BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 63, No. 4, 2015
DOI: 10.1515/bpasts-2015-0107

Robust stability of positive linear time delay systems under time-varying perturbations

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Abstract. By a novel approach, we get explicit robust stability bounds for positive linear time-invariant time delay differential systems subject to time-varying structured perturbations or non-linear time-varying perturbations. Some examples are given to illustrate the obtained results. To the best of our knowledge, the results of this paper are new.

Key words: robust stability, delay system, time-varying perturbation.

1. Introduction

Roughly speaking, a dynamical system is called *positive* if for any non-negative initial condition, the corresponding solution of the system is also non-negative. Positive dynamical systems play an important role in modelling of dynamical phenomena whose variables are restricted to be non-negative. They are often encountered in applications [1–38], for example, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers, distillation columns, storage systems, hierarchical systems, compartmental systems used for modelling transport and accumulation phenomena of substances, see e.g. [1, 7, 12, 25]. Recently, problems of stability and robust stability of positive systems have attracted a lot of attention from researchers, see e.g. [2, 11, 12, 23–33] and references therein.

In this paper, we give explicit robust stability bounds for positive linear time-invariant time delay differential systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-h_k), \qquad t \geq 0, \tag{1}$$

subject to the time-varying structured perturbations

$$A_k \rightsquigarrow A_k + D_k(t)\Delta_k(t)E_k(t), \qquad k \in \{0, 1, ..., m\}, \tag{2}$$

or the non-linear time-varying perturbations

$$\begin{aligned} A_0 x(t) &\rightsquigarrow A_0 x(t) + f(t; x(t)); \\ \sum_{k=1}^{m} A_k x(t-h_k) &\rightsquigarrow \sum_{k=1}^{m} A_k x(t-h_k) \\ &+F(t; x(t-\tau_1(t)), ..., x(t-\tau_m(t))). \end{aligned} \tag{3}$$

Motivated by many applications in control engineering, problems of robust stability of dynamical systems have attracted a lot of attention from researchers during the last twenty years, see e.g. [15–19, 31, 34–36] and references therein. In particular, problems of robust stability of linear differential systems without delay $\dot{x}(t) = Ax(t)$, $t \geq 0$, under the structured perturbations have been studied extensively for a long time, see e.g. [15–17, 34–36]. Moreover, robust stability of the linear differential systems with delay (1) under time-invariant structured perturbations has been dealt with in [19, 32, 37]. Some problems of robust stability of linear differential systems with delay under non-linear perturbations have been considered in [4, 9, 14, 21, 38] and most of obtained results have been given in terms of linear matrix inequalities (LMIs).

Although there are many works devoted to the study of robust stability of differential systems with delay, however, to the best of our knowledge, the problems of robust stability of the positive linear time delay differential system (1) under the time-varying structured perturbations (2) or the non-linear time-varying perturbations (3) have not yet been studied in the literature and the main purpose of this paper is to fill this gap.

In contrast to the traditional approaches to stability analysis of time-varying differential systems with delay (Lyapunov's method and its variants such as Razumikhin-type theorems, Lyapunov-Krasovskii functional techniques, see e.g. [13, 22]), we present in this paper a novel approach to the problems of robust stability of positive systems of the form (1) under the time-varying perturbations (2) and (3). To the best of our knowledge, the obtained results of this paper (Theorems 3.3, 3.9) are really new.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let us denote $\underline{m} := \{1, 2, ..., m\}$ and $\underline{m}_0 := \{0, 1, 2, ..., m\}$. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. For integers $l, q \geq 1$, $\mathbb{K}^l$ denotes the l-dimensional vector space over $\mathbb{K}$ and $\mathbb{K}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors are understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we

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write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1, \cdots, l$, $j = 1, \cdots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \cdots, l$, $j = 1, \cdots, q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}_+^{l \times q}$ the set of all non-negative matrices $A \geq 0$. Similar notations are adopted for vectors. For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l \times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. A norm $\| \cdot \|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{K}^n, |x| \leq |y|$. Every p-norm on $\mathbb{K}^n$ ($\|x\|_p = (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{1/p}$, $1 \leq p < \infty$ and $\|x\|_\infty = \max_{i=1,2,...,n} |x_i|$), is monotonic. Throughout the paper, if otherwise not stated, the norm of vectors on $\mathbb{K}^n$ is monotonic and the norm of a matrix $P \in \mathbb{K}^{l \times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^q$, that is $\|P\| = \max\{\|Py\| : \|y\| = 1\}$. Note that

$$P \in \mathbb{K}^{l \times q}, Q \in \mathbb{R}_+^{l \times q}, |P| \leq Q \Rightarrow \|P\| \leq \| \, |P| \, \| \leq \|Q\|, \quad (4)$$

see, e.g. [36]. In particular, if $\mathbb{K}^n$ is endowed with $\| \cdot \|_1$ or $\| \cdot \|_\infty$ then $\|A\| = \||A|\|$ for any $A = (a_{ij}) \in \mathbb{K}^{n \times n}$. More precisely, one has

$$\|A\|_1 = \||A|\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|,$$

$$\|A\|_\infty = \||A|\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Let $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$, for given $r > 0$. For any matrix $M \in \mathbb{C}^{n \times n}$ the *spectral abscissa* of M is denoted by $\mu(M) = \max\{\Re\lambda : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of M. A matrix $A \in \mathbb{R}^{n \times n}$ is called Hurwitz stable if $\mu(A) < 0$.

A matrix $M \in \mathbb{R}^{n \times n}$ is called a *Metzler matrix* if all off-diagonal elements of M are non-negative. We now summarize in the following theorem some properties of Metzler matrices.

Theorem 2.1. [36] Suppose that $M \in \mathbb{R}^{n \times n}$ is a Metzler matrix. Then

(i) (Perron-Frobenius) $\mu(M)$ is an eigenvalue of M and there exists a non-negative eigenvector $x \neq 0$ such that $Mx = \mu(M)x$.

(ii) Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Mx \geq \alpha x$ if and only if $\mu(M) \geq \alpha$.

(iii) $(tI_n - M)^{-1}$ exists and is non-negative if and only if $t > \mu(M)$.

(iv) Given $B \in \mathbb{R}_+^{n \times n}$, $C \in \mathbb{C}^{n \times n}$. Then

$$|C| \leq B \implies \mu(M + C) \leq \mu(M + B).$$

Let J be an interval of $\mathbb{R}$. Denote $C(J, \mathbb{R}^n)$ the set of all continuous functions on J with values in $\mathbb{R}^n$. In particular, if $\mathbb{R}^n$ is endowed with the norm $\| \cdot \|$ then $C([\alpha, \beta], \mathbb{R}^n)$ is a Banach space with the maximum norm $\|\phi\| = \max_{\theta \in [\alpha, \beta]} \|\phi(\theta)\|$. In what follows, $\phi \geq 0$ means that $\phi(\theta) \geq 0, \forall \theta \in [\alpha, \beta]$.

3. Robust stability of positive linear time delay differential systems

Consider a linear time-invariant time delay differential system of the form (1) where $h_k > 0$ ($k \in \underline{m}$) are given positive numbers and $A_k \in \mathbb{R}^{n \times n}$ ($k \in \underline{m}_0$) are given matrices.

For a given $\phi \in C([-h, 0], \mathbb{R}^n)$, (1) has a unique solution satisfying the initial value condition $x(s) = \phi(s)$, $s \in [-h, 0]$, where $h := \max\{h_k : k \in \underline{m}\}$, see e.g. [13]. This solution is denoted by $x(\cdot; \phi)$. Then (1) is said to be (globally) exponentially stable if, and only if, there are positive numbers α, M such that

$$\forall t \in \mathbb{R}_+, \forall \phi \in C([-h, 0], \mathbb{R}^n) \ : \quad \|x(t; \phi)\| \leq M e^{-\alpha t} \|\phi\|.$$

It is well-known that (1) is exponentially stable if, and only if,

$$\det \left(zI_n - A_0 - \sum_{k=1}^{m} A_k e^{-h_k z} \right) \neq 0, \ \ \forall z \in \mathbb{C}_+,$$

where $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathbb{R}_z \geq 0\}$, see e.g. [13, Chapter 7].

Definition 3.1. The system (1) is said to be positive if $x(t; \phi) \geq 0, \forall t \in \mathbb{R}_+$ for any $\phi \in C([-h, 0], \mathbb{R}^n), \phi \geq 0$.

The following is well-known in the theory of positive linear systems.

Theorem 3.2 (a). The system (1) is positive if, and only if, $A_0 \in \mathbb{R}^{n \times n}$ is a Metzler matrix and $A_k \in \mathbb{R}_+^{n \times n}$, for each $k \in \underline{m}$.

(b) Let (1) be positive. Then the following statements are equivalent

(i) (1) is exponentially stable;

(ii) $\mu \left(\sum_{k=0}^{m} A_k \right) < 0$;

(iii) $\left(\sum_{k=0}^{m} A_k \right) p \ll 0$ for some $p \in \mathbb{R}^n, p \gg 0$.

Proof. The proof of (a) can be found in [12], [30]. The statement (i) $\Leftrightarrow$ (ii) has been proven in [30, Theorem 4.4] (see also [29, Theorem 4.1]) and (ii) $\Leftrightarrow$ (iii) has been shown in [10, Theorem 3.1] (see also [20, Theorem 3]).

We now deal with robust stability of the positive linear time-invariant time delay differential system (1) under the time-varying structured perturbations (2) and the non-linear time-varying perturbations (3).

3.1. Time-varying structured perturbations.

Suppose (1) is exponentially stable. Consider perturbed systems of the form

$$\dot{x}(t) = (A_0 + D_0(t)\Delta_0(t)E_0(t))x(t)$$
$$+ \sum_{k=1}^{m} \big(A_k x(t - h_k) + (D_k(t)\Delta_k(t)E_k(t))x(t - \tau_k(t)) \big), \quad (5)$$
$$t \geq \sigma,$$

where $\tau_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R})$ ($k \in \underline{m}$) satisfy $0 \leq \tau_k(t) \leq \tau_k$, $\forall t \in \mathbb{R}_+$ for some $\tau_k > 0$ and $D_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R}^{n \times l_k})$, $E_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R}^{q_k \times n})$ ($k \in \underline{m}_0$) are given and $\Delta_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R}^{l_k \times q_k})$ $(k \in \underline{m}_0)$ are unknown perturbations.

Let $h := \max\{\tau_k, h_k : k \in \underline{m}\}$. Note that (5) is now a linear time-varying time delay differential system. For a fixed $\sigma \geq 0$ and a given $\phi \in C([-h, 0], \mathbb{R}^n)$, (5) has a unique solution satisfying the initial value condition

$$x(s+\sigma) = \phi(s), \quad s \in [-h, 0], \tag{6}$$

see e.g. [13]. This solution is now denoted by $x(\cdot; \sigma, \phi)$. Recall that $x(\cdot; \sigma, \phi)$ is continuously differentiable on $[\sigma, \infty)$ and satisfies (5) for any $t \in [\sigma, \infty)$. Then (5) is said to be (globally) exponentially stable if, and only if, there exist $M, \beta > 0$ such that

$$\forall \phi \in C([-h, 0], \mathbb{R}^n), \forall\, t \geq \sigma \geq 0 :$$
$$\|x(t; \sigma, \phi)\| \leq M e^{-\beta(t-\sigma)} \|\phi\|.$$

We are now in the position to state the first result of this paper whose proof is given in Appendix in a more general setting.

Theorem 3.3. Let (1) be positive and exponentially stable. Suppose that there exist $D_k \in \mathbb{R}_+^{n \times l_k}$, $E_k \in \mathbb{R}_+^{q_k \times n}$ and $\Delta_k \in \mathbb{R}_+^{l_k \times q_k}$ for $k \in \underline{m}_0$ such that $|D_k(t)| \leq D_k$, $|E_k(t)| \leq E_k$ and $|\Delta_k(t)| \leq \Delta_k$ for any $t \in \mathbb{R}_+$ and any $k \in \underline{m}_0$. Then the perturbed system (5) remains exponentially stable provided

$$\sum_{k=0}^{m} \|\Delta_k\| < \frac{1}{\max_{i,j \in \underline{m}_0} \left\| E_i \left(\sum_{k=0}^{m} A_k \right)^{-1} D_j \right\|}. \tag{7}$$

Remark 3.4. (i) In particular, the problem of robust stability of the positive linear time-invariant differential system without delay

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \tag{8}$$

under the time-invariant structured perturbations

$$A \to A + D\Delta E,$$

has been studied in [35,36]. More precisely, it has been shown that if (8) is exponentially stable and positive and D, E are given non-negative matrices then a perturbed system of the form

$$\dot{x}(t) = (A + D\Delta E)x(t), \qquad t \geq 0,$$

remains exponentially stable whenever

$$\|\Delta\| < \frac{1}{\|EA^{-1}D\|},$$

see [35, 36]. However, this problem is still open in the case of time-varying structured perturbations. Thus, Theorem 3.3 is new even for the case of systems without delay.

On the other hand, the problems of robust stability of the linear time-invariant differential system with delay (1) (not necessary positive) under time-invariant structured perturbations have been addressed in [19,37]. An upper stability bound for (1) subject to the time-invariant structured perturbations was presented in [19] in terms of solutions of an global optimization problem in $\mathbb{R}^2$ while a particular case of Theorem 3.3 with $D_k(\cdot) \equiv D_k \in \mathbb{R}_+^{n \times l_k}$, $E_k(\cdot) \equiv E_k \in \mathbb{R}_+^{q_k \times n}$ $(k \in \underline{m}_0)$ can be found in [37]. However, in the general case, a result like Theorem 3.3 cannot be found in the literature.

(ii) Note that (7) is independent of delays and so the result of Theorem 3.3 holds for perturbed systems of the form (5) with bounded delays.

The following is immediate from Theorem 3.3.

Corollary 3.5. Let $A \in \mathbb{R}^{n \times n}$ and $A_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R}^{n \times n})$ $(k \in \underline{m}_0)$ be given and let $\tau_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(k \in \underline{m})$ be given bounded continuous functions. If A is a Hurwitz stable Metzler matrix then the linear time delay differential system

$$\dot{x}(t) = (A + A_0(t))x(t) + \sum_{k=1}^{m} A_k(t)x(t - \tau_k(t)), \tag{9}$$

is exponentially stable provided there exist $A_k \in \mathbb{R}_+^{n \times n}, k \in \underline{m}_0$ such that

$$|A_k(t)| \leq A_k,\ t \in \mathbb{R}_+,\ k \in \underline{m}_0$$

and

$$\sum_{k=0}^{m} \|A_k\| < \frac{1}{\|A^{-1}\|}.$$

We illustrate the obtained results by two examples.

Example 3.6. Consider the time delay differential equation

$$\dot{x}(t) = -ax(t) + b(t)x(t-h), \qquad t \in \mathbb{R}_+, \tag{10}$$

where $a > 0$, $h > 0$ and $b(\cdot)$ is a bounded continuous function on $\mathbb{R}_+$. By applying a Razumikhin-type theorem to (10), it has been shown in *[22, Example 5.1, page 74]* that (10) is exponentially stable if $sup_{t \in \mathbb{R}_+} |b(t)| < a$. Note that this assertion is immediate from Corollary 3.5.

Moreover, a differential equation with time-varying delays of the form

$$\dot{x}(t) = -ax(t) + b(t)x(t - h_1(t)) + c(t)x(t - h_2(t)), \quad t \in \mathbb{R}_+, \tag{11}$$

is exponentially stable provided $a > 0$ and $h_1(\cdot)$, $h_2(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $b(\cdot)$, $c(\cdot) \in C(\mathbb{R}_+, \mathbb{R})$ are bounded such that

$$\sup_{t \in \mathbb{R}_+} |b(t)| + \sup_{t \in \mathbb{R}_+} |c(t)| < a.$$

Example 3.7. Consider a linear time delay differential equation in $\mathbb{R}^2$ defined by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad t \in \mathbb{R}_+, \tag{12}$$

where $h > 0$ and

$$A_0 := \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that (12) is positive and exponentially stable, by Theorem 3.2. Consider a perturbed system given by

$$\begin{aligned} \dot{x}(t) &= \big(A_0 + D_0(t)\Delta_0(t)E_0(t)\big)x(t) \\ &+ A_1 x(t-h) + D_1(t)\Delta_1(t)E_1(t)x(t - \tau(t)), \end{aligned} \tag{13}$$

where $\tau(\cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and

$$D_0(t) := \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}, \qquad D_1(t) := \begin{pmatrix} 0 \\ 1 \\ \dfrac{}{t+1} \end{pmatrix},$$

$$E_0(t) := \begin{pmatrix} e^{-t} & 0 \\ 0 & -\dfrac{2t}{t^2+1} \end{pmatrix},$$

$$E_1(t) := \begin{pmatrix} 1 & 0 \\ 0 & -\dfrac{1}{t+1} \end{pmatrix},$$

for $t \in \mathbb{R}_+$, and

$$\Delta_0(t) := \begin{pmatrix} a(t) & b(t) \end{pmatrix}, \quad \Delta_1(t) := \begin{pmatrix} c(t) & d(t) \end{pmatrix}$$

with $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot) \in C(\mathbb{R}_+, \mathbb{R})$ are unknown perturbations.

Note that for any $t \in \mathbb{R}_+$, we have

$$|D_0(t)| \leq D_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |D_1(t)| \leq D_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\|E_0(t)| \leq E_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |E_1(t)| \leq E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$E_0(A_0 + A_1)^{-1} D_0 = E_1(A_0 + A_1)^{-1} D_0$$
$$= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix},$$
$$E_0(A_0 + A_1)^{-1} D_1 = E_1(A_0 + A_1)^{-1} D_1$$
$$= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Let $\mathbb{R}^2$ be endowed with 2-norm. By Theorem 3.3, (13) is exponentially stable if $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $d(\cdot)$ are bounded and satisfy

$$\sqrt{(\sup_{t \in \mathbb{R}_+} |a(t)|)^2 + (\sup_{t \in \mathbb{R}_+} |b(t)|)^2} + \sqrt{(\sup_{t \in \mathbb{R}_+} |c(t)|)^2 + (\sup_{t \in \mathbb{R}_+} |d(t)|)^2} < \frac{1}{\sqrt{5}}.$$

3.2. Non-linear time-varying perturbations. Assume that (1) is exponentially stable and we now consider perturbed systems of the form

$$\dot{x}(t) = A_0 x(t) + f(t, x(t)) + \sum_{k=1}^{m} A_k x(t - h_k) + F\big(t; x(t - \tau_1(t)), ..., x(t - \tau_m(t))\big), \tag{14}$$

where $t \geq \sigma \geq 0$ and

(i) $\tau_k(\cdot) \in C(\mathbb{R}_+, \mathbb{R})$ ($k \in \underline{m}$) are given such that $0 < \tau_k(t) \leq \tau_k, \forall t \in \mathbb{R}_+$ for some $\tau_k > 0$, $k \in \underline{m}$;

(ii) $f(\cdot;\cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, is an unknown continuous function such that $f(t;0) = 0, \forall t \in \mathbb{R}_+$ and $f(t;u)$ is (locally) Lipschitz continuous with respect to u on each compact subset of $\mathbb{R}_+ \times \mathbb{R}^n$.

(iii) $F(\cdot\ ;\ \cdot\ , ..., \cdot\ ) : \mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times .... \times \mathbb{R}^n}^{m\ times} \to \mathbb{R}^n$, is an unknown continuous function such that $F(t\ ; 0, ..., 0) = 0, \forall t \in \mathbb{R}_+$ and $F(t; u_1, u_2, ..., u_m)$ is (locally) Lipschitz continuous with respect to $u_1, u_2, ..., u_m$ on each compact subset of $\mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times .... \times \mathbb{R}^n}^{m\ times}$.

Note that (i), (ii) and (iii) ensure that for a fixed $\sigma \geq 0$ and a given $\phi \in C([-h, 0], \mathbb{R}^n)$ ($h := \max\{\tau_k, h_k : k \in \underline{m}\}$), there exists a unique local solution of (14) satisfying the initial condition (6). This solution is defined and continuous on $[\sigma - h, \gamma)$ for some $\gamma > \sigma$ and satisfies (14) for each $t \in [\sigma, \gamma)$ see e.g. [6, 13]. It is denoted by $x(\cdot; \sigma, \phi)$. Furthermore, if the interval $[\sigma - h, \gamma)$ is the maximum interval of existence of the solution $x(\cdot; \sigma, \phi)$ then $x(\cdot; \sigma, \phi)$ is said to be *non-continuable*. The existence of a non-continuable solution follows from Zorn's lemma and the maximum interval of existence must be open.

Definition 3.8. (i) The zero solution of (14) is said to be *locally exponentially stable* if there exist positive numbers r, K, β such that for each $\sigma \in \mathbb{R}_+$ and each $\phi \in C([-h, 0], \mathbb{R}^n)$, $\|\phi\| \leq r$, the solution $x(\cdot; \sigma, \phi)$ of (14) and (6) exists on $[\sigma - h, \infty)$ and furthermore satisfies

$$\|x(t; \sigma, \phi)\| \leq K e^{-\beta(t-\sigma)}, \ \forall t \geq \sigma.$$

(ii) The zero solution of (14) is said to be *globally exponentially stable* if there exist positive numbers K, β such that for each $\sigma \in \mathbb{R}_+$ and each $\phi \in C([-h, 0], \mathbb{R}^n)$, the solution $x(\cdot; \sigma, \phi)$ of (14) and (6) exists on $[\sigma - h, \infty)$ and furthermore satisfies

$$\|x(t; \sigma, \phi)\| \leq K e^{-\beta(t-\sigma)} \|\phi\|, \ \forall t \geq \sigma.$$

When the zero solution of (14) is locally exponentially stable, globally exponentially stable then we also say that (14) is locally exponentially stable, globally exponentially stable, respectively.

The following whose proof is given in the Appendix, is an extension of Theorem 3.3 to non-linear time-varying perturbations.

Theorem 3.9. Let (1) be positive and exponentially stable. Suppose that there exist $D_k \in \mathbb{R}_+^{n \times l_k}$; $E_k \in \mathbb{R}_+^{q_k \times n}$; $\Delta_k \in \mathbb{R}_+^{l_k \times q_k}, k \in \underline{m}_0$ such that

$$|f(t; u)| \leq D_0 \Delta_0 E_0 |u|, \ \forall t \in \mathbb{R}_+, \ \forall u \in \mathbb{R}^n; \tag{15}$$

$$|F(t; u_1, ..., u_m)| \leq \sum_{k=1}^{m} D_k \Delta_k E_k |u_k|, \quad \forall t \in \mathbb{R}_+;\ \forall u_1, ..., u_m \in \mathbb{R}^n. \tag{16}$$

Then the perturbed system (14) is globally exponentially stable provided (7) holds.

Remark 3.10. It is important to note that if $f(t; u)$ is (globally) Lipschitz continuous with respect to u on $\mathbb{R}_+ \times \mathbb{R}^n$ and $f(t; 0) = 0, \forall t \in \mathbb{R}_+$ and $F(t; u_1, u_2, ..., u_m)$ is (globally) Lipschitz continuous with respect to $u_1, u_2, ..., u_m$ on

$\mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times .... \times \mathbb{R}^n}^{m\ times}$ and $F(t;0,0,...,0)=0, \forall t\in\mathbb{R}_+$, then (15) and (16) hold automatically for some $D_k, \Delta_k, E_k,\ k \in \underline{m}_0$.

We illustrate Theorem 3.9 by a couple of examples.

Example. Consider the non-linear differential equation with delay

$$\begin{gathered}\dot{x}(t) = \left(-3 + \frac{e^{-t^2}\sin t}{2(x^2(t)+1)}\right)x(t) \\ +\big(x(t-h) + e^{-(t^2+x^2(t-\tau(t)))}x(t-\tau(t))\big), \\ t \geq \sigma \geq 0,\end{gathered} \tag{17}$$

where $h > 0$ is given and $\tau(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a given bounded continuous function.

First, the linear time delay differential equation

$$\dot{x}(t) = -3x(t) + x(t-h),\ t \in \mathbb{R}_+,$$

is positive and exponentially stable, by Theorem 3.2. On the other hand, (17) can be represented as

$$\dot{x}(t) = -3x(t) + f(t;x(t)) + x(t-h) + F(t;x(t-\tau(t))),$$

where

$$f(t;u) := \frac{e^{-t^2}\sin t}{2(u^2+1)}u;$$

$$F(t;u) := e^{-(t^2+u^2)}u, \quad t\in\mathbb{R}_+,\ u\in\mathbb{R}.$$

Clearly,

$$|f(t;u)| \leq \frac{1}{2}|u|; \quad |F(t;u)| \leq |u|, \quad \forall t\in\mathbb{R}_+, u\in\mathbb{R}.$$

Theorem 3.9 implies that (17) is globally exponentially stable.

Example. Consider the non-linear differential equation with delay in $\mathbb{R}^2$ given by

$$\begin{gathered}\dot{x}(t) = A_0x(t) + A_1x(t-h) \\ +F(t;x(t-h_1(t)), x(t-h_2(t)), \\ t\geq\sigma\geq 0,\end{gathered} \tag{18}$$

where $h > 0$ is given and $h_1(\cdot), h_2(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ are given bounded continuous functions and

$$A_0 := \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$F(t;u,v) := \begin{pmatrix} \sqrt{a(1+\sin^2 t)u_1^2 + b2^{-t}v_1^2} \\ \sqrt{c(1+\cos^2 t)u_2^2 + d2^{1-t}v_2^2} \end{pmatrix},$$

$$t\in\mathbb{R}_+, \quad u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2.$$

and $a, b, c, d \geq 0$ are parameters.

By Theorem 3.2, it is easy to check that the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h), \qquad t\in\mathbb{R}_+,$$

is positive and exponentially stable. On the other hand, F is globally Lipschitz continuous with respect to u, v on $\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$ and satisfies

$$|F(t;u,v)| \leq \max\{\sqrt{2a}, \sqrt{2c}\}|u| + \max\{\sqrt{b}, \sqrt{2d}\}|v|, \\ \forall u,v\in\mathbb{R}^2.$$

Note that

$$(A_0+A_1)^{-1} = \begin{pmatrix} -1 & -1 \\ -1/2 & -1 \end{pmatrix},$$

$$\|(A_0+A_1)^{-1}\|_p = 2, \quad \text{for} \quad p = 1,\infty.$$

By Theorem 3.9, (18) is globally exponentially stable provided

$$\max\{\sqrt{2a}, \sqrt{2c}\} + \max\{\sqrt{b}, \sqrt{2d}\} < \frac{1}{2}.$$

4. Concluding remarks

By a novel approach, we present explicit robust stability bounds for positive linear time-invariant time delay differential systems subject to time-varying structured perturbations and non-linear time-varying perturbations.

It is important to note that the approach of this paper can be applied to study problems of stability of various classes of differential systems such as: linear (non-linear) time-varying ordinary differential systems, time-varying differential systems with finite (infinite) delay, time-varying Volterra differential systems (with delay), time-varying Volterra-Stieltjes differential systems, etc. These works will be done in the near future.

Appendix

Proofs of Theorems 3.3, 3.9.

It is clear that Theorem 3.3 is just a particular case of Theorem 3.9. So it remains to prove Theorem 3.9.

We divide the proof into four steps.

`Step 1`: We claim that

$$\mu\left((A_0 + D_0\Delta_0E_0) + \sum_{k=1}^{m}(A_k + D_k\Delta_kE_k)\right) < 0.$$

Since (1) is positive, it follows that A_0 is a Metzler matrix and $A_k \geq 0$ for any $k \in \underline{m}$, by Theorem 3.2 (a). Thus, $(A_0 + D_0\Delta_0E_0) + \sum_{k=1}^{m}(A_k + D_k\Delta_kE_k)$ is also a Metzler matrix because D_k, E_k, Δ_k are non-negative for any $k \in \underline{m}_0$. We show that $\mu_0 := \mu\left((A_0+D_0\Delta_0E_0) + \sum_{k=1}^{m}(A_k+D_k\Delta_kE_k)\right) < 0$. Assume on the contrary that $\mu_0 \geq 0$. By the Perron-Frobenius theorem (Theorem 2.1 (i)), there exists $x_0 \in \mathbb{R}^n_+$, $x_0 \neq 0$ such that

$$\left((A_0 + D_0\Delta_0E_0) + \sum_{k=1}^{m}(A_k + D_k\Delta_kE_k)\right)x_0 = \mu_0x_0.$$

Let $Q(t) = tI_n - A_0 - \sum_{k=1}^{m} A_k$, $t\in\mathbb{R}$. Since (1) is positive and exponentially stable, $\mu\left(A_0 + \sum_{k=1}^{m} A_k\right) < 0$, by Theorem 3.2 (b). Thus $Q(\mu_0) := \mu_0 I_n - A_0 - \sum_{k=1}^{m} A_k$ is invertible and this implies

$$Q(\mu_0)^{-1}\left(D_0\Delta_0E_0x_0 + \sum_{k=1}^{m} D_k\Delta_kE_kx_0\right) = x_0. \quad (19)$$

Let i_0 be an index such that $\|E_{i_0}x_0\| = \max_{i\in\underline{m}_0}\|E_ix_0\|$. It follows from (19) that $\|E_{i_0}x_0\| > 0$. Multiply both sides of (19) from the left by E_{i_0} to get

$$E_{i_0}Q(\mu_0)^{-1}D_0\Delta_0E_0x_0 + \sum_{k=1}^{m} E_{i_0}Q(\mu_0)^{-1}D_k\Delta_kE_kx_0 = E_{i_0}x_0.$$

This gives

$$\|E_{i_0}Q(\mu_0)^{-1}D_0\|\|\Delta_0\|\|E_0x_0\| + \sum_{k=1}^{m}\|E_{i_0}Q(\mu_0)^{-1}D_k\|\|\Delta_k\|\|E_kx_0\| \geq \|E_{i_0}x_0\|.$$

Thus,

$$\max_{i,j\in\underline{m}_0}\|E_iQ(\mu_0)^{-1}D_j\|\left(\sum_{k=0}^{m}\|\Delta_k\|\right)\|E_{i_0}x_0\| \geq \|E_{i_0}x_0\|,$$

or equivalently,

$$\max_{i,j\in\underline{m}_0}\|E_iQ(\mu_0)^{-1}D_j\|\sum_{k=0}^{m}\|\Delta_k\| \geq 1. \quad (20)$$

On the other hand, the resolvent identity gives

$$Q(0)^{-1} - Q(\mu_0)^{-1} = \mu_0 Q(0)^{-1}Q(\mu_0)^{-1}.$$

Since $A_0 + \sum_{k=1}^{m} A_k$ is a Metzler matrix with $\mu\left(A_0 + \sum_{k=1}^{m} A_k\right) < 0$ and $\mu_0 \geq 0$, Theorem 2.1 (iii) yields $Q(0)^{-1} \geq 0$ and $Q(\mu_0)^{-1} \geq 0$. Therefore,

$$Q(0)^{-1} \geq Q(\mu_0)^{-1} \geq 0.$$

This gives, $E_iQ(0)^{-1}D_j \geq E_iQ(\mu_0)^{-1}D_j \geq 0$, for any $i,j \in \underline{m}_0$. By (4), we have $\|E_iQ(0)^{-1}D_j\| \geq \|E_iQ(\mu_0)^{-1}D_j\|$, for any $i,j\in\underline{m}_0$. Then (20) implies

$$\sum_{k=0}^{m}\|\Delta_k\| \geq \frac{1}{\max_{i,j\in\underline{m}_0}\|E_iQ(0)^{-1}D_j\|}.$$

However, this conflicts with (7).

`Step 2`: Let $\phi \in C([-h,0],\mathbb{R}^n)$ be given and let $x(t) := x(t;\sigma,\phi), t\in[\sigma-h,\gamma)$ be a non-continuable solution of (14) and (6). We show that there exists $\beta > 0$ such that for any $\sigma \geq 0$ and any $r > 0$ and any $\phi \in C([-h,0],\mathbb{R}^n)$ with $\|\phi\| \leq r$,

$$\|x(t;\sigma,\phi)\| \leq Ke^{-\beta(t-\sigma)},\ \forall t\in[\sigma,\gamma), \quad (21)$$

where K depends on β, r.

By `Step 1`, $\mu\left((A_0+D_0\Delta_0E_0)+\sum_{k=1}^{m}(A_k+D_k\Delta_kE_k)\right) < 0$. Thus,

$$\left((A_0+D_0\Delta_0E_0)+\sum_{k=1}^{m}(A_k+D_k\Delta_kE_k)\right)p \ll 0, \quad (22)$$

for some $p := (\alpha_1,\alpha_2,...,\alpha_n)^T, \alpha_i > 0, \forall i\in\underline{n}$, by Theorem 3.2 (b). By continuity, (22) implies that

$$\left((A_0+D_0\Delta_0E_0)+\sum_{k=1}^{m}(A_k+D_k\Delta_kE_k)e^{h\beta}\right)p \ll -\beta(\alpha_1,...,\alpha_n)^T, \quad (23)$$

for some sufficiently small $\beta > 0$. Fix $r > 0$ and choose $K > 0$ such that $|\phi(t)| \ll Ke^{-\beta t}p$ for any $t\in[-h,0]$ and for any $\phi\in C([-h,0],\mathbb{R}^n)$ with $\|\phi\|\leq r$. Define $u(t) := Ke^{-\beta(t-\sigma)}p,\ t\in[\sigma-h,\infty)$. Set $x(t) := x(t;\sigma,\phi),\ t\in[\sigma-h,\gamma)$. Then, we have $|x(t)| \ll u(t), \forall t\in[\sigma-h,\sigma]$. We claim that $|x(t)| \leq u(t)$ for any $t\in[\sigma,\gamma)$.

Assume on the contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| \not\leq u(t_0)$. Set $t_1 := \inf\{t\in(\sigma,\gamma) : |x(t)| \not\leq u(t)\}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in \underline{n}$ such that

$$\begin{gathered}|x(t)| \leq u(t), \quad \forall t\in[\sigma,t_1),\\ |x_{i_0}(t_1)| = u_{i_0}(t_1), \quad |x_{i_0}(t)| > u_{i_0}(t),\\ \forall t\in(t_1,t_1+\epsilon),\end{gathered} \quad (24)$$

for some $\epsilon > 0$. Let $A_k := (a_{ij}^{(k)}),\ D_k\Delta_kE_k = (b_{ij}^{(k)})$, for $k\in\underline{m}_0$. Since A_0 is a Metzler matrix and $A_k \geq 0$ for $k\in\underline{m}$ and $D_k\Delta_kE_k \geq 0$ for $k\in\underline{m}_0$, we have for any $i\in\underline{n}$,

$$\begin{aligned}\frac{d}{dt}|x_i(t)| = sgn(x_i(t))\dot{x}_i(t) \leq a_{ii}^{(0)}|x_i(t)| &+ \sum_{j=1,j\neq i}^{n} a_{ij}^{(0)}|x_j(t)| + |f_i(t,x(t))|\\ &+ \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}^{(k)}|x_j(t-h_k)|\\ &+ |F_i\big(t;x(t-\tau_1(t)),...,x(t-\tau_m(t))\big)|,\end{aligned}$$

for almost any $t\in[\sigma,\gamma)$. Then (15), (16) imply that

$$\begin{aligned}\frac{d}{dt}|x_i(t)| \leq{}& a_{ii}^{(0)}|x_i(t)| + \sum_{j=1,j\neq i}^{n} a_{ij}^{(0)}|x_j(t)| + \sum_{j=1}^{n} b_{ij}^{(0)}|x_j(t)|\\ &+ \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}^{(k)}|x_j(t-h_k)| + \sum_{k=1}^{m}\sum_{j=1}^{n} b_{ij}^{(k)}|x_j(t-\tau_k(t))|,\end{aligned}$$

for almost any $t\in[\sigma,\gamma)$. Thus we have for any $t\in[\sigma,\gamma)$,

$$\begin{aligned}D^+|x_i(t)| :={}& \limsup_{h\to0^+}\frac{|x_i(t+h)|-|x_i(t)|}{h}\\ ={}& \limsup_{h\to0^+}\frac{1}{h}\int_t^{t+h}\frac{d}{ds}|x_i(s)|ds\\ \leq{}& a_{ii}^{(0)}|x_i(t)| + \sum_{j=1,j\neq i}^{n} a_{ij}^{(0)}|x_j(t)| + \sum_{j=1}^{n} b_{ij}^{(0)}|x_j(t)|\\ &+ \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}^{(k)}|x_j(t-h_k)| + \sum_{k=1}^{m}\sum_{j=1}^{n} b_{ij}^{(k)}|x_j(t-\tau_k(t))|,\end{aligned}$$

where D^+ denotes the Dini upper-right derivative. In particular, it follows from (23) and (24) that

$$\begin{aligned} D^+|x_{i_0}(t_1)| &\overset{(24)}{\leq} a^{(0)}_{i_0 i_0} K e^{-\beta(t_1-\sigma)}\alpha_{i_0} \\ &+ \sum_{j=1, j\neq i_0}^{n} a^{(0)}_{i_0 j} K e^{-\beta(t_1-\sigma)}\alpha_j + \sum_{j=1}^{n} b^{(0)}_{i_0 j} K e^{-\beta(t_1-\sigma)}\alpha_j \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{n} a^{(k)}_{i_0 j} K e^{-\beta(t_1-\sigma)} e^{\beta h}\alpha_j \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{n} b^{(k)}_{i_0 j} K e^{-\beta(t_1-\sigma)} e^{\beta h}\alpha_j \\ &= K e^{-\beta(t_1-\sigma)}\Bigg(\sum_{j=1}^{n} a^{(0)}_{i_0 j}\alpha_j + \sum_{j=1}^{n} b^{(0)}_{i_0 j}\alpha_j \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{n} a^{(k)}_{i_0 j} e^{\beta h}\alpha_j + \sum_{k=1}^{m}\sum_{j=1}^{n} b^{(k)}_{i_0 j} e^{\beta h}\alpha_j\Bigg) \\ &\overset{(23)}{<} -\beta K e^{-\beta(t_1-\sigma)}\alpha_{i_0} = D^+ u_{i_0}(t_1). \end{aligned}$$

However, this conflicts with (24). Therefore, we have for any $\sigma \geq 0$ and any $\phi \in C([-h,0],\mathbb{R}^n)$ with $\|\phi\| \leq r$,

$$|x(t;\sigma,\phi)| \leq u(t) = K e^{-\beta(t-\sigma)} p, \forall t \in [\sigma,\gamma).$$

By the monotonicity of vector norms, this yields

$$\|x(t;\sigma,\phi)\| \leq K_1 e^{-\beta(t-\sigma)}, \forall t \in [\sigma,\gamma),$$

for some $K_1 > 0$.

`Step 3`: We claim that $\gamma = \infty$ and so (14) is locally exponentially stable.

Seeking a contradiction, we assume that $\gamma < \infty$. Then it follows from (21) that $x(\cdot;\sigma,\phi)$ is bounded on $[\sigma,\gamma)$. Furthermore, this together with (14) and (15), (16) imply that $\dot{x}(\cdot)$ is bounded on $[\sigma,\gamma)$. Thus $x(\cdot)$ is uniformly continuous on $[\sigma,\gamma)$. Therefore, $\lim_{t\to\gamma^-} x(t)$ exists and $x(\cdot)$ can be extended to a continuous function on $[\sigma,\gamma]$. Moreover the closure of $\{x_t \ : \ t \in [\sigma,\gamma)\}$ is a compact set in $C([-h,0],\mathbb{R}^n)$, by Arzéla-Ascoli theorem [5]. Note that

$$\{(t,x_t) \ : \ t \in [\sigma,\gamma)\} \subset [\sigma,\gamma] \times \text{the closure of } \{x_t \ : \ t \in [\sigma,\gamma)\}.$$

Thus, the closure of $\{(t,x_t) \ : \ t \in [\sigma,\gamma)\}$ is a compact set in $\mathbb{R}_+ \times C([-h,0],\mathbb{R}^n)$. Since (γ, x_γ) belongs to this compact set, one can find a solution of (14) through this point to the right of γ. This contradicts the non-continuability hypothesis on $x(\cdot)$. Thus γ must be equal to ∞.

`Step 4`: Finally, we show that (14) is globally exponentially stable.

By `Step 3`, in particular, the linear system

$$\begin{aligned} \dot{y}(t) = (A_0 + D_0\Delta_0 E_0)y(t) + \sum_{k=1}^{m} A_k y(t-h_k) \\ + \sum_{k=1}^{m} D_k\Delta_k E_k y(t-\tau_k(t)), \end{aligned} \tag{25}$$

is locally exponentially stable provided (7) holds. Because of linearity, (25) is globally exponentially stable provided (7) holds.

Fix $\phi \in C([-h,0],\mathbb{R}^n)$ and let $x(t) := x(t;\sigma,\phi), t \in [\sigma-h,\infty)$ be the solution of (14) and (6). Denote $y(\cdot) := y(\cdot;|\phi|)$, the solution of (25) satisfying the initial condition $y(t) = |\phi|(t), t \in [-h,0]$ where $|\phi|(t) := |\phi(t)|, t \in [-h,0]$. As shown in `Step 1`, (7) implies that $\mu(A_0 + D_0\Delta_0 E_0 + \sum_{k=1}^{m} A_k + D_k\Delta_k E_k) < 0$. Furthermore, it is shown in `Step 2` that

$$\left(A_0 + D_0\Delta_0 E_0 + \sum_{k=1}^{m} A_k + D_k\Delta_k E_k\right)p \ll 0,$$

for some $p := (p_1, p_2, ..., p_n) \in \mathbb{R}^n, p \gg 0$. Thus there is $\epsilon_0 > 0$ such that

$$\left(A_0 + D_0\Delta_0 E_0 + \sum_{k=1}^{m} A_k + D_k\Delta_k E_k\right)p \ll -\epsilon p,$$

for any $\epsilon \in (0,\epsilon_0]$. Note that

$$y(t) = |\phi(t)| \gg |\phi(t)| - \epsilon p = |x(\sigma+t)| - \epsilon p, \quad t \in [-h,0],$$

for any $\epsilon \in (0,\epsilon_0]$. By similar arguments as in `Step 2`, we can show that

$$y(t) \geq |x(\sigma+t)| - \epsilon p, \quad t \in [0,\infty).$$

By the monotonicity of vector norms,

$$\|y(t)\| + \epsilon\|p\| \geq \|x(\sigma+t)\|, \quad t \in [0,\infty).$$

Letting ϵ tend to zero, we get

$$\|y(t)\| \geq \|x(\sigma+t)\|, \quad t \in [0,\infty). \tag{26}$$

Since (25) is exponentially stable, (26) implies that (25) is globally exponentially stable. This completes the proof.

Acknowledgements. This research is funded by the Vietnam National University Ho Chi Minh City (VNU-HCM) under the grant number B2015-28-01/HD-KHCN.

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