

## LIQUEFACTION OF SATURATED SOIL AND THE DIFFUSION EQUATION

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**Abstract:** The paper deals with the diffusion equation for pore water pressures with the source term, which is widely promoted in the marine engineering literature. It is shown that such an equation cannot be derived in a consistent way from the mass balance and the Darcy law. The shortcomings of the artificial source term are pointed out, including inconsistencies with experimental data. It is concluded that liquefaction and the preceding process of pore pressure generation and the weakening of the soil skeleton should be described by constitutive equations within the well-known framework of applied mechanics. Relevant references are provided.

*Key words:* liquefaction, diffusion equation, saturated soil, seabed

### 1. INTRODUCTION

The aim of this paper is to analyze the mathematical formalism applied in the description of the liquefaction phenomenon, with particular attention devoted to the diffusion equation, which is applied as a basic tool in marine engineering, see [4], [6], [17]. Liquefaction of saturated soils is an unusual phenomenon that does not occur in other materials. Therefore, its mathematical description cannot be based on classical models applied in the mechanics of materials, and new equations should be proposed. These equations should be in conformity with experimental data and the basic principles of mechanics. It is of particular interest to check whether the diffusion equation, which is widely promoted by the above-mentioned authors, really satisfies the basic requirements of applied sciences.

Note that the saturated soil consists of the soil skeleton, the pores of which are filled with water. Such a mixture behaves, under non-extreme conditions, as a solid body, as it can support external loads. Under some extreme conditions, such as earthquake excitations, severe storms, etc., the pore water pressure increases, and intergranular contacts decrease. The soil skeleton becomes increasingly weaker, so it can support smaller loads. In the extreme case, intergranular forces disappear, the soil skeleton cannot support any load and behaves like a liquid. Such a phenomenon is designated as soil liquefaction.

In geotechnical engineering, liquefaction is mainly linked to earthquake excitations, which are characterized by a small number of loading cycles and short duration. Under such conditions, the assumption of undrained behavior is justified, as the excess pore water pressure has no time to dissipate. On the basis of the above information, research programs have been developed, including laboratory investigations, such as simple shear and triaxial tests. These programs have led to various interesting discoveries in soil mechanics. For example, the definition of the initial state of soil has been generalized, and we know, at present, that soil samples, which are initially dense or loose, can also be in the dilative or contractive state. This distinction is essential, as it defines the behavior of soils under shearing. The above features of soil behavior are described in [15], [16]. Some details will be presented in subsequent sections.

Another mechanism of soil liquefaction was proposed by marine engineers, who analyze the behavior of seabeds under water wave action. They believe that it is the diffusion equation that governs the whole process, see [6], [17]. Some aspects of their approach raise questions that require detailed answers. For example, they have introduced the source term into the diffusion equation, which allegedly represents the pore pressure generation in the seabed. Sawicki in [9] has shown that such a source term can be derived only from false assumptions, i.e., a false Darcy law. The marine engineering approach to the problem of soil liquefaction gives rise to many other doubts. There-

fore, let us consider the mathematical structure of equations governing the phenomenon of soil liquefaction.

## 2. THE DIFFUSION EQUATION

There are several physical phenomena, such as heat transport, diffusion of contaminants in water, and dissipation of pore water pressure, which are described by differential equations of a similar form, see [5]. For example, consider the classical equation governing the process of pore pressure dissipation in saturated soil

$$\frac{k}{\gamma_w \kappa_s n} \nabla^2 u = \frac{\partial u}{\partial t}, \quad (1)$$

where  $k$  = coefficient of permeability;  $n$  = porosity;  $\gamma_w$  = unit weight of pore water;  $\kappa_s$  = compressibility of the soil skeleton;  $\nabla^2$  = Laplace operator. In the simple case of a soil layer, such as the seabed, there is  $\nabla^2 = \partial^2/\partial z^2$ , where  $z$  denotes the vertical co-ordinate.

Equation (1) was derived from the mass balance equation and the Darcy law, assuming that there is no source of any physical quantity. McDougal et al. [6] just add another term  $f$  to equation (1), which, in their opinion, represents the source of excess pore pressure generated in the seabed by shearing stresses. The question is whether we are allowed to add such a source term, and what is its meaning. Equations of mathematical physics, such as equation (1), should be based on clear physical assumptions and, on the other hand, should be mathematically consistent. Recall that we should satisfy both mathematical and physical requirements.

In order to better understand the origin of the source term in the diffusion-type equation, let us consider two particular cases of derivation of equation (1) with the source term for a uni-axial configuration.

### 2.1. DIFFUSION EQUATION WITH THE SOURCE OF MASS

Consider the uni-axial process of pore water flow through a deformable soil skeleton. Figure 1 shows the basic quantities influencing this phenomenon.  $V_1 = v_1 dx dy dz$  denotes the volume of pore water entering the elementary volume of saturated soil  $dx dy dz$ ,  $V_2 = V_1 + \frac{\partial v_z}{\partial z} dx dy dz dt$  = volume of water that flows out of the elementary volume,  $dS$  = volume of pore water

from the source,  $v_z$  = vertical component of the velocity of pore water flow.

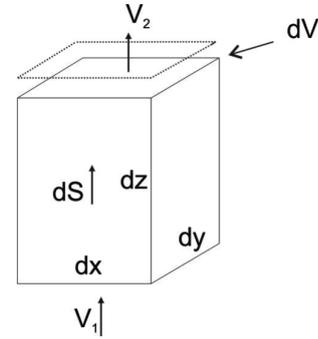


Fig. 1. Elementary volume of saturated soil

The mass balance takes the following form

$$V_1 + dV - V_2 + dS = 0. \quad (2)$$

The term  $dV$  denotes a change in the pore water volume due to the deformability of the soil skeleton

$$dV = n \frac{\partial \varepsilon}{\partial t} dx dy dz dt, \quad (3)$$

where  $n$  = porosity of the soil skeleton,  $\varepsilon = \varepsilon_z = \kappa_s \sigma'_z = -\kappa_s u$  = volumetric deformation of the soil skeleton, equal to vertical deformation in the case considered,  $\sigma'_z$  = vertical effective stress. Let us introduce the quantity  $s$ , which is a non-dimensional source of mass, such as

$$dS = \frac{\partial s}{\partial t} dx dy dz dt. \quad (4)$$

The Darcy law has the following form

$$v_z = -\frac{k}{\gamma_w} \frac{\partial u}{\partial z}. \quad (5)$$

Combination of the above equations leads to the following formula

$$\frac{\partial u}{\partial t} = \frac{k}{n \kappa_s \gamma_w} \frac{\partial^2 u}{\partial z^2} + \frac{1}{n \kappa_s} \frac{\partial s}{\partial t}, \quad (6)$$

which is a uni-axial version of equation (1) with the source term:  $f = \frac{1}{n \kappa_s} \frac{\partial s}{\partial t}$ .

Note that the source term  $f$  is physically related to the source of mass (see equation (2)), certainly not to the pore pressure. In a similar way, it is possible to derive equations describing the molecular diffusion or the heat transport, just by taking into account Fick's law or Fourier's law, instead of the Darcy law. The

mathematical structure of these laws is similar. Their respective source terms will represent a continuously distributed inflow of contaminants or heat, which is taken into account in the method of derivation equation (6).

Note that there is also another type of source, which can be imagined as a point or line in space. In such a case, the diffusion equation does not contain the source term, which is taken into account in the formulation of the initial/boundary value problem.

## 2.2. THE SOURCE TERM THAT FOLLOWS FROM THE FALSE DARCY LAW

The other method of introducing the source term into the diffusion equation is based on false assumptions. For example, Sawicki in [9], [10] has shown that the false Darcy law

$$v_z = -\frac{k}{\gamma_w} \frac{\partial u}{\partial z} + F, \quad (7)$$

where  $F$  is an arbitrary function, leads to a diffusion equation with the source term that has no physical meaning

$$\frac{\partial u}{\partial t} = \frac{k}{n\gamma_w\kappa_s} \frac{\partial^2 u}{\partial z^2} - \frac{1}{n\kappa_s} \frac{\partial F}{\partial z}. \quad (8)$$

## 3. "MARINE" INTERPRETATION OF THE SOURCE TERM

Sumer and Fredsoe [17], after McDougal et al. [6], present the diffusion equation with the source term  $f$ , in the following form

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} + f, \quad (9)$$

where  $c_v = k/n\gamma_w\kappa_s$ . They use a different notation, for example,  $\bar{p}$  instead of  $u$ , but for the sake of consistency we shall use the notation applied in this paper. The source term is given by the following formula

$$f = \frac{p'_0}{N_l T}, \quad (10)$$

where  $p'_0$  = initial mean effective stress, and

$$N_l = \left( \frac{\tau}{\alpha p'_0} \right)^{1/\beta} \quad (11)$$

denotes the number of cycles to liquefaction,  $\tau$  is the amplitude of the shear stress,  $T$  denotes the wave period.  $\alpha$  and  $\beta$  are certain empirical constants. The form of equations (10) and (11) was assumed after classical geotechnical sources, such as Peacock and Seed [7]. Equations (9)–(11) give rise to some reservations, which are summarized below:

- The method of introducing the source term into equation (9) is doubtful, as already explained in elementary derivations of the diffusion equation with the source term. It is not allowed in applied mechanics to add artificial terms to governing equations without any formal explanation. Consider, for example, Einstein's famous equation, which could be artificially "modified" as  $E = mc^2 + h(X, Y, Z)$ , where  $h$  is a certain function depending on the variables  $X, Y, Z, \dots$ . Such an approach would lead to some "new" physics, probably wrong for each set of artificial variables  $X, Y, Z, \dots$
- The form of the source term, defined by equations (10) and (11), raises many doubts. Firstly, the number of cycles to liquefaction  $N_l$  depends on a single component of the stress tensor  $\tau$ , which is not a sufficiently general formulation. Instead of  $\tau$  there should be a kind of invariant of the effective stress deviator. Secondly, equation (11) contains no parameters defining soil properties, such as the shear modulus.
- Liquefaction of the seabed depends on the loading history, which is not taken into account in equations (10) and (11).
- Equations (9)–(11) should also be valid in the case of laboratory tests, in which  $\partial u/\partial z = 0$ . In this case, these equations lead to the following formula for pore pressure generation:  $u = p'_0 \omega t / 2\pi N_l$ . Experimental results show that pore pressure generation does not depend on the frequency of cyclic loading. This means that the above equations are inconsistent with the experimental data.
- The values of parameters  $\alpha$  and  $\beta$  given by Sumer and Fredsoe [17] are questionable. For example, the source term depends on the high power of the shear stress ( $\tau^6$ ), which is unusual. Some other analyses show that in the case of simple shear under undrained conditions it should rather depend on  $\tau^2$ , which makes a large difference, see [8].
- Also note that equations (10) and (11) are based on experimental data obtained from undrained tests. Equation (9) takes into account the process of groundwater flow (Darcy law!), i.e., drained conditions. At this point, the "marine" approach is also inconsistent.

- The source term does not take into account the initial state of saturated soil, which is either contractive or dilative. This is a serious shortcoming, as it is the initial state that determines whether the soil can liquefy or not, etc.

#### 4. SOME OTHER FORMAL ERRORS

Besides the above-mentioned shortcomings, the formal derivation of equation (9), see equation (10.81), page 470 in [17], raises some doubts. This equation was derived from two differential equations numbered as (10.76) and (10.77) in [17]. Their structure is similar to the following equations

$$\frac{\partial^2 M}{\partial z^2} = \frac{\partial N}{\partial z}, \quad (12)$$

$$\frac{\partial^2 N}{\partial z^2} = \frac{\partial N}{\partial t} + \frac{\partial^2 M}{\partial z \partial t}, \quad (13)$$

where  $M$  and  $N$  are functions describing certain physical process. At present, it is not important which processes are dealt with, because we only analyze mathematical manipulations. Differentiation of equation (12) with respect to  $t$ , and of equation (13) with respect to  $z$ , makes it possible to eliminate the function  $M$  and leads to a single differential equation for  $N$

$$\frac{\partial^3 N}{\partial z^3} = 2 \frac{\partial^2 N}{\partial z \partial t}. \quad (14)$$

The next step of the derivation is integration of equation (14) with respect to  $z$ , which leads to the following formula

$$\frac{\partial^2 N}{\partial z^2} = 2 \frac{\partial N}{\partial t} + C(t). \quad (15)$$

The above procedure seems formally correct, but it raises some questions if one considers its physical interpretation. Recall that equations (12) and (13) were first differentiated and then integrated. The new function  $C(t)$  has appeared like *deus ex machina*. At the beginning, we had a system of two well-defined differential equations for two unknown functions. After some mathematical manipulations, we have a single differential equation with two unknown functions, which is ill formulated and therefore unsolved. Does it make any sense?

Recall that  $C$  is a function of time only. It does not depend on the variable  $z$ . But Sumer and Fredsoe in [17] ignore this obvious fact and introduce the “source

term” which depends on  $z$ , see their equations (10.85) and (10.86) on page 472. One can hardly accept such an obvious error.

#### 5. PHYSICS OF LIQUEFACTION

In order to construct a proper mathematical description of liquefaction, the physics of this phenomenon should be understood first. The literature on liquefaction is very rich, as it is measured in hundreds or even thousands of publications, so it is impossible in this short paper to provide a summary of the state-of-the-art. The following publications are recommended to those who wish to gain a better insight into this interesting phenomenon: [1]–[3], [11]. Earlier, a Polish book on cyclic loading liquefaction was published, [8]. However, in spite of the vast extent of the literature on the subject, the basic facts can be summarized as follows:

- Liquefaction occurs above all in sandy soils, although sands with an admixture of fines, such as silts, are also liquefiable. Highly permeable soils, such as gravels, do not liquefy.
- Liquefaction depends on the initial state of soil, defined either as contractive or dilative. Initially contractive soils do liquefy, whereas initially dilative soils do not. This problem is described in detail in [15], [16].
- The phenomenon of liquefaction is preceded by the process of pore pressure generation, which gradually reduces the mean effective stress. Physically, this means that intergranular forces decrease, and the soil skeleton becomes increasingly weaker. This phenomenon is measured by a gradual decrease in the shear modulus.
- Liquefaction depends on the intensity of external loads. For example, a single strong loading cycle may cause a sudden liquefaction of a given soil. The same soil subjected to many loading cycles of small magnitude may not liquefy.
- Liquefaction may be sudden (static liquefaction), when the external input is sufficiently strong, or may take place gradually, when consecutive loading cycles of smaller intensity gradually change the soil properties. Liquefaction takes place, in both cases, when the effective stress path reaches the instability line.
- The commonly accepted definition of liquefaction is  $p' = 0$ , where  $p'$  denotes the mean effective stress. It means that intergranular forces disappear, and the liquefied soil behaves like a liquid.

- Model conditions enabling soil liquefaction are usually identified with undrained conditions. There are some situations when this ideal model does not work – for example, under partly undrained conditions. The description of this phenomenon is a separate task. A numerical method of dealing with it was proposed by Sawicki in [8].
- There exist two types of liquefaction, as discovered by Sawicki, see [9]. The first one is caused by external loads under undrained conditions, as described above. The second depends on forced pore pressure changes, such as those caused by water waves. These two mechanisms are different. For example, laboratory investigations have shown that forced cyclic changes in pore pressure lead to unusual phenomena in saturated soil, such as apparent creep, etc., see [12].
- The source term, artificially introduced in some publications, has several shortcomings.
- The process of pore pressure generation and the phenomenon of liquefaction can be described properly within the classical framework of applied mechanics with new constitutive equations. Relevant references are provided.
- A diffusion equation without the source term can only be applied in the analysis of excess pore pressure dissipation.
- The “marine approach” is also based on elementary mathematical errors, which have been shown in this paper.

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## 6. PROPER STRUCTURE OF THE GOVERNING EQUATIONS

The above physical phenomena, characteristic of liquefaction, are not taken into account in the diffusion equation with the source term (9). Liquefaction and the preceding pore pressure generation are mechanical phenomena, and therefore should be described by methods of applied mechanics. A general structure of governing equations includes the balance of mass, the balance of momentum (equations of motions), and the constitutive equations, as well as the initial and boundary conditions when formulating a concrete problem to solve. The physics of liquefaction should be taken into account in the constitutive equations. One of possible approaches and some applications are described in the following publications: [8], [10], [11], [13]– [16], so there is no need to repeat here the results already published. Some other approaches to the problem of liquefaction can be found in [2], [3]. Empirical approaches are described by Ishihara in [1].

## 7. CONCLUSIONS

The main points presented in this paper can be summarized as follows:

- A diffusion-type equation for pore water pressures with the source term cannot be derived in a consistent way from the mass balance and the Darcy law.

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