha = 0.03$. As displayed in Figure 1, no solution curve has a zero crossing, which confirms the nonoscillatory solutions. Furthermore, the numerical results reveal that when α is small, an initial steep slope of the solution curve appears and subsequently gradually reduces. Furthermore, as α approaches 1, the slope of the solution curve increases.

Next, we determined whether the global solution of (4.1) is nonoscillatory or oscillatory for $\lambda > \lambda_*$. Specifically, we numerically simulated the global solution of (4.1) when $\alpha = 1/2$ and $\lambda = \lambda_* + 0.1$. Figure 3 displays the solution curve for $\alpha = 1/2$ and $\lambda = \lambda_* + 0.1$ starting from initial value $(x(1), D^{\alpha}x(1)) = (0, 1)$. This phenomenon is the global solution curve of (4.1) for $\alpha = 1/2$ and $\lambda = \lambda_* + 0.1$, and it shows one zero. Therefore, the global solution to (4.1) for $\alpha = 1/2$ and $\lambda = \lambda_* + 0.1$ is predicted to oscillate. Generally, all the nontrivial solutions to (4.1) for $\lambda > \lambda_*$ oscillate.

Next, consider an example in which nonzero condition (1.2) is not relaxed.

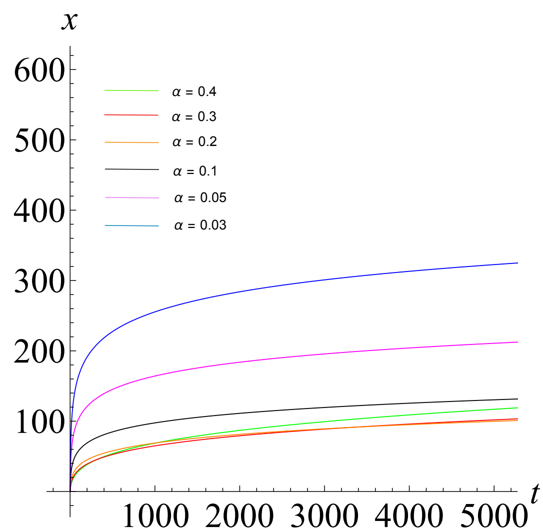

Fig. 2. Solution curves of (4.2) for $\alpha = 0.4, 0.3, 0.2, 0.1, 0.05, 0.03$

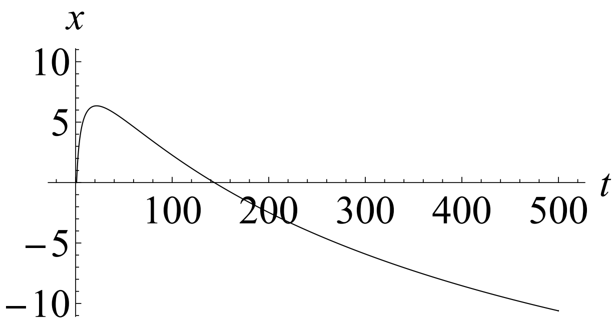

Fig. 3. Solution curve of (4.1) for $\alpha = 1/2$ and $\lambda = \lambda_* + 0.1$

Example 4.3. Consider the following conformable linear Sturm–Liouville differential equation:

$$D^\alpha\left[\frac{1}{e^{\frac{1-\alpha}{\alpha}t}}D^\alpha x\right] + \frac{\tilde{\lambda}}{e^{\frac{1-\alpha}{\alpha}t}\,t^{\alpha+1}}x = 0, \quad t \geq t_0 = 1, \tag{4.3}$$

where $\alpha \in (0,1)$, $\tilde{\lambda} \in \mathbb{R}$, $\kappa_1(\alpha,t) = 1-\alpha$, and $\kappa_0(\alpha,t) = \alpha$. If $\tilde{\lambda}_* \geq \tilde{\lambda}$, then all nontrivial solutions of (4.3) are nonoscillatory, where $\tilde{\lambda}_* = \{\alpha^2(1-\alpha)\}/2$.

Proof. To show that all nontrivial solutions of (4.3) when $\tilde{\lambda}_* \geq \tilde{\lambda}$ do not oscillate, we must show that all nontrivial solutions of the following equation are nonoscillatory:

$$D^{\alpha}\left[\frac{1}{e^{\frac{1-\alpha}{\alpha}t}}D^{\alpha}x\right]+\frac{\tilde{\lambda}_*}{e^{\frac{1-\alpha}{\alpha}t}\,t^{\alpha+1}}x=0,\quad t\geq t_0=1. \tag{4.4}$$

This result is because if we use Sturm's comparison theorem for (4.4) and (4.3), all the nontrivial solutions of (4.3) for $\tilde{\lambda}_*\geq\tilde{\lambda}$ are nonoscillatory. Therefore, we reveal that all the nontrivial solutions of (4.4) are nonoscillatory. Comparing (4.4) and (1.4) yields the following:

$$r(t)=\frac{1}{e^{\frac{1-\alpha}{\alpha}t}}\quad\text{and}\quad c(t)=\frac{\tilde{\lambda}_*}{e^{\frac{1-\alpha}{\alpha}t}\,t^{\alpha+1}}.$$

We apply Theorem 3.1 to coefficient functions r and c of (4.4). Considering $\kappa_1(\alpha,t)=1-\alpha$, $\kappa_0(\alpha,t)=\alpha$, and

$$e_0(t,s)=e^{-\int_s^t\frac{1-\alpha}{\alpha}\mathrm{d}\tau}=\frac{e^{\frac{1-\alpha}{\alpha}s}}{e^{\frac{1-\alpha}{\alpha}t}},$$

we have the following:

$$r(t)\kappa_1(\alpha,t)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s=\frac{1}{2}-\frac{e^{\frac{2(1-\alpha)}{\alpha}}}{2e^{\frac{2(1-\alpha)}{\alpha}t}}<\frac{1}{2}<1.$$

Furthermore, we have the following:

$$\begin{aligned}\lim_{t\to\infty}\int_1^t\frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s&=\lim_{t\to\infty}\frac{1}{\alpha\,e^{\frac{1-\alpha}{\alpha}t}}\int_1^t e^{\frac{2(1-\alpha)}{\alpha}s}\mathrm{d}s\\&=\lim_{t\to\infty}\frac{1}{2(1-\alpha)e^{\frac{1-\alpha}{\alpha}t}}(e^{\frac{2(1-\alpha)}{\alpha}t}-e^{\frac{2(1-\alpha)}{\alpha}})=\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\int_1^t c(s)e_0(t_0,s)\mathrm{d}_\alpha s&=\lim_{t\to\infty}\frac{\tilde{\lambda}_*}{\alpha\,e^{\frac{1-\alpha}{\alpha}}}\int_1^t\frac{1}{s^{\alpha+1}}\mathrm{d}s\\&=\lim_{t\to\infty}\frac{\tilde{\lambda}_*}{\alpha^2\,e^{\frac{1-\alpha}{\alpha}}}\left(1-\frac{1}{t^\alpha}\right)=\frac{\tilde{\lambda}_*}{\alpha^2\,e^{\frac{1-\alpha}{\alpha}}}.\end{aligned}$$

Thus, conditions (3.1) and (3.2) of Theorem 3.1 are satisfied. Function B_1 is calculated as follows:

$$\begin{aligned}B_1(t)&=\int_1^t\frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s\int_t^\infty c(s)e_0(t,s)\mathrm{d}_\alpha s\\&=\frac{\tilde{\lambda}_*}{2\alpha^2(1-\alpha)t^\alpha\,e^{\frac{2(1-\alpha)}{\alpha}t}}\left(e^{\frac{2(1-\alpha)}{\alpha}t}-e^{\frac{2(1-\alpha)}{\alpha}}\right)\\&=\frac{1}{4t^\alpha}-\frac{e^{\frac{2(1-\alpha)}{\alpha}}}{4t^\alpha\,e^{\frac{2(1-\alpha)}{\alpha}t}}\leq\frac{1}{4t^\alpha}\leq\frac{1}{4}\end{aligned}$$

and because $1/t^\alpha$ and $1/e^{\frac{2(1-\alpha)}{\alpha}t}$ are monotonically decreasing functions, we obtain the following:

$$B_1(t) > -\frac{e^{\frac{2(1-\alpha)}{\alpha}}}{4t^\alpha e^{\frac{2(1-\alpha)}{\alpha}t}} \geq -\frac{1}{4} > -\frac{3}{4} + \frac{1}{2} - \frac{e^{\frac{2(1-\alpha)}{\alpha}}}{2e^{\frac{2(1-\alpha)}{\alpha}t}}$$
$$= -\frac{3}{4} + r(t)\kappa_1(\alpha,t)\int_{t_0}^{t}\frac{e_0(t,s)}{r(s)}\mathrm{d}_\alpha s$$

from $-1/t^\alpha e^{\frac{2(1-\alpha)}{\alpha}t} \geq -1/e^{\frac{2(1-\alpha)}{\alpha}}$ for any $t \geq 1$. Therefore, B_1 satisfies condition (3.4). Therefore, because all nontrivial solutions to (4.4) are nonoscillatory, all nontrivial solutions to (4.3) with $\tilde{\lambda}_* \geq \tilde{\lambda}$ are nonoscillatory. □

We subsequently numerically simulated the global solution of (4.4). Figure 4 displays the seven solution curves for (4.4) from initial value $(x(1), D^\alpha x(1)) = (0,1)$, where the green, red, orange, black, magenta, blue, and purple curves are the solutions for $\alpha = 0.9$, $\alpha = 0.8$, $\alpha = 0.7$, $\alpha = 0.6$, $\alpha = 0.5$, $\alpha = 0.4$, and $\alpha = 0.3$, respectively. Because no solution curve has a zero crossing, we can confirm nonoscillatory solutions. Furthermore, the global solution of (4.4) from initial value $(x(1), D^\alpha x(1)) = (0,1)$ converges to a positive constant. Because the value of α changes from 0.9 to 0.6, for the equation with initial value $(x(1), D^\alpha x(1)) = (0,1)$, the global solution of (4.4) converges to smaller positive values. However, with the value of α changing from 0.5 to 0.3, for the equation with initial value $(x(1), D^\alpha x(1)) = (0,1)$, the global solution of (4.4) converges to large positive values.

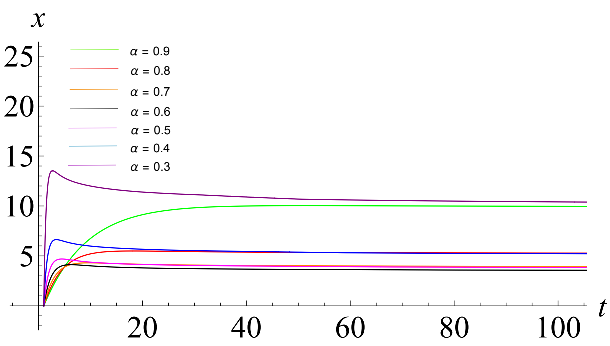

Fig. 4. Solution curves of (4.4) for $\alpha = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3$

For Example 4.1, we determined whether the behavior of the global solution of (4.3) for $\tilde{\lambda} > \tilde{\lambda}_*$ is nonoscillatory or oscillatory. Specifically, we numerically simulated the global solution of (4.3) for $\alpha = 1/2$ and $\tilde{\lambda} = \tilde{\lambda}_* + 0.7$. Figure 5 displays the solution curve for $\alpha = 1/2$ and $\tilde{\lambda} = \tilde{\lambda}_* + 0.7$ from initial value $(x(1), D^\alpha x(1)) = (0,1)$. This result is the global solution of (4.3) for $\alpha = 1/2$ and $\tilde{\lambda} = \tilde{\lambda}_* + 0.7$, and it has one zero. Figure 5 indicates that the global solution to (4.3) for $\alpha = 1/2$ and $\tilde{\lambda} = \tilde{\lambda}_* + 0.7$ probably oscillates. However, determining whether the global solution of (4.3) oscillates for $\tilde{\lambda} > \tilde{\lambda}_*$ is challenging even when using a numerical simulation.

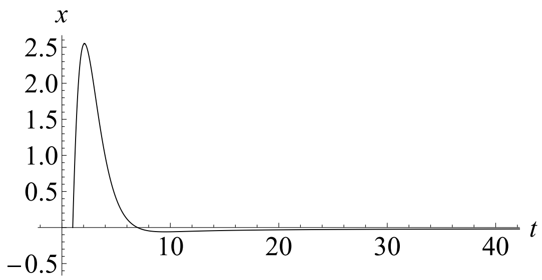

Fig. 5. Solution curve of (4.4) for $\alpha = 1/2$ and $\lambda = \lambda_* + 0.7$

5. CONCLUSION

We extended Theorems 1.4 and 1.5 in oscillation theory to obtain results that can be applied to (1.4). The main results of Theorems 3.1 and 3.3 can be proven by introducing the Riccati-type inequality corresponding to (1.4). As a simple example of (i) in Theorem 3.1, this theorem is applied to Euler-type equations. However, the oscillation constants in (4.1) and (4.3) requires discussion.

As a further development, lower bounds

$$-\frac{3}{4} + r(t)\kappa_1(\alpha, t) \int_{t_0}^{t} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s \le B_1(t)$$

and

$$-\frac{3}{4} - r(t)\kappa_1(\alpha, t) \int_{t}^{\infty} \frac{e_0(t,s)}{r(s)} \mathrm{d}_\alpha s \le B_2(t)$$

of Theorems 3.1(i) and 3.3 to be improved in the future. Similarly, Moore [22] and Wray [26] derived a nonoscillation theorem that can extend the lower bounds of Theorems 1.4 and 1.5.

6. BASIC PROPERTIES ON CONFORMABLE CALCULUS

In this section, the background on conformable calculus for (1.3) of Definition 1.1 is summarized [10].

Theorem 6.1 ([10])**.** *Let $\alpha \in (0,1]$, points $s, t \in \mathbb{R}$ with $s \le t$, and function $\phi : [s,t] \to \mathbb{R}$ be continuous. Furthermore, let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.1) *and* (1.2)*, with ϕ/κ_0 and κ_1/κ_0 being Riemann integrable on $[s,t]$. Next, the exponential function with respect to D^α in* (1.3) *is defined as follows:*

$$e_\phi(t,s) := e^{\int_s^t \frac{\phi(\tau)-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} \mathrm{d}\tau}, \quad e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} \mathrm{d}\tau}, \tag{6.1}$$

and

$$D^\alpha e_\phi(t,s) = \phi(t) e_\phi(t,s), \quad D^\alpha e_0(t,s) = 0.$$

Definition 6.2 ([10])**.** Let $\alpha \in (0,1]$ and $t_0 \in \mathbb{R}$. The antiderivative is defined as follows:

$$\int D^\alpha f(t) \mathrm{d}_\alpha t = f(t) + ce_0(t,t_0), \quad c \in \mathbb{R}.$$

Similarly, the integral of f over $[a,b]$ is defined as follows:

$$\int_a^t f(s)e_0(t,s)\mathrm{d}_\alpha s := \int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}\mathrm{d}s, \quad \mathrm{d}_\alpha s := \frac{1}{\kappa_0(\alpha,s)}\mathrm{d}s.$$

Theorem 6.3 ([10])**.** *Let conformable differential operator D^α be expressed as* (1.3)*, with $\alpha \in [0,1]$. Let function $\phi : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.1) *and* (1.2)*. We assume that functions f and g are differentiable as needed. Next,*

(i) $D^\alpha[kf(t) + lg(t)] = kD^\alpha f(t) + lD^\alpha g(t)$ *for all* $k, l \in \mathbb{R}$,
(ii) $D^\alpha k = k\kappa_1(\alpha,t)$ *for all constant* $k \in \mathbb{R}$,
(iii) $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t) - f(t)g(t)\kappa_1(\alpha,t)$,
(iv) $D^\alpha[f(t)/g(t)] = \frac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g^2(t)} + \frac{f(t)}{g(t)}\kappa_1(\alpha,t)$,
(v) *for* $\alpha \in (0,1]$ *and exponential function* e_0 *given in* (6.1)*, we have*

$$D^\alpha\left[\int_a^t f(s)e_0(t,s)\mathrm{d}_\alpha s\right] = f(t), \quad \mathrm{d}_\alpha s = \frac{1}{\kappa_0(\alpha,s)}\mathrm{d}s.$$

Acknowledgements

I would like to thank Editage (www.editage.com) for English language editing. This study was supported by JSPS KAKENHI Grant-in-Aid for Early-Career Scientists (Grant Number JP22K13933).

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Received: May 13, 2024.
Revised: May 31, 2024.
Accepted: June 1, 2024.