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A CORRECTION METHOD FOR PROJECTS OF SYSTEM SAFETY FROM THE RISK MINIMIZATION CRITERION

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Abstract. In this paper we give an analytical method of solution to the problem of correction of the technical system parameters in order to reduce the planned common risk of technical system safety violation.

In the article [1] we considered the possibility and assumptions under which the common risk (criterion) $y = f(x)$ of technical system safety violation is expressed as a sum of particular risks $u_i(x)$ on a certain threats for safety violation, i.e.

$$y = f(x) = u_1(x) + u_2(x) + \ldots + u_n(x). \tag{1}$$

Let $u_i(x)$ be a polynomial

$$u_i(x) = C_i \cdot \prod_{j=1}^{m} x_j^{a_{ij}}, \quad C_i > 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

and the vector $x = (x_1, x_2, \ldots, x_m)$ of some parameters x_j be positive. The matrix $A = (a_{ij})$ is called an exponent matrix. Suppose that the matrix $A = (a_{ij}) = \begin{pmatrix} B \\ H \end{pmatrix}$, where the basis B is an $m \times m$ matrix ($| B | \neq 0$) and the submatrix H contains d rows of matrix A, which do not belong to the basis B. The difficulty level is characterized by this number d: $d = n - m$. The coefficients a_{ij} and C_i are got by methods of linear regression analysis [2].

A selected set (vector) x will be called *a project* of system safety.

Suppose that the project x_* ensures the minimum value $y_* = f(x_*)$ of the common risk y (see [3]); then the project x_* will be called a *perfect* project.

Numerical method for obtaining the values y_* and x_* was proposed in [4,5].

Assume that the perfect project x_* is not accepted on account of costs. Therefore, a maximum acceptable risk $\bar{y}$ such that $y_* < \bar{y}$ is given, and the requirement $y = f(x) < \bar{y}$ should be fulfilled (for more details we refer the reader to [5, 6]).

Let $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots , x_m^{(1)})$ of some parameters $x_1^{(1)}$ be a project satisfying $y = f(x) \le \bar{y}$ for $x = x^{(1)}$.

The value $y^{(1)} = f(x^{(1)})$ will be called the *planned* risk of technical system safety violation. By the above, we have $y_* \le y^{(1)} \le \bar{y}$.

On the basis of project $x^{(1)}$ and x_*, recomendations on possible improvement of the planned parameters $x_j^{(1)}$, $j = 1, 2, \ldots , m$ should be given.

In other words, we should propose a new project $\tilde{x}$ such that the value $\tilde{y} = f(\tilde{x})$ of the common risk is less than the planned risk. Furthermore, it satisfies the inequality $y_* \le \tilde{y} < y^{(1)}$.

In order to solve this problem we shall need some results of the Theorem which is proved below. According to this Theorem, the function $y = f(x)$ of (1) satisfies the inequality

$$f(x_*) < f((x^{(1)})^\lambda \cdot x_*^{1-\lambda}) < f(x^{(1)}). \tag{3}$$

Here $(x^{(1)})^\lambda \cdot x_*^{1-\lambda} = \tilde{x}$ is a new project with parameters

$$\tilde{x}_j = (x_j^{(1)})^\lambda \cdot x_{j*}^{1-\lambda} \tag{4}$$

for each λ from the interval $0 < \lambda < 1$.

The number $\bar{y}$ is the value of the common risk for the *improved* project.

Substituting the different values $\lambda \in (0; 1)$ in (4), we obtain the set of new projects $\tilde{x}$, each of which satisfies $f(\tilde{x}) \le \bar{y}$.

The common risk $\tilde{y}$ for the new project $\tilde{x}$ is less than or equal to $y^{(1)}$: $f(\tilde{x}) \le f(x^{(1)})$. For this reason, it is advisable to change the project $x^{(1)}$ by the project $\tilde{x}$.

Example. Let the common risk be given as a function

$$y = f(x) = \frac{1}{100} \cdot \left(2x_1^2 \cdot x_2 + 3x_1 \cdot x_2^2 + 4\frac{1}{x_1} \cdot \frac{1}{x_2}\right)$$

of the parameters x_1, x_2. Let $\bar{y}$=0.109 be the maximum acceptable value of this risk.

Minimizing the function $y = f(x)$ according to [3], we get the minimum value y_*=0.085 of the criterium and the perfect project x_* with parameters $x_1* = 1.08$ and $x_2* = 0.72$.

Assume that the project x_* is not accepted on account of costs. Suppose that the project $x^{(1)}$ =[1.04;1.04] is brought up for discussion. Substituting $x^{(1)}$ into (1) yields the value of the common risk: $y^{(1)} = f(x^{(1)}) = 0.093$, which satisfies the inequality $y^{(1)} < \bar{y} = 0.109$.

It is necessary to propose an improved project $\tilde{x}$.

Solution. Take a number $\lambda \in (0;1)$, for example, $\lambda = 0.4$ and define the new project $\tilde{x} = [\tilde{x_1}, \tilde{x_2}]$, where $\tilde{x_1} = 1.04^{0.4} \cdot 1.08^{0.6} = 1.07$, $\tilde{x_2} = 1.04^{0.4} \cdot 0.72^{0.6} = 0.84$. It can be easily checked that the corresponding value $\tilde{y} = f(\tilde{x}) = 0.086$. If the new project satisfies us for the economic reason, then $\tilde{x}$ can be accepted because $\tilde{y} < y^{(1)}$.

Now these calculations will be substantiated for $d = 1$.

Theorem. Let $A = (a_{ij})$ be the matrix corresponding to the criterion $y = f(x)$ of (1). Further $A_0 = (A, \mathbf{1})$, where $\mathbf{1}$ is a column of ones, is a square and nonsingular matrix. Moreover a unique solution $\delta = (\delta_1, \delta_2, \ldots, \delta_n) = (\underbrace{0, 0, \ldots, 0}_{m}, 1) \cdot A_0^{-1}$ of the equality $\delta \cdot A_0 = (\underbrace{0, 0, \ldots, 0}_{m}, 1)$ is positive: $\delta > 0$. Suppose that $f(x) \leq \overline{y}$ and $y_* \leq \overline{y}$; then the following statements are true:

1. The vector x_* with the components

$$x_{j*} = \prod_{i=1}^{n} \left(\frac{\delta_i \cdot y_*}{C_i} \right)^{k_{ji}}, \quad j = 1, 2, \ldots, m, \tag{5}$$

is one of the solutions of the inequality $f(x) \leq \bar{y}$, where k_{ji} are the elements of the inverse matrix $B^{-1} = (k_{ji})$. Similarly, the vector

$$x_j = \prod_{i=1}^{n} \left(\frac{\delta_i \cdot \bar{y}}{C_i} \right)^{k_{ji}}, \quad j = 1, 2, \ldots, m, \tag{6}$$

is the solution of the same inequality $f(x) \leq \bar{y}$.

2. Suppose that $\alpha^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_n^{(1)})$ and $\alpha^{(2)} = (\alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \alpha_n^{(2)})$ are the solutions of the so called *generative inequality*

$$\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \geq r, \tag{7}$$

where $\alpha_i > 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, and $r = \frac{\prod_{i=1}^{n} C_i^{\delta_i}}{\bar{y}}$.

Substituting these solutions into (6) in place of δ_i, we obtain two solutions: $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}) > 0$ and $x^{(2)} = (x_1^{(2)}, x_2^{(2)}, \ldots, x_m^{(2)}) > 0$ of the inequality $f(x) \leq \bar{y}$.

Then for any number $\lambda \in [0;1]$ the vector $\tilde{x} = (x^{(1)})^{\lambda} \cdot (x^{(2)})^{1-\lambda}$ with the components $\bar{x}_j = (x_j^{(1)})^{\lambda} \cdot (x_j^{(2)})^{1-\lambda}$ is the solution of the same inequality $f(x) \le \bar{y}$, i.e.

$$(f(x^{(1)}) \le \bar{y},\ f(x^{(2)}) \le \bar{y}) \Rightarrow (f(\tilde{x}) \le \bar{y}). \tag{8}$$

Moreover,

$$\left(x_j \in \left[x_j^{(1)}, x_j^{(2)}\right], j = 1, 2, \ldots, m\right) \Longrightarrow \left(y = \sum_{i=1}^{n} C_i \prod_{j=1}^{m} x_j^{a_{ij}} \in [y_*, \overline{y}]\right) \tag{9}$$

Proof.

1. According to [3], the vector x_* with components from (5) guarantees that $f(x)$ of (1) has a minimum at x_*: $f(x_*) = y_*$. Under the conditions of the Theorem, it follows that x_* is the solution of $f(x) \le \bar{y}$, i.e. $f(x_*) = y_* \le \bar{y}$.

Let us consider inequality (7) and the vector $\alpha = \delta$ with components $\alpha_i = \delta_i$ for which the minimum $y_* = \prod_{i=1}^{n} (\frac{C_i}{\delta_i})^{\delta_i}$ is occurred (see [3]).

We claim that the inequality $\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \le \delta_1^{\delta_1} \cdot \delta_2^{\delta_2} \cdots \delta_n^{\delta_n}$ is fulfilled for fixed numbers $\delta_i > 0$, where $\delta_1 + \delta_2 + \ldots + \delta_n = 1$ and arbitrary numbers $\alpha_i > 0$, where $\alpha_1 + \alpha_2 + \ldots \alpha_n = 1$. Indeed, if the function $g(\alpha)$ has the form $g(\alpha) = \alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n}$, then there exists the *Lagrange* function $L = \ln g(\alpha) + \lambda(1 - (\alpha_1 + \alpha_2 + \ldots \alpha_n))$. Consider the equations $L'_{\alpha_i} = (\frac{\delta_i}{\alpha_i} - \lambda) = 0$, so $\frac{\delta_i}{\alpha_i} = \lambda$, $i = 1, 2, \ldots, n$. From these equations we obtain $\delta_i = \lambda\alpha_i$, $\sum_{i=1}^{n} \delta_i = \lambda \cdot \sum_{i=1}^{n} \alpha_i$, and $\lambda = 1$.

Thus, the function $g(\alpha)$ has an extremum at the point with components $\alpha_i = \delta_i$. Since $L''_{\alpha_i^2} = -\frac{\delta_i}{\alpha_i^2} < 0$ and $L''_{\alpha_i\alpha_j} = 0$, (i$\neq$j), we see that the matrix $(L''_{\alpha_i\alpha_j})$ of second partial derivatives for the *Lagrange* function is negative definite. This means that the function $g(\alpha)$ takes the maximum at the point with components $\alpha_i = \delta_i$. Hence $g(\alpha) = \alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \le \delta_1^{\delta_1} \cdot \delta_2^{\delta_2} \cdots \delta_n^{\delta_n}$. An equality sign is achieved if and only if $\alpha_i = \delta_i$. Now the inequality yields $(\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdots \alpha_n^{\delta_n} \ge q) \Rightarrow (\delta_1^{\delta_1} \cdot \delta_2^{\delta_2} \cdots \delta_n^{\delta_n} \ge q)$. This relationship and Theorem 1 from [3] show that the vector α $(= \delta)$ with components $\alpha_i = \delta_i$ is one of the solutions of (7) corresponding to the solution (6) of the inequality $f(x) \le \bar{y}$.

2. Suppose that $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}) > 0$ is the solution of $f(x) \le \bar{y}$. The parameters $x_j^{(1)}$ are received under change of δ_i by $\alpha_i^{(1)}$ in (6), where $\alpha^{(1)}$ is one of the solutions of the generative inequality (7).

Then, by the results of [3], there exist $x_{m+1}^{(1)} = x_{m+2}^{(1)} = \ldots = x_{m+n-1}^{(1)} = 1$, $x_{m+n}^{(1)} = \prod_{i=1}^{n} \left(\frac{\alpha_i^{(1)} \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu} \ge 1$ such that the vector $\ln(x^{(1)})' = (\ln x_1^{(1)}, \ln x_2^{(1)},$

$\ldots, \ln x^{(1)}_{m+n})$ is the solution of the equation $A_1 \cdot \ln x^{'} = \ln b(\alpha)$, where the matrix $A_1 = (A.I_n)$ and $b(\alpha) = (\ln(\frac{\alpha_1 \cdot \bar{y}}{C_1}, \ln(\frac{\alpha_2 \cdot \bar{y}}{C_2}, \ldots, \ln(\frac{\alpha_n \cdot \bar{y}}{C_n})$.

Similarly, if $x^{(2)} > 0$ is the solution of $f(x) \leq \bar{y}$ with components $x^{(2)}_j$, $j = 1, 2, \ldots, m$, which are received under substitution of δ_i by $\alpha^{(2)}_i$ in (6), where $\alpha^{(2)}$ is one of the solutions of the generative inequality (7), then there exist $x^{(2)}_{m+1} = x^{(2)}_{m+2} = \ldots = x^{(2)}_{m+n-1} = 1$, $x^{(2)} = \prod_{i=1}^n \left(\frac{\alpha^{(2)}_i \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu} \geq 1$ such that the vector $\ln(x^{(2)})' = (\ln x^{(2)}_1, \ln x^{(2)}_2, \ldots, \ln x^{(2)}_{m+n})$ is the solution of the equation $A_1 \cdot \ln x^{'} = \ln b(\alpha)$.

Suppose that $\lambda \in [0;1]$. Since $A_1(\lambda \ln(x^{(1)})' + (1-\lambda)\ln(x^{(2)})') = \lambda A_1 \cdot \ln(x^{(1)})' + (1-\lambda)A_1 \cdot \ln(x^{(2)})' = \lambda \ln b(\alpha) + (1-\lambda)\ln b(\alpha) = \ln b(\alpha)$, we see that the vector $\ln(\tilde{x})' = \lambda \ln(x^{(1)})' + (1-\lambda)\ln(x^{(2)})'$, where $\tilde{x}' = (x^{(1)})'^{\lambda} \cdot (x^{(2)})'^{(1-\lambda)}$, is the solution of the inequality $A_1 \cdot \ln x^{'} = \ln b(\alpha)$.

The elements $\tilde{x}^{'}_j = (x^{(1)}_j)'^{\lambda} \cdot (x^{(2)}_j)'^{(1-\lambda)}$ are components of the vector $\tilde{x}^{'}$. Thus, from the formulas for $(x^{(1)}_j)^{'}$ and $(x^{(2)}_j)'$, it follows that

$$\tilde{x}^{'}_j = \tilde{x}_j = (\prod_{i=1}^n \left(\frac{\alpha^{(1)}_i \cdot \bar{y}}{C_i}\right)^{k_{ij}})^{\lambda} \cdot (\prod_{i=1}^n \left(\frac{\alpha^{(2)}_i \cdot \bar{y}}{C_i}\right)^{k_{ij}})^{1-\lambda}, \quad j = 1, 2, \ldots, m,$$

$$\tilde{x}^{'}_{m+1} = \tilde{x}^{'}_{m+2} = \ldots = \tilde{x}^{'}_{m+n-1} = 1,$$

$$\tilde{x}^{'}_{m+n} = (\prod_{i=1}^n \left(\frac{\alpha^{(1)}_i \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu})^{\lambda} \cdot (\prod_{i=1}^n \left(\frac{\alpha^{(2)}_i \cdot \bar{y}}{C_i}\right)^{\delta_i \cdot \mu})^{1-\lambda} \geq 1.$$

Here μ is equal to the sum of components of the row matrix $(-a_n \cdot B^{-1}, 1)$, where a_n is the last row of the exponent matrix A. These expressions and the results of [3] allow us to reach the conclusion that the vector $\tilde{x}^{'}$ with components $\tilde{x}^{'}_j = \tilde{x}_j = (x^{(1)}_j)^{\lambda} \cdot (x^{(2)}_j)^{1-\lambda}$, $j = 1, 2, \ldots, m$ is the solution of the inequality $f(x) \leq \bar{y}$. Hence the relationship (8) holds.

Since λ is any number in [0;1], then $\min(x^{(1)}_j, x^{(2)}_j) \leq \tilde{x}_j \leq \max(x^{(1)}_j, x^{(2)}_j)$, i.e. $\tilde{x}_j \in [x^{(1)}_j, x^{(2)}_j]$.

Therefore, the inequality $f(x) \leq \bar{y}$ is fulfilled for all vectors $x = \tilde{x}$ with components $x_j = [x^{(1)}_j, x^{(2)}_j]$. Moreover, since $y_* = f(x_*)$ and $y_* \leq \bar{y}$, then $\left(x_j \in \left[x^{(1)}_j, x^{(2)}_j\right], j = 1, 2, \ldots, m\right) \Longrightarrow \left(y = \sum_{i=1}^n C_i \prod_{j=1}^m x_j^{a_{ij}} \in [y_*, \overline{y}]\right)$.

Thus, we have proved the relationship (9).

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