Opuscula Math. 44, no. 2 (2024), 267–283
https://doi.org/10.7494/OpMath.2024.44.2.267 Opuscula Mathematica

POSITIVE SOLUTIONS TO A THIRD ORDER NONLOCAL BOUNDARY VALUE PROBLEM WITH A PARAMETER

Gabriela Szajnowska and Mirosława Zima

Communicated by Marek Galewski

Abstract. We present some sufficient conditions for the existence of positive solutions to a third order differential equation subject to nonlocal boundary conditions. Our approach is based on the Krasnosel'skiĭ–Guo fixed point theorem in cones and the properties of the Green's function corresponding to the BVP under study. The main results are illustrated by suitable examples.

Keywords: boundary value problem, nonlocal boundary conditions, positive solution, cone.

Mathematics Subject Classification: 34B10, 34B15, 34B18, 34B27.

1. INTRODUCTION

We study the existence of positive solutions to the third order differential equation of the form

$$-u''' + m^2 u' = f(t, u, u'), \quad t \in [0, 1], \tag{1.1}$$

subject to the non-local boundary conditions

$$u(0) = 0, \quad u'(0) = \alpha[u], \quad u'(1) = \beta[u], \tag{1.2}$$

where m is a positive parameter and α and β are functionals (not necessarily linear) acting on the space $C^1[0, 1]$. By a positive solution to problem (1.1)–(1.2) we mean a function that satisfies the equation (1.1), the boundary conditions (1.2), and is nonnegative and nontrivial on the interval $[0, 1]$.

Theory and applications of third order differential equations in physics and engineering are widely discussed in the monograph [8]. In particular, the equation

$$-u''' + \kappa(K_2 A_e - K_1^2)u' = a \tag{1.3}$$

governs the deflection u of a three layer beam formed by parallel layers of different materials (see [1, 8, 10]). Here K_1 and K_2 are shear parameters, A_e is the area of the cross-section of the beam, and κ and a are parameters related to the elasticity of the layers. Clearly, if $\kappa(K_2A_e - K_1^2) > 0$, then equation (1.3) is a special case of (1.1).

Interesting existence results on third order boundary value problems (BVPs for short) for equations and systems can be found in a number of papers; see for example [2, 11, 13, 14, 18, 24]. Among the methods used in the mentioned papers are fixed point index, the Leray-Schauder continuation principle, Mawhin's theorem for coincidences, and the method of lower and upper solutions. Our main tool is the following Krasnosel'skiĭ-Guo fixed point theorem on cone expansion and compression.

Theorem 1.1. [9] *Let P be a cone in a Banach space X and suppose that Ω_1 and Ω_2 are bounded open sets in X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that one of the following conditions holds:*

(i) $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$,
(ii) $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in the set $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let us recall that a cone in a Banach space X is a closed, convex subset P of X such that $\lambda u \in P$ for $u \in P$ and $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. Here we work in the space $X = \mathcal{C}^1[0,1]$ endowed with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}, \tag{1.4}$$

where $\|\cdot\|_\infty$ stands for the supremum norm in the space $\mathcal{C}[0,1]$. In the sequel we exploit the following lemma which can be derived from the Mean Value Theorem (see for example [23]).

Lemma 1.2. *If $u \in \mathcal{C}^1[0,1]$ and $u(0) = 0$, then $\|u\| = \|u'\|_\infty$.*

Theorem 1.1 is a tool frequently used for studying positive solutions to BVPs or integral equations, in particular, for third order problems. In [17], the authors consider the BVP for the system of third order equations

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t)), \\ -v'''(t) = h(t, u(t), u'(t)), \\ u(0) = u'(0) = 0, \ u'(1) = \alpha\, u'(\eta), \\ v(0) = v'(0) = 0, \ v'(1) = \alpha\, v'(\eta). \end{cases}$$

They introduce the cone

$$K = \left\{ u \in \mathcal{C}^1[0,1] : u(t) \geq 0, \min_{t \in [\frac{\eta}{\alpha}, \eta]} u(t) \geq k_0 \|u\|_\infty, \min_{t \in [\frac{\eta}{\alpha}, \eta]} u'(t) \geq k_1 \|u'\|_\infty \right\}.$$

In [7], the authors use the cone

$$K = \left\{ u \in \mathcal{C}^n[0,1] : \min_{t \in [a_i, b_i]} u^{(i)}(t) \geq c_i \|u^{(i)}\|_\infty, \ i = 0, 1, \ldots, n \right\}$$

to study the existence of nontrivial solutions to the Hammerstein generalized integral equation

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s),u'(s),\dots,u^{(n)}(s))\,ds.$$

Similar cones appear for example in [2, 12, 13, 16]. A common feature of the cones used in the mentioned papers is that the minimum of the function $u^{(i)}$ on some interval is compared with its norm $\|u^{(i)}\|_\infty$.

On the other hand, in [19] and [22] the authors consider the cone

$$K = \left\{ u \in \mathcal{C}^1[0,1] : \; u(t) \geq 0, \; u'(t) \geq 0, \; \min_{t\in[a,1]} u(t) \geq b\|u\| \right\},$$

to deal with the nonlocal BVPs for the equation $u''' + f(t,u,u') = 0$. Moreover, in [23], the cone

$$K = \left\{ u \in \mathcal{C}^1[0,1] : \; u(t) \geq 0, \; u'(t) \geq 0, \; \min_{t\in[\theta,1-\theta]} u'(t) \geq b\|u\| \right\},$$

is employed to study a system of third order equations. This time the elements of cones are characterized by the inequalities involving norm (1.4). It is also worth mentioning the recent paper [1], where the authors study the existence of positive solutions to the equation on the half-line

$$-u''' + k^2u' = \phi(t)f(t,u,u'), \quad t > 0,$$

subject to local boundary conditions

$$u(0) = u'(0) = u'(\infty) = 0.$$

They apply Theorem 1.1 in the space

$$E = \left\{ u \in \mathcal{C}^1(\mathbb{R}^+,\mathbb{R}) : \lim_{t\to\infty} \mathrm{e}^{-kt}u(t) = 0, \lim_{t\to\infty} \mathrm{e}^{-kt}u'(t) = 0 \right\}$$

equipped with the norm

$$\|u\|_E = \|u\|_k + \|u'\|_k,$$

where $\|u^{(i)}\|_k = \sup\{e^{-kt}|u^{(i)}(t)| : t \geq 0\}$, $i = 0,1$. The cone in [1] is

$$K = \{ u \in E : \; u(t) \geq g(t)\|u\|_E, \; u'(t) \geq \widetilde{g}(t)\|u\|_E, \; \text{for all } t > 0 \},$$

where g and $\widetilde{g}$ are suitably chosen functions.

Theorem 1.1 is also used in the recent paper [15] to prove the existence of positive radial solutions to the nonlinear Poisson equation with some nonlocal conditions. The author exploits the cone

$$K = \left\{ u \in C[0,1] : u(t) \geq 0 \text{ for } t \in [0,1], \inf_{t\in[a,b]} u(t) \geq \min\{a, 1-b\}\|u\|_\infty \right\}$$

where $[a,b] \subset (0,1)$.

Our idea in this paper is to use a cone defined in terms of the norm (1.4). Namely, we work with the cone

$$P = \left\{ u \in \mathcal{C}^1[0,1] : \, u^{(i)}(t) \geq 0, \, \min_{t \in [\delta, 1-\delta]} u^{(i)}(t) \geq c\|u\|, \; i = 0,1 \right\}, \tag{1.5}$$

where $\delta \in (0, \frac{1}{2})$. The constant $c \in (0,1)$ depends on the parameters m and δ and is specified at the end of Section 2. The aim of this paper is to establish a few sufficient conditions for the existence of positive increasing solutions to problem (1.1)–(1.2). In comparison with the literature (see for example [12, 19]), an advantage of employing (1.5) is that it enables us to relax to some extent assumptions imposed upon nonlinearity f.

This paper is organized as follows. In Section 2, we study the properties of the Green's function corresponding to the linear local BVP

$$\begin{cases} -u''' + m^2 u' = 0, \\ u(0) = 0, \; u'(0) = 0, \; u'(1) = 0. \end{cases} \tag{1.6}$$

We also derive some inequalities for the solutions of the auxiliary linear BVPs

$$\begin{cases} -u''' + m^2 u' = 0, \\ u(0) = 0, \; u'(0) = 1, \; u'(1) = 0, \end{cases} \tag{1.7}$$

and

$$\begin{cases} -u''' + m^2 u' = 0, \\ u(0) = 0, \; u'(0) = 0, \; u'(1) = 1. \end{cases} \tag{1.8}$$

In Section 3, we apply results obtained in Section 2 to establish two theorems on the existence of positive and increasing solutions to problem (1.1)–(1.2). For this purpose, we are concerned with the perturbed Hammerstein equation

$$u(t) = \alpha[u]\gamma_1(t) + \beta[u]\gamma_2(t) + \int_0^1 G(t,s) f(s, u(s), u'(s)) \, ds. \tag{1.9}$$

Here, γ_1 and γ_2 are the solutions of (1.7), (1.8), respectively, and $G(t,s)$ is the Green's function associated with (1.6). Observe that each solution of (1.9) is a solution of BVP (1.1)–(1.2). Let us mention that the perturbed Hammerstein equations have been recently studied and applied to BVPs in several papers. In particular, we refer the reader to important contributions due to Goodrich [4, 5], Graef and Webb [6], and Webb and Infante [20].

In Section 3 we also show the applicability of our results to the nonlocal BVP for the equation $-u''' = \widetilde{f}(t,u,u')$ by considering the equivalent perturbed equation $-u''' + m^2 u' = \widetilde{f}(t,u,u') + m^2 u'$. A similar approach with the shift $m^2 u$ is used for example in [3] and [21] in order to deal with the second-order resonant BVPs. To illustrate our results, three numerical examples are included.

2. LINEAR PROBLEMS

We begin this section with a detailed analysis of the properties of the Green's function associated with the BVP (1.6). This function is given by

$$G(t,s) = \frac{1}{m^2 \sinh m} \begin{cases} \sinh(m(1-s))(\cosh(mt)-1), & t \le s, \\ \sinh m - \sinh(m(1-s)) - \sinh(ms)\cosh(m(1-t)), & t \ge s, \end{cases} \tag{2.1}$$

where $t,\ s \in [0,1]$.

Lemma 2.1. *For all* $(t,s) \in [0,1]\times[0,1]$ *function* (2.1) *has the following properties:*

$$G(t,s) \ge 0, \tag{2.2}$$

$$G(t,s) \le G(1,s), \tag{2.3}$$

$$G(t,s) \ge c_1(t)G(1,s), \tag{2.4}$$

where $c_1(t) = \dfrac{\cosh(mt)-1}{\cosh m}$.

Proof. For $t \in [0,s]$ the first inequality is obvious. For $t \in [s,1]$, it is enough to show that

$$\sinh(ms)\cosh(m(1-t)) \le \sinh m - \sinh(m(1-s)).$$

The expression $\cosh(m(1-t))$ attains its maximum at $t = s$. Thus, consider the function

$$\phi_1(s) = \sinh m - \sinh(m(1-s)) - \sinh(ms)\cosh(m(1-s)), \quad s \in [0,1],$$

and its derivative

$$\phi_1'(s) = m\cosh(m(1-s)) - m\cosh(m(2s-1)) = 2m \sinh\frac{ms}{2}\sinh\frac{m(2-3s)}{2}.$$

It is clear that ϕ_1 achieves its maximum at $s = \frac{2}{3}$. Together with $\phi_1(0) = 0$ and $\phi_1(1) = 0$, this gives (2.2). To prove (2.3) we first show that for $t \in [0,s]$

$$\sinh(m(1-s))\cosh(mt) \le \sinh m - \sinh(ms).$$

The expression $\cosh(mt)$ attains its maximum when $t = s$. Let us define

$$\phi_2(s) = \sinh(m(1-s))\cosh(ms) - \sinh m + \sinh(ms), \quad s \in [0,1].$$

In this case, we have

$$\phi_2''(s) = m^2 \sinh(ms) \ge 0,$$

which implies that ϕ_2 is convex. Moreover, $\phi_2(0) = 0$ and $\phi_2(1) = 0$. Hence, $\phi_2(s) \le 0$ for $s \in [0,1]$. If $t \in [s,1]$, inequality (2.3) is equivalent to

$$\cosh(m(1-t)) \ge 1,$$

which clearly holds for any t. For $t \in [0,s]$ inequality (2.4) reduces to

$$\cosh m \sinh(m(1-s)) \ge \sinh m - \sinh(m(1-s)) - \sinh(ms), \quad s \in [0,1].$$

Hence, for $s \in [0,1]$ it is enough to consider the function

$$\phi_3(s) = \cosh m \sinh(m(1-s)) - \sinh m + \sinh(m(1-s)) + \sinh(ms),$$

and study its derivative

$$\phi_3'(s) = -m \cosh m \cosh(m(1-s)) - m\cosh(m(1-s)) + m\cosh(ms) \leq 0.$$

Since ϕ_3 is decreasing and $\phi_3(1) = 0$, we have that $\phi_3(s) \geq 0$ for $s \in [0,1]$. For $t \in [s,1]$ let

$$H(t,s) = G(t,s) - c_1(t)G(1,s).$$

Then,

$$H(t,s) = \frac{1}{m^2 \sinh m}\Big[\sinh m - \sinh(m(1-s)) - \sinh(ms)\cosh(m(1-t)) - \frac{\cosh(mt)-1}{\cosh m}(\sinh m - \sinh(m(1-s)) - \sinh(ms))\Big].$$

Clearly, $H \in \mathcal{C}^2([0,1]\times[0,1])$. What is left is to prove that H is nonnegative. Note that

$$H(t,0) = 0, \quad H(1,s) = \frac{G(1,s)}{\cosh m} \geq 0, \quad \text{and} \quad H(s,s) \geq 0. \tag{2.5}$$

We obtain

$$H_{tt}(t,s) = -\frac{1}{\sinh m}\sinh(ms)\cosh(m(1-t)) - G(1,s)\frac{m^2\cosh(mt)}{\cosh(m)}.$$

Since $H_{tt}(t,s) \leq 0$ for $t \in (s,1)$, for each fixed s, the function H, as a function of one variable, is concave. Taking into account (2.5), we deduce that $H(t,s) \geq 0$. □

As a consequence of (2.4) we obtain the following corollary.

Corollary 2.2. *For* $(t,s) \in [\delta, 1-\delta]\times[0,1]$,

$$G(t,s) \geq c_1 G(1,s), \tag{2.6}$$

where

$$c_1 = \frac{\cosh(m\delta)-1}{\cosh m}. \tag{2.7}$$

Remark 2.3. Inequality (2.6) can be also proved by finding the global minimum and maximum of the function H in the domain $[\delta, 1-\delta]\times[0,1]$.

Now, we study the properties of the partial derivative G_t of G, that is,

$$G_t(t,s) = \frac{1}{m\sinh m}\begin{cases} \sinh(m(1-s))\sinh(mt) & \text{for } t \leq s, \\ \sinh(ms)\sinh(m(1-t)) & \text{for } t \geq s, \end{cases} \tag{2.8}$$

where $t,\ s \in [0,1]$.

Lemma 2.4. *Function* (2.8) *has the following properties:*

$$G_t(t,s) \geq 0 \qquad \textit{for all } (t,s) \in [0,1] \times [0,1], \tag{2.9}$$
$$G_t(t,s) \leq G_t(s,s) \quad \textit{for all } (t,s) \in [0,1] \times [0,1], \tag{2.10}$$
$$G_t(t,s) \geq d\, G_t(s,s) \quad \textit{for all } (t,s) \in [\delta, 1-\delta] \times [0,1], \tag{2.11}$$

where

$$d = \frac{\sinh(m\delta)}{\sinh m}. \tag{2.12}$$

Proof. The inequalities (2.9) and (2.10) are quite obvious, therefore we focus on the property (2.11). Let $t \in [\delta, 1-\delta]$. Then for $t \leq s$,

$$\sinh(mt) \geq \sinh(m\delta) \geq \frac{\sinh(m\delta)}{\sinh m} \sinh(ms),$$

while for $t \geq s$ we have

$$\sinh(m(1-t)) \geq \sinh(m\delta) \geq \frac{\sinh(m\delta)}{\sinh m} \sinh(m(1-s)),$$

and (2.11) follows. ☐

Next, we provide an interesting relation between the Green's function (2.1) and its derivative (2.8).

Lemma 2.5. *For all* $(t,s) \in [\delta, 1-\delta] \times [0,1]$ *we have*

$$G(t,s) \geq \omega\, G_t(s,s), \tag{2.13}$$

where

$$\omega = \frac{\cosh(m\delta) - 1}{m \sinh m}. \tag{2.14}$$

Proof. If $t \leq s$ and $t \in [\delta, 1-\delta]$, we get $s \in [\delta, 1]$. Then (2.13) holds if

$$\frac{\cosh(mt) - 1}{m} \geq w_1 \sinh(ms)$$

for some positive constant w_1. By minimizing left-hand side and maximizing right-hand side of the above inequality we obtain

$$w_1 = \frac{\cosh(m\delta) - 1}{m \sinh m}.$$

For $t \geq s$ we first consider the case $s \in [0, \delta]$. Then (2.13) holds if

$$\begin{aligned} &\sinh m - \sinh(m(1-s)) - \sinh(ms)\cosh(m(1-t)) \\ &\geq m w_2 \sinh(ms) \sinh(m(1-s)) \end{aligned}$$

for some positive constant w_2. By minimizing the left side we get

$$\begin{aligned} &\sinh m - \sinh(m(1-s)) - \sinh(ms)\cosh(m(1-\delta)) \\ &- m w_2 \sinh(ms) \sinh(m(1-s)) \geq 0. \end{aligned}$$

Obviously, the above inequality holds for $s = 0$. Hence, for $s \in (0, \delta]$, consider

$$\begin{aligned}\phi_4(s) &= \sinh m - \sinh\left(m(1-s)\right) - \sinh\left(ms\right)\cosh(m(1-\delta)) \\ &\quad - mw_2 \sinh\left(ms\right)\sinh\left(m(1-s)\right)\end{aligned}$$

and

$$\psi(s) = \frac{\phi_4(s)}{\sinh(ms)\sinh(m(1-s))}.$$

Using the identity

$$\frac{\sinh m - \sinh\left(m(1-s)\right) - \sinh\left(ms\right)\cosh(m(1-s))}{\sinh(ms)\sinh(m(1-s))} = \tanh\left(\frac{ms}{2}\right),$$

we obtain

$$\begin{aligned}\psi(s) &= \frac{\phi_4(s) - \sinh(ms)\cosh(m(1-s)) + \sinh(ms)\cosh(m(1-s))}{\sinh(ms)\sinh(m(1-s))} \\ &= \tanh\left(\frac{ms}{2}\right) + \frac{\sinh\left(ms\right)(\cosh(m(1-s)) - \cosh(m(1-\delta))}{\sinh(ms)\sinh(m(1-s))} - mw_2 \\ &= \tanh\left(\frac{ms}{2}\right) + \frac{\cosh(m(1-s)) - \cosh(m(1-\delta))}{\sinh(m(1-s))} - mw_2.\end{aligned}$$

Observe that for $s \in [0, \delta]$,

$$\tanh\left(\frac{ms}{2}\right) \geq 0 \quad \text{and} \quad \frac{\cosh(m(1-s)) - \cosh(m(1-\delta))}{\sinh(m(1-s))} \geq 0.$$

Moreover,

$$\lim_{s\to 0} \psi(s) = \frac{\cosh m - \cosh(m(1-\delta))}{\sinh m} - mw_2$$

and

$$\psi(\delta) = \tanh\frac{m\delta}{2} - mw_2.$$

Therefore, to get $\phi_4(s) \geq 0$ for $s \in [0, \delta]$ it is enough to set

$$w_2 = \min\left\{\frac{\cosh m - \cosh(m(1-\delta))}{m\sinh m}, \frac{1}{m}\tanh\frac{m\delta}{2}\right\}.$$

If $\delta \leq s \leq t \leq 1-\delta$, in a similar fashion we obtain

$$w_2 = \frac{1}{m}\tanh\frac{m\delta}{2}.$$

As a result,

$$\omega = \min\{w_1, w_2\} = w_1 = \frac{\cosh(m\delta) - 1}{m\sinh m},$$

which completes the proof. □

For the convenience of the reader, we provide here the values of the integrals employed in Section 3:

$$\int_0^1 G_t(s,s)ds = \frac{m\cosh m - \sinh m}{2m^2\sinh m},$$

$$\int_\delta^{1-\delta} G(1,s)ds = \frac{(1-2\delta)m\sinh m - 2\cosh m(1-\delta) + 2\cosh(m\delta)}{m^3\sinh m},$$

and

$$\int_\delta^{1-\delta} G_t(s,s)ds = \frac{(1-2\delta)m\cosh m - \sinh(m(1-2\delta))}{2m^2\sinh m}.$$

In the remainder of this section we study the properties of the unique solutions to the BVPs (1.7) and (1.8). These solutions are

$$\gamma_1(t) = \frac{\cosh m - \cosh\left(m(1-t)\right)}{m\sinh m} \quad \text{and} \quad \gamma_2(t) = \frac{\cosh(mt) - 1}{m\sinh m},$$

for (1.7) and (1.8), respectively. Hence

$$\gamma_1'(t) = \frac{\sinh\left(m(1-t)\right)}{\sinh m}, \quad \gamma_2'(t) = \frac{\sinh(mt)}{\sinh m},$$

and

$$\|\gamma_1\|_\infty = \|\gamma_2\|_\infty = \frac{\cosh m - 1}{m\sinh m}, \quad \|\gamma_1'\|_\infty = \|\gamma_2'\|_\infty = 1.$$

By Lemma 1.2 we get

$$\|\gamma_1\| = \|\gamma_2\| = 1. \tag{2.15}$$

The following lemma deals with the inequalities for γ_1, γ_2 and their derivatives.

Lemma 2.6. *Let $t \in [\delta, 1-\delta]$, $i = 1,2$. Then:*

$$\gamma_i(t) \geq c_{\gamma_i}\|\gamma_i\|_\infty, \quad \gamma_i'(t) \geq d_{\gamma_i}\|\gamma_i'\|_\infty, \quad \text{and} \quad \gamma_i(t) \geq a_{\gamma_i}\|\gamma_i'\|_\infty,$$

where

$$c_{\gamma_1} = \frac{\cosh m - \cosh(m(1-\delta))}{\cosh m - 1}, \quad c_{\gamma_2} = \frac{\cosh(m\delta) - 1}{\cosh m - 1},$$

$$d_{\gamma_1} = d_{\gamma_2} = \frac{\sinh(m\delta)}{\sinh m},$$

$$a_{\gamma_1} = \frac{\cosh m - \cosh(m(1-\delta))}{m\sinh m}, \quad a_{\gamma_2} = \frac{\cosh(m\delta) - 1}{m\sinh m}.$$

The proofs of all above inequalities are straightforward so we omit them. Note that $d = d_{\gamma_1} = d_{\gamma_2}$ and $\omega = a_{\gamma_2}$, where d and ω are given by (2.12) and (2.14), respectively. To construct a suitable cone in $\mathcal{C}^1[0,1]$ we take

$$c = \min\{c_1, d, \omega, c_{\gamma_1}, c_{\gamma_2}, a_{\gamma_1}\},$$

where c_1 is given by (2.7). After a thorough comparison of all above constants we select the smallest possible, that is,

$$c = \min\{c_1, d, \omega, c_{\gamma_1}, c_{\gamma_2}, a_{\gamma_1}\} = \min\{c_1, \omega\} = \begin{cases} c_1, & \text{if } m \in (0, m_0], \\ \omega, & \text{if } m \in [m_0, \infty), \end{cases} \tag{2.16}$$

where $m_0 \approx 1.19968$ is the unique positive solution of equation $\tanh m = \frac{1}{m}$. Recall that the constant c is the one that appears in definition of cone (1.5).

3. EXISTENCE RESULTS

Let P be the cone defined by (1.5). Throughout this section we make the following assumptions:

(C1) $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous,
(C2) the functionals α, $\beta : P \to [0,\infty)$ are continuous and map bounded sets into bounded sets.

In the sequel, for a given $r > 0$ we use the notation

$$f^r = \max\{f(t,u,v) : (t,u,v) \in [0,1] \times [0,r]^2\},$$

$$f_r = \min\{f(t,u,v) : (t,u,v) \in [\delta, 1-\delta] \times [cr, r]^2\},$$

$$A^r = \sup\{\alpha[u] : u \in P \text{ and } \|u\| = r\}, \quad B^r = \sup\{\beta[u] : u \in P \text{ and } \|u\| = r\},$$

and

$$A_r = \inf\{\alpha[u] : u \in P \text{ and } \|u\| = r\}, \quad B_r = \inf\{\beta[u] : u \in P \text{ and } \|u\| = r\}.$$

With above assumptions we can finally state our existence results.

Theorem 3.1. *Assume that there exist r_1, $r_2 > 0$, $r_1 < r_2$ such that:*

(C3) $A^{r_2} + B^{r_2} + f^{r_2} \int_0^1 G_t(s,s)\, ds \le r_2$,

and either

(C4) $c_{\gamma_1} A_{r_1} \|\gamma_1\|_\infty + c_{\gamma_2} B_{r_1} \|\gamma_2\|_\infty + c_1 f_{r_1} \int\limits_{\delta}^{1-\delta} G(1,s)\, ds \ge r_1$

or

(C5) $d(A_{r_1} + B_{r_1} + f_{r_1} \int\limits_{\delta}^{1-\delta} G_t(s,s)\, ds) \ge r_1$

are satisfied. Then problem (1.1)–(1.2) *has a positive increasing solution* u^* *such that* $r_1 \leq \|u^*\| \leq r_2$*. Moreover,* u^* *satisfies the Harnack inequalities*

$$\min\{u^*(t) : t \in [\delta, 1-\delta]\} \geq c r_1$$

and

$$\min\{(u^*)'(t) : t \in [\delta, 1-\delta]\} \geq c r_1.$$

Proof. For $t \in [0,1]$ and $u \in P$ consider the operator

$$Tu(t) = \alpha[u]\gamma_1(t) + \beta[u]\gamma_2(t) + \int\limits_0^1 G(t,s) f(s, u(s), u'(s))\, ds.$$

Then,

$$(Tu)'(t) = \alpha[u]\gamma_1'(t) + \beta[u]\gamma_2'(t) + \int\limits_0^1 G_t(t,s) f(s, u(s), u'(s))\, ds.$$

It is clear that the fixed points of T are the solutions of equation (1.9), and hence of problem (1.1)–(1.2). In order to apply Theorem 1.1, we first show that $T(P) \subset P$. From (C1), (C2), (2.2), and (2.9) it follows that $Tu \in \mathcal{C}^1[0,1]$, $Tu(t) \geq 0$ and $(Tu)'(t) \geq 0$. Moreover, by (2.10), we get

$$\|(Tu)'\|_\infty \leq \alpha[u]\|\gamma_1'\|_\infty + \beta[u]\|\gamma_2'\|_\infty + \int\limits_0^1 G_t(s,s) f(s, u(s), u'(s))\, ds. \tag{3.1}$$

Thus, by Lemma 1.2

$$\|Tu\| \leq \alpha[u]\|\gamma_1\| + \beta[u]\|\gamma_2\| + \int\limits_0^1 G_t(s,s) f(s, u(s), u'(s))\, ds.$$

On the other hand, for $t \in [\delta, 1-\delta]$ we have by Lemma 2.5, Lemma 2.6, (2.11) and (2.16)

$$\begin{aligned}
Tu(t) &\geq \alpha[u] a_{\gamma_1} \|\gamma_1'\|_\infty + \beta[u] a_{\gamma_2} \|\gamma_2'\|_\infty + \omega \int\limits_0^1 G_t(s,s) f(s, u(s), u'(s))\, ds \\
&\geq \alpha[u] c \|\gamma_1\| + \beta[u] c \|\gamma_2\| + c \int\limits_0^1 G_t(s,s) f(s, u(s), u'(s))\, ds \\
&\geq c\|(Tu)'\|_\infty = c\|Tu\|
\end{aligned}$$

and

$$\begin{aligned}(Tu)'(t) &\geq \alpha[u]d_{\gamma_1}\|\gamma_1'\|_\infty + \beta[u]d_{\gamma_2}\|\gamma_2'\|_\infty + d\int_0^1 G_t(s,s)f(s,u(s),u'(s))\,ds\\ &\geq \alpha[u]c\|\gamma_1\| + \beta[u]c\|\gamma_2\| + c\int_0^1 G_t(s,s)f(s,u(s),u'(s))\,ds\\ &\geq c\|(Tu)'\|_\infty = c\|Tu\|.\end{aligned}$$

Therefore

$$\min\{Tu(t) : t \in [\delta, 1-\delta]\} \geq c\|Tu\|$$

and

$$\min\{(Tu)'(t) : t \in [\delta, 1-\delta]\} \geq c\|Tu\|$$

which implies that $T(P) \subset P$.

Let

$$\Omega_1 = \{u \in \mathcal{C}^1[0,1] : \|u\| < r_1\} \quad \text{and} \quad \Omega_2 = \{u \in \mathcal{C}^1[0,1] : \|u\| < r_2\}.$$

Then $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Under the assumptions (C1) and (C2), applying the Ascoli–Arzelà theorem, we can prove that T is a completely continuous operator on $P \cap \Omega_2$.

For $u \in P \cap \partial\Omega_2$ and $t \in [0,1]$ we have $\|u\| = r_2$, $u(t) \geq 0$ and $u'(t) \geq 0$. Hence, by Lemma 1.2, (2.15), (3.1), and (C3) we obtain

$$\begin{aligned}\|Tu\| = \|(Tu)'\|_\infty &\leq \alpha[u]\|\gamma_1\| + \beta[u]\|\gamma_2\| + \int_0^1 G_t(s,s)f(s,u(s),u'(s))\,ds\\ &\leq A^{r_2} + B^{r_2} + f^{r_2}\int_0^1 G_t(s,s)\,ds \leq r_2 = \|u\|.\end{aligned}$$

For $u \in P \cap \partial\Omega_1$ and $t \in [\delta, 1-\delta]$, we have $\|u\| = r_1$, $u(t) \geq cr_1$, and $u'(t) \geq cr_1$. If (C4) holds, then from Corollary 2.2 and Lemma 2.6, we get for $t \in [\delta, 1-\delta]$,

$$\begin{aligned}Tu(t) &\geq \alpha[u]c_{\gamma_1}\|\gamma_1\|_\infty + \beta[u]c_{\gamma_2}\|\gamma_2\|_\infty + c_1\int_0^1 G(1,s)f(s,u(s),u'(s))\,ds\\ &\geq c_{\gamma_1}A_{r_1}\|\gamma_1\|_\infty + c_{\gamma_2}B_{r_1}\|\gamma_2\|_\infty + c_1\int_\delta^{1-\delta} G(1,s)f(s,u(s),u'(s))\,ds\\ &\geq c_{\gamma_1}A_{r_1}\|\gamma_1\|_\infty + c_{\gamma_2}B_{r_1}\|\gamma_2\|_\infty + c_1 f_{r_1}\int_\delta^{1-\delta} G(1,s)\,ds \geq r_1.\end{aligned}$$

If (C5) is satisfied, then by Lemma 2.6 and (2.11), we obtain

$$\begin{aligned}(Tu)'(t) &\geq \alpha[u]d_{\gamma_1}\|\gamma_1'\|_\infty + \beta[u]d_{\gamma_2}\|\gamma_2'\|_\infty + d\int_0^1 G_t(s,s)f(s,u(s),u'(s))\,ds\\ &\geq d_{\gamma_1}A_{r_1}\|\gamma_1'\|_\infty + d_{\gamma_2}B_{r_1}\|\gamma_2'\|_\infty + d\int_\delta^{1-\delta} G_t(s,s)f(s,u(s),u'(s))\,ds\\ &\geq d_{\gamma_1}A_{r_1} + d_{\gamma_2}B_{r_1} + d\,f_{r_1}\int_\delta^{1-\delta} G_t(s,s)\,ds\\ &= d(A_{r_1} + B_{r_1} + f_{r_1}\int_\delta^{1-\delta} G_t(s,s)\,ds) \geq r_1.\end{aligned}$$

Thus, $\|Tu\|_\infty \geq r_1$ or $\|(Tu)'\|_\infty \geq r_1$, which gives $\|Tu\| \geq r_1 = \|u\|$. An application of Theorem 1.1(i) completes the proof. □

In a similar manner the following existence result can be proved using Theorem 1.1(ii).

Theorem 3.2. *Assume that there exist* r_1, $r_2 > 0$, $r_1 < r_2$ *such that:*

(C6) $A^{r_1} + B^{r_1} + f^{r_1}\int_0^1 G_t(s,s)\,ds \leq r_1$,

and either

(C7) $c_{\gamma_1}A_{r_2}\|\gamma_1\|_\infty + c_{\gamma_2}B_{r_2}\|\gamma_2\|_\infty + c_1 f_{r_2}\int_\delta^{1-\delta} G(1,s)\,ds \geq r_2$

or

(C8) $d(A_{r_2} + B_{r_2} + f_{r_2}\int_\delta^{1-\delta} G_t(s,s)\,ds) \geq r_2$

are satisfied. Then problem (1.1)–(1.2) *has a positive increasing solution* u^* *such that* $r_1 \leq \|u^*\| \leq r_2$. *Moreover,* u^* *satisfies the Harnack inequalities*

$$\min\{u^*(t) : t \in [\delta, 1-\delta]\} \geq cr_1$$

and

$$\min\{(u^*)'(t) : t \in [\delta, 1-\delta]\} \geq cr_1.$$

Remark 3.3. It is worth noting that working with the cone (1.5) makes it possible to involve in conditions (C4) and (C5) of Theorem 3.1 the minimum of the function f over the set $[\delta, 1-\delta] \times [cr_1, r_1]^2$, which is clearly less restrictive than taking into account the behaviour of f on $[\delta, 1-\delta] \times [0, r_1]^2$. The analogous comment applies to (C7) and (C8) of Theorem 3.2.

We complete this section with providing three examples obtained with the help of the *Mathematica* software.

Example 3.4. Consider the boundary value problem

$$\begin{cases} -u''' + 4u' = h(t)u^2(2 + \arctan(u')), \\ u(0) = 0,\ u'(0) = \frac{1}{3}(u(\xi))^2,\ u'(1) = \frac{1}{5}(u'(\eta))^3, \end{cases} \tag{3.2}$$

where $\xi, \eta \in [\frac{1}{4}, \frac{3}{4}]$ and $h : [0, 1] \to [0, \infty)$ is a continuous function. In this case, we have $m = 2$ and $f(t, u, v) = h(t)u^2(2 + \arctan v)$, $\alpha[u] = \frac{1}{3}(u(\xi))^2$ and $\beta[u] = \frac{1}{5}(u'(\eta))^3$. We set $\delta = \frac{1}{4}$. Assume that $\max\{h(t) : t \in [0, 1]\} = 3$ and $\min\{h(t) : t \in [\frac{1}{4}, \frac{3}{4}]\} = 2.7$. Then,

$$f^{r_1} = 3r_1^2(2 + \arctan r_1)$$

and

$$f_{r_2} = 2.7(cr_2)^2(2 + \arctan(cr_2)).$$

Moreover,

$$c = \omega \approx 0.0175946, \quad d \approx 0.143677,$$

$$\int_0^1 G_t(s, s)\, ds \approx 0.134329, \qquad \text{and} \qquad \int_{\frac{1}{4}}^{\frac{3}{4}} G_t(s, s)\, ds \approx 0.0891609.$$

Assumption (C6) becomes

$$\frac{1}{3}r_1^2 + \frac{1}{5}r_1^3 + 3r_1^2(2 + \arctan r_1) \int_0^1 G_t(s, s)\, ds \leq r_1$$

and holds for $r_1 = 0.6$ while (C8) takes the form

$$d \left(\frac{1}{3}(cr_2)^2 + \frac{1}{5}(cr_2)^3 + 2.7(cr_2)^2(2 + \arctan(cr_2)) \int_{\frac{1}{4}}^{\frac{3}{4}} G_t(s, s)\, ds \right) \geq r_2$$

and is satisfied for $r_2 = 2365$. By Theorem 3.2, the BVP has a positive solution u^* such that $r_1 \leq \|u^*\| \leq r_2$. Moreover, $\min\{u^*(t) : t \in [\frac{1}{4}, \frac{3}{4}]\} \geq cr_1$ and $\min\{(u^*)'(t) : t \in [\frac{1}{4}, \frac{3}{4}]\} \geq cr_1$. Observe that the BVP has also a trivial solution.

As mentioned in the Introduction, we apply our results also to equations of the form $-u''' = \widetilde{f}(t, u, u')$, which is shown below.

Example 3.5. Consider the boundary value problem

$$\begin{cases} -u''' = \sqrt{u}(1 - t\sin(u + u')), \\ u(0) = 0,\ u'(0) = 0,\ u'(1) = \frac{1}{10}u'(\eta), \end{cases} \tag{3.3}$$

with $\eta \in [\frac{1}{3}, \frac{2}{3}]$.

Similar problems are considered for example in [2], [7] and [24]. In (3.3) we have $\widetilde{f}(t,u,v)=\sqrt{u}(1-t\sin(u+v))$, $\alpha[u]=0$, $\beta[u]=\frac{1}{10}u'(\eta)$, and $\delta=\frac{1}{3}$.

We put $f(t,u,v)=\sqrt{u}(1-t\sin(u+v))+m^2v$, and consider the equivalent problem

$$\begin{cases} -u'''+m^2u'=\sqrt{u}(1-t\sin(u+u'))+m^2u', \\ u(0)=0,\ u'(0)=0,\ u'(1)=\frac{1}{10}u'(\eta). \end{cases} \tag{3.4}$$

Fix $m=1$. Then,

$$f^{r_2}=2\sqrt{r_2}+r_2, \quad f_{r_1}=\frac{1}{3}\sqrt{cr_1}+cr_1,$$

$$c=c_1\approx 0.0363376, \quad d\approx 0.288921,$$

$$\int_0^1 G_t(s,s)\,ds\approx 0.156518, \quad \text{and} \quad \int_{\frac{1}{3}}^{\frac{2}{3}} G_t(s,s)\,ds\approx 0.0743786.$$

Conditions (C3) and (C5) required by Theorem 3.1 become

$$\frac{1}{10}r_2+(2\sqrt{r_2}+r_2)\int_0^1 G_t(s,s)\,ds\le r_2,$$

$$d\left(\frac{1}{10}cr_1+\left(\frac{1}{3}\sqrt{cr_1}+cr_1\right)\int_{\frac{1}{3}}^{\frac{2}{3}} G_t(s,s)\,ds\right)\ge r_1,$$

and they are met for $r_2=0.177274$ and $r_1=1.87137\cdot 10^{-6}$, respectively. Therefore, problem (3.3) has a positive increasing solution u^* such that $r_1\le\|u^*\|\le r_2$. Moreover, $\min\{u^*(t):t\in[\frac{1}{3},\frac{2}{3}]\}\ge cr_1$ and $\min\{(u^*)'(t):t\in[\frac{1}{3},\frac{2}{3}]\}\ge cr_1$.

Let us mention that Theorem 2.1 in [12], which is a general result for the perturbed Hammerstein equation, cannot be applied here as $\min\{\widetilde{f}(t,u,v):(t,u,v)\in[0,1]\times[0,r]^2\}=0$ for any $r>0$. The approach developed in [2] cannot be used either. The key requirement for the boundary condition $u'(1)=\beta u'(\eta)$ in [2] is $\beta>1$. On the other hand, Theorems 3.1 and 3.2 are applicable if $\beta\in[0,1)$. In this way our results complement and improve to some extent the ones from the cited literature.

Remark 3.6. Observe that for problem (3.4) condition (C5) gives a better estimate for r_1 than (C4). Indeed, (C4) holds for $r_1=7.04994\cdot 10^{-9}$. However, it need not be always the case as the next example indicates.

Example 3.7. Consider the boundary value problem

$$\begin{cases} -u'''=\sqrt{u}(1-t\sin(u+u')), \\ u(0)=0,\ u'(0)=\frac{1}{10}u'(\eta),\ u'(1)=0, \end{cases}$$

with $\eta\in[\frac{1}{3},\frac{2}{3}]$. Application of (C4) gives $r_1=7.06113\cdot 10^{-9}$ while (C5) holds for $r_1=7.04994\cdot 10^{-9}$.

Acknowledgements

The authors are grateful to the anonymous referees for their comments that led to an improved version of this paper. G. Szajnowska and M. Zima were partially supported by the Centre for Innovation and Transfer of Natural Science and Engineering Knowledge of the University of Rzeszów.

REFERENCES

[1] A. Benmezaï, E.-D. Sedkaoui, *Positive solution for singular third-order BVPs on the half line with first-order derivative dependence*, Acta Univ. Sapientiae **13** (2021), 105–126.

[2] A. Cabada, L. López-Somoza, F. Minhós, *Existence, non-existence and multiplicity results for a third order eigenvalue three-point boundary value problem*, J. Nonlinear Sci. Appl. **10** (2017), 5445–5463.

[3] J.A. Cid, G. Infante, M. Tvrdý, M. Zima, *A topological approach to periodic oscillations related to the Liebau phenomenon*, J. Math. Anal. Appl. **423** (2015), 1546–1556.

[4] C.S. Goodrich, *On a nonlocal BVP with nonlinear boundary conditions*, Results Math. **63** (2013), 1351–1364.

[5] C.S. Goodrich, *New Harnack inequalities and existence theorems for radially symmetric solutions of elliptic PDEs with sign changing or vanishing Green's function*, J. Differential Equations **264** (2018), 236–262.

[6] J. Graef, J.R.L. Webb, *Third order boundary value problems with nonlocal boundary conditions*, Nonlinear Anal. **71** (2009), 1542–1551.

[7] J. Graef, L. Kong, F. Minhós, *Generalized Hammerstein equations and applications*, Results Math. **72** (2017), 369–383.

[8] M. Greguš, *Third Order Linear Differential Equations*, Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht, 1987.

[9] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[10] Ch.P. Gupta, *On a third-order three-point boundary value problem at resonance*, Differential Integral Equations **2** (1989), 1–12.

[11] B. Hopkins, N. Kosmatov, *Third-order boundary value problems with sign-changing solutions*, Nonlinear Anal. **67** (2007), 126–137.

[12] G. Infante, *Positive and increasing solutions of perturbed Hammerstein integral equations with derivative dependence*, Discrete Contin. Dyn. Syst. B **25** (2020), 691–699.

[13] G. Infante, F. Minhós, *Nontrivial solutions of systems of Hammerstein integral equations with first derivative dependence*, Mediterr. J. Math. (2017) 14:242.

[14] W. Jiang, N. Kosmatov, *Solvability of a third-order differential equation with functional boundary conditions at resonance*, Bound. Value Probl. **2017** (2017), Article no. 81.

[15] I. Kossowski, *Radial solustions for nonlinear elliptic equation with nonlinear elliptic equation nonlocal boundary conditions*, Opuscula Math. **43** (2023), no. 5, 675–687.

[16] L. López-Somoza, F. Minhós, *Existence and multiplicity results for some generalized Hammerstein equations with a parameter*, Adv. Differ. Equ. **2019** (2019), Article no. 423.

[17] F. Minhós, R. de Sousa, *On the solvability of third-order three point systems of differential equations with dependence on the first derivative*, Bull. Braz. Math. Soc. **48** (2017), 485–503.

[18] I. Rachůnková, *On some three-point problems for third-order differential equations*, Math. Bohemica **117** (1992), 98–110.

[19] J.-P. Sun, H.-B. Li, *Monotone positive solution of nonlinear third-order BVP with integral boundary conditions*, Bound. Value Probl. **2010** (2010), Article no. 874959.

[20] J.R.L. Webb, G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc. **74** (2006), 673–693.

[21] J.R.L. Webb, M. Zima, *Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems*, Nonlinear Anal. **71** (2009), 1369–1378.

[22] H.-E. Zhang, J.-P. Sun, *Existence and iteration of monotone positive solutions for third-order nonlocal BVPs involving integral conditions*, Electron. J. Qual. Theory Differ. Equ. 2012, no. 18, 1–9.

[23] H.-E. Zhang, *Multiple positive solutions of nonlinear BVPs for differential systems involving integral conditions*, Bound. Value Probl. **2014** (2014), Article no. 61.

[24] L. Zhao, W. Wang, C. Zhai, *Existence and uniqueness of monotone positive solutions for a third-order three-point boundary value problem*, Differ. Equ. Appl. **10** (2018), 251–260.

Gabriela Szajnowska
gszajnowska@ur.edu.pl
https://orcid.org/0000-0002-5257-9435

University of Rzeszów
Institute of Mathematics
Pigonia 1, Rzeszów, 35-959, Poland

Mirosława Zima (corresponding author)
mzima@ur.edu.pl
https://orcid.org/0000-0002-6152-4962

University of Rzeszów
Institute of Mathematics
Pigonia 1, Rzeszów, 35-959, Poland

Received: July 19, 2023.
Revised: November 17, 2023.
Accepted: November 22, 2023.