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Application of isogeometric approach to dynamics of curved beam

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Abstract In the paper dynamics of a free-form Timoshenko curved beam is investigated. The considered problem is solved using isogeometric analysis. Non-uniform rational B-spline (NURBS) basis functions are applied to describe both geometry and displacement field of the considered beam. The Timoshenko beam theory is used to derive the element stiffness and mass matrices. The application of the presented method is shown in numerical examples. The correctness of the presented approach is proved by comparing the obtained results to those available in the literature and calculated by the finite element method. Analysis of convergence is presented for different orders of NURBS basis functions.

Keywords: isogeometric analysis, free-form curved beam, Timoshenko beam theory, vibration analysis.

1. Introduction

Curved beams are used in many engineering structures such as arches, underground buildings, covering of large-scale buildings and bridges. However, the analysis, especially the analysis of the dynamic response of the structure, is not simple due to the coupling between the equations describing the tangential and radial displacement by curvature. This topic is analyzed in the literature both in analytical and numerical terms.

In [1] the authors derived the differential governing equations for free in-plane vibrations of non-circular arches, in which the effects of rotatory inertia, shear deformation and axial deformation are taken into considerations. The analytical solution for curved beams including these effects is also presented in [2]. The authors considered in-plane free vibrations of laminated curved beams. They used the dynamic stiffness method with series solution. In [3] the analytical solution is presented for elliptical arches. The axial deformation is taken into account, but the shear deformation is neglected.

In many cases of analysis of natural vibrations, the finite element method is used. Yang et al. [4] analyzed the different types of non-circular arches by use of finite element analysis. In [5] the authors derived the simple implicit shape functions associated with the tangential, radial and rotational displacements and presented them in matrix form.

The finite element method is widely used, however requires transformation of the computer aided design (CAD) geometry to an appropriate computational model. It can be difficult with free-form curved beams geometry because in the case of CAD the non-uniform rational B-splines (NURBS) are used to describe it, while in the finite element method, Lagrange interpolations are used to approximate the geometry. For this reason, the isogeometric approach (IGA) can be used, in which NURBS are applied to describe both geometry and displacement field. IGA has been developed by Hughes et al. in [6] and many authors successfully applied it to dynamics. In [7] the free vibrations of Euler-Bernoulli beam, elastic membrane and Poisson-Kirchhoff plate are considered, whereas the free vibrations of Timoshenko beam are analyzed in [8-9]. An extensive discussion of statics and dynamics for straight and curved beams using the isogeometric approach can be found in [10].

The paper presents an analysis of free vibrations of a free-form Timoshenko curved beam by making use of isogeometric analysis. The presented examples prove that the applied method is correct and efficient due to the very good convergence of the results.

The paper is organized as follows: first, in Section 2 a brief presentation of basic concepts related to the isogeometric approach is presented, then in Section 3 the free-form curved Timoshenko beam element is derived by use of the isogeometric approach, next two examples are presented in Section 4, and the paper is ended with conclusions.

2. Isogeometric approach

2.1. B-spline basis functions

In order to construct B-spline basis functions it is necessary to define the knot vector as a set of coordinates:

$$\Xi = [\xi_1, \xi_2, \dots, \xi_{n+p+1}], \tag{1}$$

where ξ_i is a knot value, n is the number of basis functions and p is the order of the basis functions. The knot vector is called open knot vector if the first and the last knot are repeated $p+1$ times. In this paper only open knot vectors are used.

After determining the knot vector it is possible to construct the B-spline basis functions [11] using the following formula for $p = 0$:

$$N_{i,0}(\xi) = \begin{Bmatrix} 1 & \text{if } \xi_i \le \xi \le \xi_{i+1} \\ 0 & \text{otherwise} \end{Bmatrix}, \tag{2}$$

and for higher order $p = 1, 2, 3, \dots$:

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi). \tag{3}$$

On the basis of [12], the quotient $\frac{0}{0}$ is taken as 0. In Fig. 1 the B-spline basis functions of the order $p = 3$ are shown with the knot vector $\Xi = [-1 -1 -1 -1 -0.5\ 0\ 0.5\ 1\ 1\ 1\ 1]$.

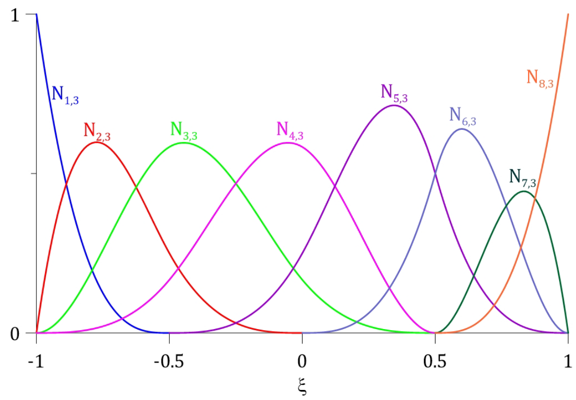

Figure 1. B-spline basis functions of the order p = 3 with the knot vector $\Xi = [-1 -1 -1 -1 -0.5\ 0\ 0.5\ 0.5\ 1\ 1\ 1\ 1]$.

2.2. Non-Uniform Rational B-Splines – NURBS

NURBS (non-uniform rational B-splines) curve is defined as a set of functions with n control points $P_i = [P_{xi},\ P_{yi}]$ and corresponding weights w_i [10]:

$$D(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi) P_i = \frac{\sum_{i=1}^{n} N_{i,p}(\xi) w_i}{\sum_{j=1}^{n} N_{j,p}(\xi) w_j} P_i, \tag{4}$$

where $D(\xi) = [x(\xi), y(\xi)]$ is a vector function defined along the curve represented by the parameter $\xi_1 \le \xi \le \xi_{n+p+1}$. Weights are responsible for the influence of a given control point on the curve. In this paper weights are taken equal to 1 and then the NURBS are simplified into the B-splines, which can be expressed as:

$$D(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) P_i. \tag{5}$$

The B-spline of order $p = 3$ with the same knot vector as in Fig. 1 and 8 control points is shown in Fig.2.

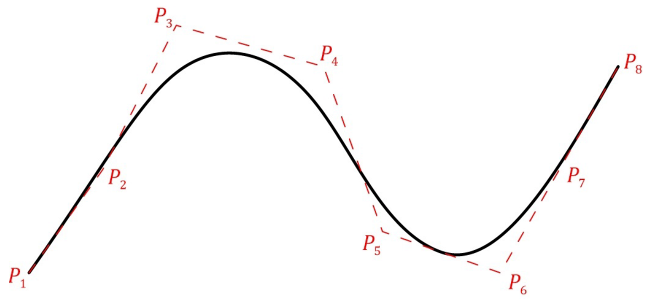

Figure 2. B-spline basis functions of the order p = 3 with 8 control points.

In engineering practice, it is usually possible to obtain the geometry of the considered structure by curve fitting technique [13]. If the data points describing the curve are known, then control points can be determined using Eq. (5). Thus, it is necessary to find a solution of the set of linear equations, which can be written in the matrix form:

$$\mathbf{D} = \mathbf{NP}, \tag{6}$$

where

$$\mathbf{D} = [x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n]^T$$

$$\mathbf{P} = \left[P_{x1}, P_{x2}, \dots, P_{xn}, P_{y1}, P_{y2}, \dots, P_{yn}\right]^T$$

$$\mathbf{N} = \begin{bmatrix} N_{1,p}(\bar{\xi}_1) & N_{2,p}(\bar{\xi}_1) & \dots & N_{n,p}(\bar{\xi}_1) & 0 & 0 & 0 & 0 \\ N_{1,p}(\bar{\xi}_2) & N_{2,p}(\bar{\xi}_2) & \dots & N_{n,p}(\bar{\xi}_2) & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ N_{1,p}(\bar{\xi}_n) & N_{2,p}(\bar{\xi}_n) & \dots & N_{n,p}(\bar{\xi}_n) & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & N_{1,p}(\bar{\xi}_1) & N_{2,p}(\bar{\xi}_1) & \dots & N_{n,p}(\bar{\xi}_1) \\ 0 & 0 & \dots & 0 & N_{1,p}(\bar{\xi}_2) & N_{2,p}(\bar{\xi}_2) & \dots & N_{n,p}(\bar{\xi}_2) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & N_{1,p}(\bar{\xi}_n) & N_{2,p}(\bar{\xi}_n) & \dots & N_{n,p}(\bar{\xi}_n) \end{bmatrix}. \tag{7}$$

The data points, which are input data to get the solution of the system of equations, correspond to the parameters $\bar{\xi}_i$ and can be determined by different techniques [14].

3. Free-form curved Timoshenko beam

3.1. Description of the beam model

Figure 3 presents the geometry of a free-form curved Timoshenko beam. Coordinate s corresponds to the midline of considered beam and $R(s)$ denotes the radius of curvature. The displacement field of an arbitrary point of free-form curved beam can be expressed as [15]:

$$\bar{u}(s,r,t) = u(s,t) - r\theta(s,t) \qquad \bar{v}(s,r,t) = v(s,t), \tag{8}$$

where u is the tangential displacement and v is the normal displacement on the midline of the beam; θ is the rotation of the cross-section and t is the time. Therefore, the strain-displacement relations can be written as follows:

$$\varepsilon = \frac{du}{ds} + \frac{v}{R} \qquad \chi = \frac{d\theta}{ds} \qquad \gamma = -\theta + \frac{dv}{ds} + \frac{u}{R}, \tag{9}$$

where ε and χ are the membrane strain and the curvature strain, respectively; γ is the transverse shear strain.

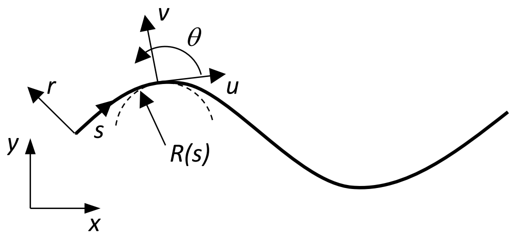

Figure 3. Free-form curved beam.

The governing equations can be derived using the Hamilton's principle:

$$\delta H = \delta \int_{t1}^{t2} (S - T - W)dt = 0, \tag{10}$$

where S, T and W are the elastic strain energy, the kinetic energy, and the external work, respectively:

$$S = \frac{1}{2}\int_0^l \left[EA\left(\frac{du}{ds} + \frac{v}{R}\right)^2 + EI\left(\frac{d\theta}{ds}\right)^2 + \kappa GA\left(-\theta + \frac{dv}{ds} + \frac{u}{R}\right)^2\right] ds, \tag{11}$$

$$T = \frac{1}{2}\int_0^l \left[\rho A\left(\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2\right) + \rho I\left(\frac{d\theta}{dt}\right)^2\right] ds, \tag{12}$$

$$W = \int_0^l (pu + qv + m\theta)ds, \tag{13}$$

where A is the area of cross-section, I – moment of inertia, E – Young's modulus, G – Kirchhoff's modulus, κ – shear factor, ρ is the mass density, and p, q and m are distributed tangential load, radial load, and moment and l is the length of beam. In the case of the considered eigenproblem, the external load W is zero and substituting Eqs. (11) and (12) into (10), the following formula is obtained:

$$0=\int_{t1}^{t2}\int_0^l \left[EA\left(u' + \frac{v}{R}\right)\delta u' + \frac{\kappa GA}{R}\left(-\theta + v' + \frac{u}{R}\right)\delta u - \rho A\dot{u}\delta\dot{u} + \frac{EA}{R}\left(u' + \frac{v}{R}\right)\delta v + \kappa GA\left(-\theta + v' + \frac{u}{R}\right)\delta v' - \rho A\dot{v}d\dot{v} + EI\theta'\delta\theta' - \kappa GA\left(-\theta + v' + \frac{u}{R}\right)\delta\theta - \rho I\dot{\theta}\delta\dot{\theta}\right] dsdt \tag{14}$$

In order to write Eq. (14) in a more compact form, the notation $(\blacksquare)'$ relating to the differentiation with respect to s and $(\dot{\blacksquare})$ relating to differentiation with respect to time are introduced.

3.2. Matrix formulation

In the isogeometric approach the NURBS are used to describe both geometry and displacement field. Since the shape functions used in the classical finite element method are replaced by NURBS basis functions, the displacement field can be expressed as:

$$u = \sum_{i=1}^{nu} S_{i,pu}(\xi)u_i \qquad v = \sum_{i=1}^{nv} S_{i,pv}(\xi)v_i \qquad \theta = \sum_{i=1}^{n\theta} S_{i,p\theta}(\xi)\theta_i, \tag{15}$$

where u_i, v_i and θ_i can be understood as control variables and the number of control points for displacements are nu, nv and $n\theta$. It is worth noting that the increasing the number of control points or degree of B-spline, or both, specifies the mesh size of the beam. These refinements are called h-, p-, and k-refinements, respectively [9].

In the isogeometric analysis, it is necessary to transform the curvilinear coordinate s to the parameter of curve ξ. It can be performed by use of the Jacobian:

$$J = \frac{ds}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}. \tag{16}$$

The curvature $\varkappa$ can be calculated as:

$$\varkappa = \frac{1}{R} = \frac{1}{J^3}\left(\frac{dx}{d\xi}\cdot\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\cdot\frac{d^2x}{d\xi^2}\right). \tag{17}$$

The stiffness and mass matrix are obtained by substitution of Eqs. (15) into (14) and integration. Finally, the stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} \\ \mathbf{k}_{21} & \mathbf{k}_{22} & \mathbf{k}_{23} \\ \mathbf{k}_{31} & \mathbf{k}_{32} & \mathbf{k}_{33} \end{bmatrix}, \tag{18}$$

in which

$$k_{11}^{ij} = \int_0^l \left(EAS'_{i,pu}S'_{j,pu} + \frac{\kappa GA}{R^2}S_{i,pu}S_{j,pu}\right)ds \qquad k_{12}^{ij} = \int_0^l \left(\frac{EA}{R}S'_{i,pu}S_{j,pv} - \frac{\kappa GA}{R^2}S_{i,pu}S'_{j,pv}\right)ds$$

$$k_{13}^{ij} = \int_0^l \left(\frac{\kappa GA}{R}S_{i,pu}S_{j,p\theta}\right)ds \qquad k_{22}^{ij} = \int_0^l \left(\frac{EA}{R}S_{i,pv}S_{j,pv} + \kappa GAS'_{i,pv}S'_{j,pv}\right)ds \tag{19}$$

$$k_{23}^{ij} = \int_0^l \left(\kappa GAS'_{i,pv}S_{j,p\theta}\right)ds \qquad k_{33}^{ij} = \int_0^l \left(\kappa GAS_{i,p\theta}S_{j,p\theta} + EIS'_{i,p\theta}S'_{j,p\theta}\right)ds.$$

and the mass matrix:

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{m}_{33} \end{bmatrix}, \tag{20}$$

with

$$m_{11}^{ij} = \int_0^l \rho AS_{i,pu}S_{j,pu}ds \qquad m_{22}^{ij} = \int_0^l \rho AS_{i,pv}S_{j,pv}ds \qquad m_{33}^{ij} = \int_0^l \rho AS_{i,p\theta}S_{j,p\theta}ds. \tag{21}$$

The derivative of NURBS in (19) can be obtained via the derivative of the B-spline curves. The k-th derivative of basis functions and the derivatives of NURBS are [10]:

$$N_{i,p}^k(\xi) = \frac{p}{\xi_{max} - \xi_{min}}\left(\frac{N_{i,p-1}^{k-1}(\xi)}{\xi_{i+p} - \xi_i} - \frac{N_{i+1,p-1}^{k-1}(\xi)}{\xi_{i+p+1} - \xi_{i+1}}\right) \qquad S_{i,p}^k(\xi) = \frac{N_{i,p}^k(\xi)w_i}{\sum_{j=1}^n N_{i,p}^k w_j}. \tag{22}$$

Each component of the stiffness and mass matrices can be integrated numerically using the standard Gauss-Legendre quadrature technique. The eigenvalue problem can be expressed in the well-known form:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} = \mathbf{0}, \tag{23}$$

where ω is a set of natural frequencies and $\mathbf{q}$ denotes eigenvectors.

4. Numerical examples

4.1. Circular arch

First, the circular arch is considered, which can be treated as a particular case of the free-form curved beam (after assuming radius of curvature $R = const$). The following data are taken into account [4]: $E = 70$ GPa, $\nu = 5/12$, $\kappa = 0.85$, $A = 1$ m^2, $I = 0.0016$ m^4, $\rho = 2777$ kg/m^3, the radius R = 0.6363 m and arch angle $\pi/2$. The analyzed circular arch which is clamped on its ends is shown in Fig. 4.

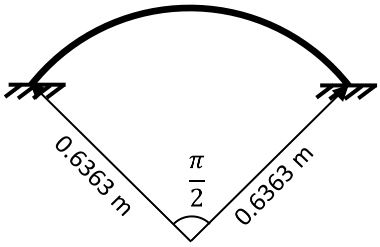

Figure 4. Circular arch.

In order to check the proposed formulation the considered arch is calculated using the isogeometric approach and the finite element method. In the isogeometric analysis the orders of basis functions used for NURBS are $p_u = p_v = p_\theta = 3$. In the finite element method the stiffness and mass matrices are derived using the exact shape functions for a circular element expressed as trigonometric functions [16]. In Tab. 1 the comparison of the natural frequencies computed by the two methods and those obtained in [4] are shown. Both in the isogeometric approach and finite element method the circular arch is divided into 50 elements. The compared non-dimensional natural frequencies are:

$$\lambda_i = \omega_i \left(\frac{\pi}{2} R\right)^2 \sqrt{\frac{\rho A}{EI}}. \tag{24}$$

Table 1. Comparison of non-dimensional natural frequencies λ_i for circular arch.

mode	IGA	FEM	Yang et al. [4]
1	36.715	36.707	36.657
2	42.299	42.268	42.289
3	82.328	82.278	82.228
4	84.545	84.514	84.471

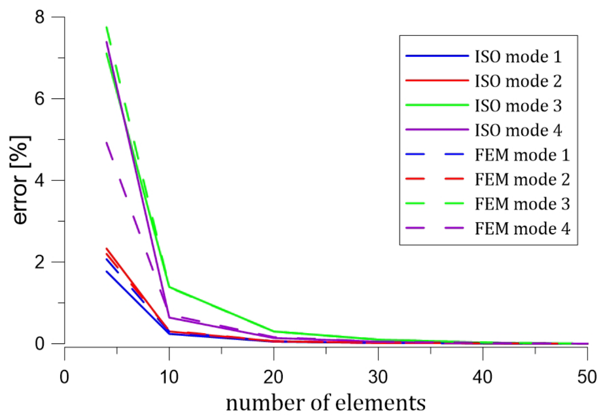

Figure 5. Comparison of convergence for the isogeometric approach and finite element method.

In Figure 5 the comparison of convergence for the isogeometric approach and finite element method is presented. The relative error displayed in Fig. 5 was calculated as the difference of the solutions related to the reference solution gathered in Tab. 1 which was obtained for 50 elements, respectively.

4.2. Tschirnhausen beam

As the second example a Tschirnhausen beam as free-form curved beam is considered. The geometry of the beam is described as follows:

$$x = -3(t^2 - 3) \qquad y = t(t^2 - 3). \tag{25}$$

The span of the structure is 9 m and the other data are: $E = 30$ GPa, $b = 0.4$ m, $h = 0.6$ m, $\nu = 0.2$, $\kappa = 5/6$, $\rho = 2500$ kg/m^3. The geometry is constructed by 30 control points (Fig. 6). In Tab. 2 the results obtained for two different orders of NURBS basis functions are compared. The considered beam is divided into 50 elements.

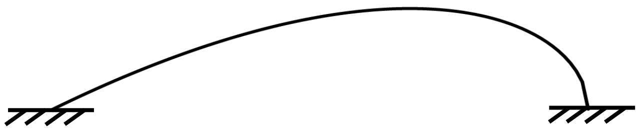

Figure 6. Geometry of Tschirnhausen beam.

Table 2. Comparison of natural frequencies ω_i [rad/s] for $p = 3$ and $p = 6$.

mode	$p = 3$	$p = 6$
1	241.676	241.471
2	436.596	436.473
3	656.460	656.625
4	862.053	861.109

In Figure 7 the convergence for various numbers of elements is shown for both considered orders $p = 3$ and $p = 6$. The relative error was calculated as a difference between solutions obtained and solutions computed for 50 elements. The diagrams prove that the convergence is very good for both orders of basis functions.

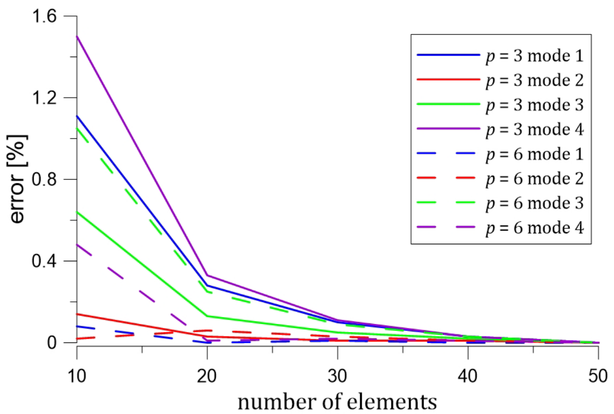

Figure 7. Comparison of convergence for $p = 3$ and $p = 6$.

4. Conclusions

In the paper the isogeometric approach is applied to analysis of dynamics of the free-form curved Timoshenko beam. The NURBS are used to describe both the geometry and displacement field of the considered structure. The Timoshenko beam theory is used to derive the element stiffness and mass matrices. Having obtained these matrices we are in a position to apply the well-known finite dimensional solution algorithms.

The circular arch and the Tschirnhausen beam are analyzed. In the first example the obtained results are compared to those available in the literature and computed by finite element method. The agreement of the results proves the correctness of the developed approach. In the second example the obtained results for two different orders of NURBS basis functions are almost the same. The presented analyses show that the convergence of the developed formulation is very good in both considered cases. The research results available in the literature were extended to include a convergence analysis of obtained solutions and use of other orders of polynomials.

The derived formulas can be effectively and simply coded. The isogeometric approach seems to be a flexible and efficient tool for analysis of curved structures (bars, shells).

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Additional information

The authors declare: no competing financial interests and that all material taken from other sources (including their own published works) is clearly cited and that appropriate permits are obtained.

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