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AN INVERSE BACKWARD PROBLEM FOR DEGENERATE TWO-DIMENSIONAL PARABOLIC EQUATION

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Abstract. This paper deals with the determination of an initial condition in the degenerate two-dimensional parabolic equation

$$\partial_t u - \operatorname{div}(a(x,y)I_2\nabla u) = f, \quad (x,y)\in\Omega,\ t\in(0,T),$$

where Ω is an open, bounded subset of $\mathbb{R}^2$, $a \in C^1(\bar{\Omega})$ with $a \geqslant 0$ everywhere, and $f \in L^2(\Omega\times(0,T))$, with initial and boundary conditions

$$u(x,y,0) = u_0(x,y), \quad u\mid_{\partial\Omega}=0,$$

from final observations. This inverse problem is formulated as a minimization problem using the output least squares approach with the Tikhonov regularization. To show the convergence of the descent method, we prove the Lipschitz continuity of the gradient of the Tikhonov functional. Also we present some numerical experiments to show the performance and stability of the proposed approach.

Keywords: data assimilation, adjoint method, regularization, heat equation, inverse problem, degenerate equations, optimization.

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1. INTRODUCTION

In many modern applications, it is necessary to estimate the initial state of a system (typically a system governed by a partial derivative equation (PDE) of evolution) from the partial knowledge of the system in a limited time interval.

This type of identification problem is applied in many areas. In medicine, the thermoacoustic tomography tumor detection can be reduced to the initial data reconstruction problems [7]. In cosmology, to have a good understanding of the cosmos formation or its evolution, the estimation of initial conditions is necessary [6]. Other application where the initial identification requirement is an essential procedure is the data assimilation [8], it is used for the numerical weather forecast, ocean circulation, and environmental prediction.

There are several approaches to identify the initial condition for non-degenerate parabolic heat transfer, for example, [9] apply genetic algorithm with epidemic operator, and [4] apply Tikhonov regularization.

Until the writing of these lines, all the works that deal with the degenerate problems are made for the one-dimensional case. Among the most recent works we can mention the work done by [10] to identify the initial condition of degenerate one-dimensional parabolic problem.

This work is the continuation of [1–3], in which we identify the initial condition and we study numerically the null controllability of a degenerate/singular parabolic problem in one-dimensional case. In [3], we solve an inverse backward problem for degenerate hyperbolic equation from final observations, and to reduce the execution time, we propose a new approach based on double regularization: a Tikhonov's regularization and regularization in equation by viscose-elasticity.

In the present paper, we study the inverse problem of determining the initial state in a degenerate two-dimensional parabolic equation from the theoretical analysis and numerical computation angles. More precisely, we consider the following problem:

$$\begin{cases} \partial_t u + A(u) = f & \text{in } \Omega \times (0,T), \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega, t \in (0,T), \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{1.1}$$

A is the operator defined as

$$A(u) = -\operatorname{div}\left(a(x,y) I_2 \nabla u\right),$$

where Ω is an open, bounded subset of $\mathbb{R}^2$, $a \in C^1(\bar{\Omega})$ with $a \geqslant 0$ everywhere ($a(\cdot,\cdot)$ can be equal to zero at any point in Ω), and $f \in L^2(\Omega \times (0,T))$.

We now specify some notations we shall use. Let introduce the following functional spaces

$$H_a^1(\Omega) = \left\{ u \in L^2(\Omega) \,:\, \sqrt{a}\nabla u \in L^2(\Omega) \text{ and } u(x,y) = 0 \text{ for all } (x,y) \in \partial\Omega \right\},$$

with

$$\|u\|^2_{H_a^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\sqrt{a}\nabla u\|^2_{L^2(\Omega)}.$$

The weak formulation of problem (1.1) is

$$\int\limits_\Omega \partial_t u v \, dxdy + \int\limits_\Omega a(x,y) \nabla u \nabla v \, dxdy = \int\limits_\Omega f v \, dxdy, \quad v \in H_0^1(\Omega).$$

Let us define the bilinear form

$$B[u,v] = \int\limits_\Omega a(x,y) \nabla u \nabla v \, dxdy.$$

This form is noncoercive.

We put

$$A_{ad} = \{h \in H^1_a(\Omega) \,:\, \|h\|_{H^1_a(\Omega)} \leq r\},$$

where r is a real strictly positive constant. Evidently, the set A_{ad} is a bounded, closed, and convex subset of $L^2(\Omega)$.

Let us define our inverse problem.

Inverse Source Problem (ISP). Let u be the solution to (1.1). Determine the initial state u_0 from the measured data at the final time $u(T, \cdot)$.

Remark 1.1. It should be mentioned that we do not need the supplement distributed measurements to obtain the numerical solution of the inverse problem.

We treat Problem (ISP) by interpreting its solution as a minimizer of the following problem

$$\text{find } u_0^\star \in A_{ad} \text{ such that } E(u_0^\star) = \min_{u_0 \in A_{ad}} E(u_0), \tag{1.2}$$

where the cost function E is defined as follows

$$E(u_0) = \frac{1}{2T} \left\| u(T) - u^{obs} \right\|^2_{L^2(\Omega)},$$

subject to u is the weak solution of the parabolic problem (1.1) with initial state u_0. $u^{obs} \in L^2(\Omega)$ is the observation data with noise.

Problem (1.2) is ill-posed in the sense of Hadamard, some regularization technique is needed in order to guarantee numerical stability of the computational procedure even with noisy input data. The problem thus consists in minimizing a functional of the form

$$J(u_0) = \frac{1}{2T} \left\| u(T) - u^{obs} \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \left\| u_0 - u^b \right\|^2_{L^2(\Omega)},$$

here, ε being a small positive regularizing coefficient that provides extra convexity to the functional J. u^b an a priori (background state) knowledge of the state u_0^{exact}. The background error is then defined as

$$err = \|u_0^{\text{exact}} - u^b\|_2.$$

u_0^{exact} is called the true state, and is the state to estimate.

Firstly, we present a new theorem which gives the existence and uniqueness of the weak solutions of problem (1.1). Secondly, with the aim of showing that the minimization problem and the direct problem are well-posed, we prove that the solution's behavior changes continuously with the initial conditions. For this we prove the Lipschitz continuity of the input-output operator $\varphi : u_0 \longmapsto u$, where u is the weak solution of (1.1) with initial state u_0. Thirdly, we prove the differentiability of the functional J, which gives the existence of the gradient of J, that is computed using the adjoint state method. Finally, to show the convergence of the descent method, we prove that the gradient of J is Lipschitz continuous, this gives that

$$\lim_{k \to \infty} \|\nabla J(u_0^k)\|_{L^2(\Omega)} = 0$$

and $\left(J(u_0^k)\right)_k$ is a monotone decreasing sequence, where $(u_0^k)_k$ is the sequence of iterations obtained by the Landweber iteration algorithm

$$u_0^{k+1} = u_0^k - t_k \nabla J(u_0^k)$$

and t_k is chosen by the inaccurate linear search by the Armijo–Goldstein Rule. Also we present some numerical experiments to study the noise resistance and the performance of this approach.

2. WELL-POSEDNESS

In case that there exists a constant $c > 0$ such that $a > c$ we recall the following theorem.

Theorem 2.1 ([5, p. 360])**.** *Assume* $u_0 \in H_0^1(\Omega)$ *and* $f \in L^2(0,T;L^2(\Omega))$. *Then there exists a unique weak solution which solves problem* (1.1) *such that*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \ , \quad \partial_t u \in L^2(0,T;L^2(\Omega)),$$

and we have the estimate

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\|_{H_0^1(\Omega)} + \|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u\|_{L^2(0,T;L^2(\Omega))} \\ &\le C_1 \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)} \right), \end{aligned}$$

where the constant C_1 *depends on* Ω *and* T.

In the case when $a \geq 0$ we have the following result.

Theorem 2.2. *For all* $f \in L^2(\Omega \times (0,T))$ *and* $u_0 \in H_a^1(\Omega)$, *there exists a unique weak solution which solves problem* (1.1) *such that*

$$u \in L^2\left(0,T;H_a^1(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right), \quad \partial_t u \in L^2(0,T;L^2(\Omega)),$$

and we have the estimate

$$\begin{aligned} &\sup_{t\in[0,T]} \|u(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\partial_t u\|_{L^2(\Omega)}^2 dt + \int_0^T \|\sqrt{a}\nabla u\|_{L^2(\Omega)}^2 \\ &\le C \left(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|u_0\|_{H_a^1(\Omega)}^2 \right), \end{aligned}$$

where the constant C *depends on* Ω *and* T.

Proof. For any positive integer n, consider the perturbed problem

$$\begin{cases} \partial_t u^n - \operatorname{div}\left(\left(a(x,y) + \frac{1}{n}\right) I_2 \nabla u^n\right) = f & \text{in } \Omega \times (0,T), \\ u^n(x,y,t) = 0, & (x,y) \in \partial\Omega,\ t \in (0,T), \\ u^n(x,y,0) = u_0, & (x,y) \in \Omega. \end{cases} \tag{2.1}$$

The weak formulation of problem (2.1) is

$$\int_\Omega \partial_t u^n v\, dxdy + \int_\Omega \left(a(x,y) + \frac{1}{n}\right) \nabla u^n \nabla v \; dxdy = \int_\Omega f v\, dxdy, \quad \forall v \in H_0^1(\Omega). \quad (2.2)$$

The bilinear form

$$B[u,v] = \int_\Omega \left(a(x,y) + \frac{1}{n}\right) \nabla u \nabla v \, dxdy$$

is continuous and coercive. By Theorem 2.1, problem (2.1) has a unique weak solution such that

$$u^n \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \quad \partial_t u^n \in L^2(0,T;L^2(\Omega)).$$

In equation (2.2) we take $v = u^n$, which gives

$$\int_\Omega \partial_t u^n u^n \, dxdy + \int_\Omega \left(a(x,y) + \frac{1}{n}\right) (\nabla u^n)^2 \; dxdy = \int_\Omega f u^n \, dxdy,$$

$$\frac{1}{2}\frac{d}{dt} \int_\Omega (u^n)^2 dxdy + \int_\Omega \left(a(x,y) + \frac{1}{n}\right) (\nabla u^n)^2 \; dxdy = \int_\Omega f u^n \, dxdy.$$

By integrating between 0 and t_1 with $t_1 \in [0,T]$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u^n(t_1)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u^n(0)\|^2_{L^2(\Omega)} + \int_0^{t_1}\int_\Omega \left(a(x,y) + \frac{1}{n}\right) (\nabla u^n)^2 \, dxdydt \\ &= \int_0^{t_1}\int_\Omega f u^n \, dxdydt, \end{aligned} \quad (2.3)$$

and so

$$\frac{1}{2}\|u^n(t_1)\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^{t_1}\int_\Omega f u^n \, dxdydt,$$

$$\frac{1}{2}\|u^n(t_1)\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^{t_1} \|f\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^{t_1} \|u^n\|^2_{L^2(\Omega)} dt.$$

Let

$$M = \|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f\|^2_{L^2(\Omega)} dt.$$

Then we have

$$\|u^n(t_1)\|^2_{L^2(\Omega)} \leq M + \int_0^{t_1} \|u^n\|^2_{L^2(\Omega)} dt.$$

Gronwall's Lemma gives

$$\|u^n(t_1)\|^2_{L^2(\Omega)} \leq M \exp\left(\int_0^{t_1} dt\right),$$

$$\|u^n(t_1)\|^2_{L^2(\Omega)} \leq M \exp(T), \tag{2.4}$$

$$\|u^n(t_1)\|_{L^2(\Omega)} \leq \sqrt{M \exp(T)}. \tag{2.5}$$

Then the sequence $(u^n)_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Consequently, there exists a subsequence $(u^n)_n$ such that

$$u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Returning to equation (2.3) with $t_1 = T$ we get

$$\frac{1}{2}\|u^n(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega \left(a(x,y) + \frac{1}{n}\right)(\nabla u^n)^2 \, dxdydt$$
$$= \int_0^T \int_\Omega f u^n \, dxdydt,$$

$$\int_0^T \int_\Omega \left(a(x,y) + \frac{1}{n}\right)(\nabla u^n)^2 \; dxdydt \leq \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega f u^n \, dxdydt,$$

$$\int_0^T \int_\Omega \left(a(x,y) + \frac{1}{n}\right)(\nabla u^n)^2 \; dxdydt \leq \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T \|f\|^2_{L^2(\Omega)} dt$$
$$+ \frac{1}{2}\int_0^T \|u^n\|^2_{L^2(\Omega)} dt.$$

Hence, by (2.4), we obtain

$$\int_0^T \int_\Omega \left(a(x,y) + \frac{1}{n}\right)(\nabla u^n)^2 \; dxdydt \leq \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T \|f\|^2_{L^2(\Omega)} dt$$
$$+ \frac{1}{2} T M \exp(T).$$

Let

$$M_2 = \left(\frac{1}{2} + \frac{1}{2}T\exp(T)\right)\left(\|u(0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right).$$

Then it follows that

$$\int_0^T \int_\Omega \left(a(x,y) + \frac{1}{n}\right)(\nabla u^n)^2 \, dxdydt \le M_2,$$

and

$$\int_0^T \|\sqrt{a(x,y) + \frac{1}{n}}\nabla u^n\|^2_{L^2(\Omega)} dt \le M_2, \tag{2.6}$$

$$\int_0^T \|\sqrt{a(x,y)}\nabla u^n\|^2_{L^2(\Omega)} dt \le M_2. \tag{2.7}$$

From (2.4) and (2.7) we have

$$\int_0^T \|u^n(t)\|^2_{L^2(\Omega)} dt + \int_0^T \|\sqrt{a(x,y)}\nabla u^n\|^2_{L^2(\Omega)} dt \le TM\exp(T) + M_2.$$

We deduce that

$$\|u^n\|^2_{L^2(0,T;H^1_a(\Omega))} \le TM\exp(T) + M_2,$$

$$\|u^n\|^2_{L^2(0,T;H^1_a(\Omega))} \le \left(\frac{1}{2} + \frac{3}{2}T\exp(T)\right)\left(\|u(0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right), \tag{2.8}$$

and, since $a \in L^\infty(\Omega)$, we obtain

$$\|\nabla u^n\|^2_{L^2(0,T;L^2(\Omega))} \le TM\exp(T) + M_2.$$

Then $(u^n)_n$ is bounded in $L^2(0,T;H^1_a(\Omega))$ and ∇u^n is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, there exists a subsequence $(u^n)_n$ such that

$$u^n \rightharpoonup u \quad \text{weakly in} \quad L^2(0,T;H^1_a(\Omega)),$$
$$\nabla u^n \rightharpoonup \nabla u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

In addition,

$$\int_0^T \left\| \left(a(x,y) + \frac{1}{n}\right)\nabla u^n \right\|^2_{L^2(\Omega)} dt$$
$$\le \|(a(x,y)+1)\|_{L^\infty(\Omega)} \int_0^T \left\| \sqrt{\left(a(x,y) + \frac{1}{n}\right)}\nabla u^n \right\|^2_{L^2(\Omega)} dt,$$

from (2.6)

$$\int_0^T \left\|\Big(a(x,y)+\frac{1}{n}\Big)\nabla u^n\right\|^2_{L^2(\Omega)} dt \le \|(a(x,y)+1)\|_{L^\infty(\Omega)} M_2. \tag{2.9}$$

Then $((a(x,y)+\frac{1}{n})\nabla u^n)_n$ is bounded in $L^2(0,T;L^2(\Omega))$. Since $\nabla u^n \rightharpoonup \nabla u$ weakly in $L^2(0,T;L^2(\Omega))$ and $a(x,y)+\frac{1}{n}\to a(x,y)$ in $L^2(0,T;L^2(\Omega))$. Consequently, there exists a subsequence $((a(x,y)+\frac{1}{n})\nabla u^n)_n$ such that

$$\left(a(x,y)+\frac{1}{n}\right)\nabla u^n \rightharpoonup a(x,y)\nabla u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

We return to the equation (2.2) with $v\in H_0^1(\Omega)$ and $\|v\|_{H_0^1(\Omega)}\le 1$:

$$\int_\Omega \partial_t u^n v\,dxdy + \int_\Omega \left(a(x,y)+\frac{1}{n}\right)\nabla u^n \nabla v\,dxdy = \int_\Omega fv\,dxdy,$$
$$\int_\Omega \partial_t u^n v\,dxdy - \int_\Omega \left|\left(a(x,y)+\frac{1}{n}\right)\nabla u^n \nabla v\right| dxdy \le \int_\Omega fv\,dxdy,$$
$$\int_\Omega \partial_t u^n v\,dxdy \le \int_\Omega \left|\left(a(x,y)+\frac{1}{n}\right)\nabla u^n \nabla v\right| dxdy + \int_\Omega fv\,dxdy.$$

By Hölder's inequality, we obtain

$$\langle \partial_t u^n, v\rangle_{L^2(\Omega)} \le \left\|\Big(a(x,y)+\frac{1}{n}\Big)\nabla u^n\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)},$$

since $\|v\|_{H_0^1(\Omega)}\le 1$. Thus

$$\|\partial_t u^n\|_{H^{-1}(\Omega)} \le \left\|\Big(a(x,y)+\frac{1}{n}\Big)\nabla u^n\right\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}.$$

By the inequality $(c+b)^2\le 2(c^2+b^2)$, we obtain

$$\|\partial_t u^n\|^2_{H^{-1}(\Omega)} \le 2\Big(\left\|\Big(a(x,y)+\frac{1}{n}\Big)\nabla u^n\right\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\Big).$$

We have in (2.9)

$$\int_0^T \left\|\Big(a(x,y)+\frac{1}{n}\Big)\nabla u^n\right\|^2_{L^2(\Omega)} dt \le \|(a(x,y)+1)\|_{L^\infty(\Omega)} M_2,$$

which implies that

$$\int_0^T \|\partial_t u^n\|^2_{H^{-1}(\Omega)} dt < 2\left(\|(a(x,y)+1)\|_{L^\infty(\Omega)} M_2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right).$$

Consequently, there exists a subsequence $(\partial_t u^n)_n$ such that

$$(\partial_t u^n)_n \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)).$$

Returning to equation (2.2) with $v = \partial_t u^n$ we get

$$\int\limits_\Omega \partial_t u^n \partial_t u^n \, dxdy + \int\limits_\Omega \left(a(x,y) + \frac{1}{n}\right) \nabla u^n \nabla \partial_t u^n \, dxdy = \int\limits_\Omega f \partial_t u^n \, dxdy,$$

$$\|\partial_t u^n\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt} \int\limits_\Omega \left(a(x,y) + \frac{1}{n}\right) (\nabla u^n)^2 \, dxdy = \int\limits_\Omega f \partial_t u^n \, dxdy,$$

$$\int\limits_0^T \|\partial_t u^n\|^2_{L^2(\Omega)} dt + \frac{1}{2}\left\|\sqrt{a(x,y) + \frac{1}{n}}(\nabla u^n(T))\right\|^2_{L^2(\Omega)}$$
$$= \frac{1}{2}\left\|\sqrt{a(x,y) + \frac{1}{n}}(\nabla u^n(0))\right\|^2_{L^2(\Omega)} + \int\limits_0^T \int\limits_\Omega f \partial_t u^n \, dxdydt,$$

$$\int\limits_0^T \|\partial_t u^n\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|\sqrt{a(x,y)}\nabla u^n(T)\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2}\|\sqrt{a(x,y)+1}\nabla u^n(0)\|^2_{L^2(\Omega)} + \int\limits_0^T \int\limits_\Omega f \partial_t u^n \, dxdydt,$$

$$\int\limits_0^T \|\partial_t u^n\|^2_{L^2(\Omega)} dt \leq \int\limits_0^T \int\limits_\Omega f \partial_t u^n \, dxdydt + \frac{1}{2}\|\sqrt{a(x,y)+1}\|^2_{L^\infty(\Omega)} \|u_0\|^2_{H^1(\Omega)},$$

$$\int\limits_0^T \|\partial_t u^n\|^2_{L^2(\Omega)} dt \leq \frac{1}{2}\int\limits_0^T \|f\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int\limits_0^T \|\partial_t u^n\|^2_{L^2(\Omega)} dt$$
$$+ \frac{1}{2}\|\sqrt{a(x,y)+1}\|^2_{L^\infty(\Omega)} \|u_0\|^2_{H^1(\Omega)},$$

$$\|\partial_t u^n\|^2_{L^2(0,T;L^2(\Omega))} \leq \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\sqrt{a(x,y)+1}\|^2_{L^\infty(\Omega)} \|u_0\|^2_{H^1(\Omega)}. \tag{2.10}$$

Then $(\partial_t u^n)_n$ is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, there exists a subsequence $(\partial_t u^n)_n$, such that

$$\partial_t u^n \rightharpoonup \partial_t u \quad \text{weakly in } \; L^2(0,T;L^2(\Omega)).$$

We find upon passing to weak limits that

$$\int\limits_0^T \int\limits_\Omega \partial_t u v \, dxdydt + \int\limits_0^T \int\limits_\Omega a(x,y)\nabla u \nabla v \, dxdydt = \int\limits_0^T \int\limits_\Omega f v \, dxdydt, \quad v \in H^1_0(\Omega).$$

Hence, u is the weak solution of (1.1).

Next, we prove the existence of weak solutions of (1.1) for any $u_0 \in H_a^1(\Omega)$ and $f \in L^2(0;T;L^2(\Omega))$. Let $(u_0^m)_m$ and $(f^m)_m$ be Cauchy sequences of smooth functions, respectively, such that as $m \longrightarrow \infty$,

$$u_0^m \longrightarrow u_0 \text{ in } H_a^1(\Omega) \quad \text{and} \quad f^m \longrightarrow f \text{ in } L^2(0,T;L^2(\Omega)).$$

Denote by u^m the solution of (1.1) associated to u_0^m and f^m, and u^n the solution of (1.1) associated to u_0^n and f^n.

We have the following variational problem

$$\begin{cases} \int\limits_\Omega \partial_t(u^n - u^m)v dxdy + \int\limits_\Omega a\nabla(u^n ! u^m)\nabla v dxdy = \int\limits_\Omega (f^n - f^m)v dxdy, & v \in H_0^1(\Omega), \\ (u^n - u^m)(x,y,t) = 0, & (x,y) \in \partial\Omega,\ t \in (0,T), \\ (u^n - u^m)(x,y,0) = (u_0^n - u_0^m), & (x,y) \in \Omega. \end{cases}$$

Similarly to inequality (2.10), we obtain

$$\begin{aligned} &\|\partial_t u^n - \partial_t u^m\|^2_{L^2(0,T;L^2(\Omega))} \\ &\leq \|f^n - f^m\|^2_{L^2(0,T;L^2(\Omega))} + \|u_0^n - u_0^m\|^2_{H^1(\Omega)} \|\sqrt{a(x,y)+1}\|^2_{L^\infty(\Omega)}, \end{aligned}$$

and same to obtain inequality (2.5), we have

$$\|u^n - u^m\|^2_{L^\infty(0,T;L^2(\Omega))} \leq T\exp(T)\left[\|u_0^n - u_0^m\|^2_{L^2(\Omega)} + \|f^n - f^m\|^2_{L^2(0,T;L^2(\Omega))}\right].$$

In addition, similarly to obtain inequality (2.8), we have

$$\begin{aligned} &\|u^n - u^m\|^2_{L^2(0,T;H_a^1(\Omega))} \\ &\leq \left(\frac{1}{2} + \frac{3}{2}T\exp(T)\right)\left[\|u_0^n - u_0^m\|^2_{L^2(\Omega)} + \|f^n - f^m\|^2_{L^2(0,T;L^2(\Omega))}\right]. \end{aligned} \tag{2.11}$$

Therefore, there exist

$$u \in L^2(0,T;H_a^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \partial_t u \in L^2(0,T;L^2(\Omega))$$

such that as $m \longrightarrow \infty$

$$u^m \longrightarrow u \text{ in } L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \partial_t u^m \longrightarrow \partial_t u \text{ in } L^2(0,T;L^2(\Omega)).$$

Now, we prove that the weak solution of problem (1.1) is unique. Let u_1, u_2 two weak solutions of problem (1.1), and $\delta u = u_1 - u_2$. Consequently, δu verifies

$$\begin{cases} \int\limits_\Omega \partial_t \delta u v\, dxdy + \int\limits_\Omega a(x,y)\nabla\delta u \nabla v\, dxdy = 0, & v \in H_0^1(\Omega), \\ \delta u(x,y,t) = 0, & (x,y) \in \partial\Omega,\ t \in (0,T), \\ \delta u(x,y,0) = 0, & (x,y) \in \Omega. \end{cases}$$

From (2.11) we get

$$\|\delta u\|^2_{L^2(0,T;H^1_a(\Omega))} = 0.$$

Therefore,

$$\begin{aligned}\|\delta u\|_{H^1_a(\Omega)} = 0, \text{ a.e. } t \in [0,T],\\ \delta u = 0, \text{ a.e. } t \in [0,T],\end{aligned}$$

which implies that

$$u_1 = u_2, \text{ a.e. } t \in [0,T]. \qquad \square$$

Now we show the existence of minimizers to problem (1.2). To do so, we need the following lemma.

Lemma 2.3. *Let u be the weak solution of* (1.1) *corresponding to a given initial state u_0. Then the input-output operator*

$$\varphi : H^1_a(\Omega) \to L^2\left(0,T;H^1_a(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right)$$

defined as $\varphi(u_0) := u$ is Lipschitz continuous.

Proof. Let $\delta u_0 \in L^2(\Omega)$ be a small variation such that $u_0 + \delta u_0 \in A_{ad}$. Consider $\delta u = u^\delta - u$, where u is the weak solution of (1.1) with initial state u_0 and u^δ is the weak solution of (1.1) with initial state $u^\delta_0 = u_0 + \delta u_0$. Consequently, δu is the solution of

$$\begin{cases} \int\limits_\Omega \partial_t \delta u v \, dxdy + \int\limits_\Omega a(x,y)\nabla\delta u \nabla v \, dxdy = 0, & v \in H^1_0(\Omega),\\ \delta u(x,y,t) = 0, & (x,y) \in \partial\Omega,\ t \in (0,T),\\ \delta u(x,y,0) = \delta u_0(x,y), & (x,y) \in \Omega. \end{cases}$$

Hence, δu is a weak solution of (1.1) with $f = 0$. We apply the estimate in Theorem 2.2, we obtain

$$\|\delta u\|^2_{L^2(0,T;H^1_a(\Omega))} \le C\|\delta u_0\|^2_{H^1_a(\Omega)},$$

and

$$\|\delta u\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\|\delta u_0\|^2_{H^1_a(\Omega)},$$

the constant C depending only on Ω and T. This implies the Lipschitz continuity of the input-output operator

$$\begin{aligned}\varphi : H^1_a(\Omega) &\longrightarrow L^2\left(0,T;H^1_a(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right),\\ u_0 &\longmapsto u,\end{aligned}$$

and hence the cost function J is continuous. $\square$

As an immediate consequence of Lemma 2.3 we get the following result.

Proposition 2.4. *The functional J is continuous on A_{ad} and there exists a unique minimizer $u_0^\star \in A_{ad}$ of $J(u_0)$, i.e.*

$$J(u_0^\star) = \min_{u_0 \in A_{ad}} J(u_0).$$

The differentiability of the functional J is deduced from the differentiability of the input-output operator

$$\varphi : u_0 \longmapsto u,$$

where u is the weak solution of (1.1) with initial state u_0.

We have the following result.

Proposition 2.5. *Let u the weak solution of* (1.1) *with initial state u_0. The input-output operator*

$$\begin{aligned} \varphi : H_a^1(\Omega) &\longrightarrow L^2\left(0,T; H_a^1(\Omega)\right) \cap L^\infty\left(0,T; L^2(\Omega)\right), \\ u_0 &\longmapsto u \end{aligned}$$

is G-derivable.

Proof. Let $u_0 \in A_{ad}$ and $\delta u_0 \in H_0^1(\Omega)$ a small variation such that $u_0 + \delta u_0 \in A_{ad}$. We define the function

$$\varphi'(u_0) : A_{ad} \ni \delta u_0 \longmapsto \delta u,$$

where δu is the solution of the variational problem

$$\begin{cases} \displaystyle\int_\Omega \partial_t \delta u v \; dxdy + \int_\Omega a(x,y) \nabla \delta u \nabla v \, dxdy = 0, & v \in H_0^1(\Omega), \\ \delta u(x,y,t) = 0, & (x,y) \in \partial\Omega, \; t \in (0,T), \\ \delta u(x,y,0) = \delta u_0, & (x,y) \in \Omega, \end{cases}$$

and we pose

$$\phi(u_0) = \varphi(u_0 + \delta u_0) - \varphi(u_0) - \varphi'(u_0)\delta u_0.$$

We want to show that

$$\phi(u_0) = o(\delta u_0).$$

It is easy to verify that the function ϕ is solution of following variational problem

$$\begin{cases} \displaystyle\int_\Omega \partial_t \phi v \; dxdy + \int_\Omega a(x,y) \nabla \phi \nabla v dxdy = 0, & v \in H_0^1(\Omega), \\ \phi(x,y,t) = 0, & (x,y) \in \partial\Omega, \; t \in (0,T), \\ \phi(x,y,0) = \delta u_0 - (\delta u_0)^2, & (x,y) \in \Omega. \end{cases}$$

By the same way as that used in the proof of continuity, we deduce that

$$\|\phi\|^2_{L^2(0,T;H_a^1(\Omega))} \leq C \|\delta u_0 - (\delta u_0)^2\|^2_{H_a^1(\Omega)},$$

and

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C \|\delta u_0 - (\delta u_0)^2\|^2_{H_a^1(\Omega)}.$$

Hence, the input-output operator $\varphi : u_0 \longmapsto u$ is G-derivable, and we deduce the existence of the gradient of the functional J. □

Now, we are going to compute the gradient of J with the adjoint state method.

3. GRADIENT OF J

We define the Gâteaux derivative of u at u_0 in the direction $h \in L^2(\Omega)$ by

$$\hat{u} = \lim_{s\to 0} \frac{u(u_0 + sh) - u(u_0)}{s},$$

$u(u_0 + sh)$ is the weak solution of (1.1) with initial state $u_0 + sh$, and $u(u_0)$ is the weak solution of (1.1) with initial state u_0.

We compute the Gâteaux (directional) derivative of (1.1) at u_0 in some direction $h \in L^2(\Omega)$, and we get the so-called tangent linear model:

$$\begin{cases} \partial_t \hat{u} + A\hat{u} = 0 & \text{in } \Omega \times (0,T), \\ \hat{u}(x,y,t) = 0, & (x,y) \in \partial\Omega,\ t \in (0,T), \\ \hat{u}(x,y,0) = h, & (x,y) \in \Omega. \end{cases}$$

We introduce the adjoint variable P, and we integrate:

$$\int_\Omega \int_0^T \partial_t \hat{u}\ P dt dx dy + \int_\Omega \int_0^T A\hat{u} P\, dt dx dy = 0,$$

$$\int_\Omega \left([\hat{u}P]_0^T - \int_0^T \hat{u} \partial_t P dt \right) dx dy + \int_0^T \langle A\hat{u}, P\rangle_{L^2(\Omega)}\, dt = 0,$$

$$\int_\Omega [\hat{u}(T)P(T) - \hat{u}(0)P(0)]\, dx dy - \int_0^T \langle \hat{u}, \partial_t P\rangle_{L^2(\Omega)}\, dt + \int_0^T \langle A\hat{u}, P\rangle_{L^2(\Omega)}\, dt = 0. \quad (3.1)$$

Let us take $P(x,y) = 0$ for all (x,y) in $\partial\Omega$, then we may write

$$\langle \hat{u}, AP\rangle_{L^2(\Omega)} = \langle A\hat{u}, P\rangle_{L^2(\Omega)}.$$

And with $P(T) = 0$ we may now rewrite equation (3.1) as

$$\int_\Omega \hat{u}(0)P(0)\, dx dy + \int_0^T \langle \hat{u}, \partial_t P - AP\rangle_{L^2(\Omega)}\, dt = 0.$$

This gives

$$\begin{cases} \displaystyle\int_0^T \langle \hat{u}, \partial_t P - AP\rangle_{L^2(\Omega)}\, dt = \langle -P(0), h\rangle_{L^2(\Omega)}, \\ P(x,y) = 0,\ \forall (x,y) \in \partial\Omega \text{ and } P(T) = 0. \end{cases} \quad (3.2)$$

The discretization in time of (3.2), using the rectangular integration method, gives

$$\begin{cases} \sum_{j=0}^{M+1} \langle \hat{u}(t_j), \partial_t P(t_j) - AP(t_j) \rangle_{L^2(\Omega)} \Delta t = \langle -P(0), h \rangle_{L^2(\Omega)}, \\ P(x,y) = 0, \ \forall (x,y) \in \partial\Omega \text{ and } P(T) = 0, \end{cases} \tag{3.3}$$

with

$$t_j = j\Delta t, \quad j \in \{0, 1, \dots, M+1\},$$

where Δt is the steps in time and $T = (M+1)\Delta t$.

The Gâteaux derivative of the cost function J at u_0 in the direction $h \in L^2(\Omega)$ is given by

$$\hat{J}(h) = \lim_{s \to 0} \frac{J(u_0 + sh) - J(u_0)}{s}.$$

After some calculations, we arrive at

$$\hat{J}(h) = \left\langle u(T) - u^{obs}, \hat{u}(T) \right\rangle_{L^2(\Omega)} + \left\langle \varepsilon(u_0 - u^b), h \right\rangle_{L^2(\Omega)}. \tag{3.4}$$

The adjoint model is

$$\begin{cases} \begin{cases} \partial_t P(T) - AP(T) = \frac{1}{\Delta t}(u(T) - u^{obs}), \\ \partial_t P(t_j) - AP(t_j) = 0, \quad \forall t_j \neq T, \end{cases} \\ P(x,y) = 0, \ \forall (x,y) \in \partial\Omega, \forall t \in (0,T), \\ P(T) = 0. \end{cases} \tag{3.5}$$

From equations (3.2), (3.4), and (3.5), the gradient of J is

$$\frac{\partial J}{\partial u_0} = -P(0) + \varepsilon(u_0 - u^b).$$

Problem (3.5) is retrograde. We make the change of variable $t_j \longleftrightarrow T - t_j$.

The main steps for descent method at each iteration are the following:

- Calculate u^k solution of (1.1) with initial condition u_0.
- Calculate P^k solution of the adjoint problem.
- Calculate the descent direction $d_k = -\nabla J(u_0)$.
- Find $t_k = \text{argmin}_{t>0} J(u_0 + td_k)$.
- Update the variable $u_0 = u_0 + t_k d_k$.

The algorithm ends when $\mid J(u_0) \mid < \mu$, where μ is a given small precision.

t_k is chosen by the inaccurate linear search by the Armijo–Goldstein Rule as follows:

let $\alpha_i, \beta \in [0,1)$ and $\alpha > 0$
if $J(u_0^k + \alpha_i d_k) \leq J(u_0^k) + \beta \alpha_i d_k^T d_k$
  $t_k = \alpha_i$ and stop
if not
  $\alpha_i = \alpha \alpha_i$.

4. LIPSCHITZ CONTINUITY OF THE GRADIENT

The most important issue in numerical solutions of inverse problems is the Lipschitz continuity of the gradient, which ensures the convergence of the method of descent, for that we have the follows result.

Proposition 4.1. *Let u_0 and δu_0 be such that $u_0 + \delta u_0 \in A_{ad}$. Then ∇J is Lipschitz continuous*

$$\|\nabla J(u_0 + \delta u_0) - \nabla J(u_0)\|_{L^2(\Omega)} \leq L_1 \|\delta u_0\|_{H^1_a(\Omega)},$$

with the Lipschitz constant $L_1 > 0$.

Proof. In Section 3, we have $\nabla J(u_0) = -P_1(T) + \varepsilon(u_0 - u^b)$ with P_1 is the solution of the adjoint model (with change of variable $t_j \longleftrightarrow T - t_j$)

$$\begin{cases} \begin{cases} \partial_t P_1(0) + AP_1(0) = \frac{1}{\Delta t}(u^{obs} - u_1(T)), \\ \partial_t P_1(t_j) + AP_1(t_j) = 0, \quad \forall t_j \neq 0, \end{cases} \\ P_1(x, y, t) = 0, \ \forall (x, y) \in \partial\Omega, \forall t \in (0, T), \\ P_1(x, y, 0) = 0, \end{cases}$$

where u_1 is the weak solution of (1.1) with initial state u_0, and

$$\nabla J(u_0 + \delta u_0) = -P_2(T) + \varepsilon(u_0 + \delta u_0 - u^b),$$

with P_2 is the solution of the adjoint model (with change of variable $t_j \longleftrightarrow T - t_j$)

$$\begin{cases} \begin{cases} \partial_t P_2(0) + AP_2(0) = \frac{1}{\Delta t}(u^{obs} - u_2(T)), \\ \partial_t P_2(t_j) + AP_2(t_j) = 0, \quad \forall t_j \neq 0, \end{cases} \\ P_2(x, y, t) = 0, \ \forall (x, y) \in \partial\Omega, \forall t \in (0, T), \\ P_2(x, y, 0) = 0, \end{cases}$$

where u_2 is the weak solution of (1.1) with initial state $u_0 + \delta u_0$.

Let $\delta P = P_1 - P_2$. We easily verify that δP is the solution of the variational problem

$$\begin{cases} \int\limits_\Omega \partial_t \delta P v \, dxdy + \int\limits_\Omega \nabla \delta P \nabla v dxdy = & \dfrac{1}{\Delta t} \int\limits_\Omega (u_2(T) - u_1(T)) \mathbb{1}_0 dx, \ v \in H^1_0(\Omega), \\ \delta P(x, y, t) = 0, & (x, y) \in \partial\Omega, \ t \in (0, T), \\ \delta P(x, y, 0) = 0, & (x, y) \in \Omega, \end{cases}$$

where $\mathbb{1}_0(t = 0) = 1$ and $\mathbb{1}_0(t \neq 0) = 0$, hence, δP is weak solution of (1.1) with $f = (u_2(T) - u_1(T))\mathbb{1}_0$. We apply the estimate in Theorem 2.2 to obtain

$$\|\delta P\|^2_{L^2(0,T;H^1_a(\Omega))} \leq C \left(\|(u_2(T) - u_1(T))\mathbb{1}_0\|^2_{L^2(0,T;L^2(\Omega))} \right),$$

and

$$\|\delta P\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C \left(\|(u_2(T) - u_1(T))\mathbb{1}_0\|^2_{L^2(0,T;L^2(\Omega))} \right),$$

the constant C depending only on Ω and T.

We showed above the Lipschitz continuity of the input-output operator

$$\begin{aligned}\varphi : H_a^1(\Omega) &\longrightarrow L^2\left(0,T;H_a^1(\Omega)\right)\cap L^\infty\left(0,T;L^2(\Omega)\right)\\ u_0 &\longmapsto u,\end{aligned}$$

from where

$$\left(\|(u_2(T)-u_1(T))\mathbb{1}_0\|^2_{L^2(0,T;L^2(\Omega))}\right)\le C\|\delta u_0\|^2_{H_a^1(\Omega)}.$$

Therefore

$$\|\delta P\|^2_{L^\infty(0,T;L^2(\Omega))}\le C\|\delta u_0\|^2_{H_a^1(\Omega)}. \tag{4.1}$$

We have

$$\begin{aligned}\|\nabla J(u_0+\delta u_0)-\nabla J(u_0)\|_{L^2(\Omega)} &= \|\delta P(T)+\varepsilon\delta u_0\|_{L^2(\Omega)}\\ &\le \|\delta P(T)\|_{L^2(\Omega)}+\|\varepsilon\delta u_0\|_{L^2(\Omega)}.\end{aligned}$$

From inequality (4.1), we obtain

$$\|\nabla J(u_0+\delta u_0)-\nabla J(u_0)\|_{L^2(\Omega)}\le(\sqrt{C}+\varepsilon)\|\delta u_0\|_{H_a^1(\Omega)}.$$

This completes the proof of the theorem. □

5. DISCRETIZATION OF PROBLEM

Step 1. Full discretization

Discrete approximations of these problems need to be made for numerical implementation. To resolve problem (1.1) and the adjoint problem, we use the method θ-schema in time. This method is unconditionally stable for $1>\theta\ge\frac{1}{2}$.

Let h_x, h_y be the steps in space and Δt the steps in time, and let

$$\begin{aligned}x_i &= ih_x, \quad i\in\{0,1,\dots,N+1\},\\ y_i &= sh_y, \quad s\in\{0,1,\dots,R+1\},\\ b_{i,s} &= a(x_i,y_s),\\ t_j &= j\Delta t, \quad j\in\{0,1,\dots,M+1\},\\ f_{i,s}^j &= f(x_i,y_s,t_j).\end{aligned}$$

We put

$$u_{i,s}^j = u(x_i,y_s,t_j).$$

Let

$$\begin{aligned}g_1(x_i,y_s) &= -\frac{b_{i,s}\theta\Delta t}{h_x^2},\\ g_2(x_i,y_s) &= -\frac{b_{i,s}\theta\Delta t}{h_y^2},\end{aligned}$$

$$g_3(x_i,y_s) = 1 + 2\theta\Delta t b_{i,s}\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) + \theta\Delta t\left(\frac{da_x(x_i,y_s)}{h_x}+\frac{da_y(x_i,y_s)}{h_y}\right),$$
$$g_4(x_i,y_s) = -\left(\frac{\theta\Delta t}{h_x^2}b_{i,s} + da_x(x_i,y_s)\frac{\theta\Delta t}{h_x}\right),$$
$$g_5(x_i,y_s) = -\left(\frac{\theta\Delta t}{h_y^2}b_{i,s} + da_y(x_i,y_s)\frac{\theta\Delta t}{h_y}\right)$$

and

$$k_1(x_i,y_s) = \frac{b_{i,s}\,(1-\theta)\,\Delta t}{h_x^2},$$
$$k_2(x_i,y_s) = \frac{b_{i,s}\,(1-\theta)\,\Delta t}{h_y^2},$$
$$k_3(x_i,y_s) = 1 - 2(1-\theta)\Delta t b_{i,s}\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) - (1-\theta)\Delta t\left(\frac{da_x(x_i,y_s)}{h_x}+\frac{da_y(x_i,y_s)}{h_y}\right),$$
$$k_4(x_i,y_s) = \frac{(1-\theta)\Delta t}{h_x^2}b_{i,s} + da_x(x_i,y_s)\frac{(1-\theta)\Delta t}{h_x},$$
$$k_5(x_i,y_s) = \frac{(1-\theta)\Delta t}{h_y^2}b_{i,s} + da_y(x_i,y_s)\frac{(1-\theta)\Delta t}{h_y}.$$

Then the equation $\partial_t u + Au = f$ is approximated by

$$\begin{aligned}
&g_1(x_i,y_s)u_{i-1,s}^{j+1} + g_2(x_i,y_s)u_{i,s-1}^{j+1} + g_3(x_i,y_s)u_{i,s}^{j+1} + g_4(x_i,y_s)u_{i+1,s}^{j+1} + g_5(x_i,y_s)u_{i,s+1}^{j+1}\\
&= k_1(x_i,y_s)u_{i-1,s}^{j} + k_2(x_i,y_s)u_{i,s-1}^{j} + k_3(x_i,y_s)u_{i,s}^{j} + k_4(x_i,y_s)u_{i+1,s}^{j}\\
&\quad + k_5(x_i,y_s)u_{i,s+1}^{j} + \Delta t[(1-\theta)\,f_{i,s}^j + \theta f_{i,s}^j].
\end{aligned}$$

We have

$$u_{0,s}^{j+1} = u_{0,s}^{j} = u_{N+1,s}^{j+1} = u_{N+1,s}^{j} = 0, \quad s \in \{0,1,\ldots,R+1\},$$

and

$$u_{i,0}^{j+1} = u_{i,0}^{j} = u_{i,R+1}^{j+1} = u_{i,R+1}^{j} = 0, \quad i \in \{0,1,\ldots,N+1\}.$$

Let $V^j = (V_k^j)_{k\in\{1,2,\ldots R\times N\}}$ with $V_k^j = u_{i,s}^j$. This gives the equation system

$$\begin{cases} AV^{j+1} = BV^j + S^j & \text{with } j \in \{1,2,\ldots,M\},\\ V^0 = (u_0(ih_x, sh_y))_k & \text{with } k \in \{1,2,\ldots,R\times N\}. \end{cases} \tag{5.1}$$

Step 2. Discretization of the functional J:

$$J(u) = \frac{\varepsilon}{2}\int_0^1\int_0^1 (u(x,y)-u^b(x,y))^2dxdy + \frac{1}{2T}\int_0^1\int_0^1 (u(x,y,T)-u^{obs}(x,y))^2dxdy.$$

We recall that the method of Thomas Simpson to calculate an integral is

$$\int_a^b f(x)\ dx \simeq \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{\frac{N+1}{2}-1} f(x_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}} f(x_{2i+1}) + f(x_{N+1})\right],$$

with $x_0 = a$, $x_{N+1} = b$, $x_i = a + ih$, $i \in \{1, \dots, N+1\}$.

Let us define the following functions:

$$\phi(x,y) = (u(x,y) - u^b(x,y))^2, \quad (x,y) \in \Omega,$$
$$S(y) = \int_0^1 \phi(x,y)\ dx,$$
$$R = \int_0^1 \int_0^1 \phi(x,y)\ dxdy = \int_0^1 S(y)\ dy.$$

This gives

$$S(y) \simeq \frac{h}{2}\left[\phi(0,y) + 2\sum_{i=1}^{\frac{N+1}{2}-1} \phi(x_{2i},y) + 4\sum_{i=1}^{\frac{N+1}{2}} \phi(x_{2i+1},y) + \phi(1,y)\right],$$

and

$$R \simeq \frac{h}{2}\left[S(0) + 2\sum_{i=1}^{\frac{N+1}{2}-1} S(y_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}} S(y_{2i+1}) + S(1)\right].$$

Next, let us define the following functions:

$$\varphi(x,y) = (u(x,y,T) - u^{obs}(x,y))^2, \quad (x,y) \in \Omega,$$
$$W_1(y) = \int_0^1 \varphi(x,y)\, dx,$$
$$W_2 = \int_0^1 W_1(y)\, dy.$$

This gives

$$W_1(y) \simeq \frac{h}{2}\left[\varphi(0,y) + 2\sum_{i=1}^{\frac{N+1}{2}-1} \varphi(x_{2i},y) + 4\sum_{i=1}^{\frac{N+1}{2}} \varphi(x_{2i+1},y) + \varphi(1,y)\right],$$
$$W_2 \simeq \frac{h}{2}\left[W_1(0) + 2\sum_{i=1}^{\frac{N+1}{2}-1} W_1(y_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}} W_1(y_{2i+1}) + W_1(1)\right],$$

therefore

$$J(u) \simeq \tfrac{\varepsilon}{2} R + \frac{1}{2T} W_2.$$

6. NUMERICAL EXPERIMENTS

Now, we assume that we have an a priori knowledge of the state u_0^{exact}, under the form of a vector u^b of the same dimension as u_0^{exact}. This is the background state. The background error is then defined as

$$err = \|u_0^{\text{exact}} - u^b\|_2.$$

u_0^{exact} is called the true state, and is the state to estimate.

In the following tests we study the noise resistance of the proposed method. We take

$$u_0^{\text{exact}} = \frac{x(x-1)y(y-1)}{T} \quad \text{and} \quad a(x,y) = \sqrt{(x-0.5)^2 + (y-0.5)^2}.$$

The true state and the result of a test without regularization are presented in Figures 1 and 2, respectively.

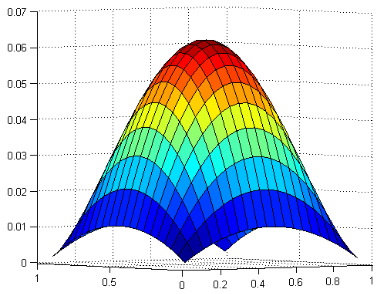

Fig. 1. Graph of u_0^{exact}

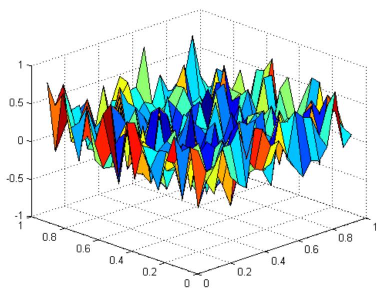

Fig. 2. Initial temperature in a test without regularisation. We have $\|u_0 - u_0^{\text{exact}}\|_2 = 2.9776$

The tests (Figs 3 and 4) show that the proposed algorithm is uniformly stable to noise. The tolerable percentage of err to rebuild the initial state is $err = 0.1\|u_0^{\text{exact}}\|_2$. The convergence of the descent method in all tests is proved by Figures 5 and 6.

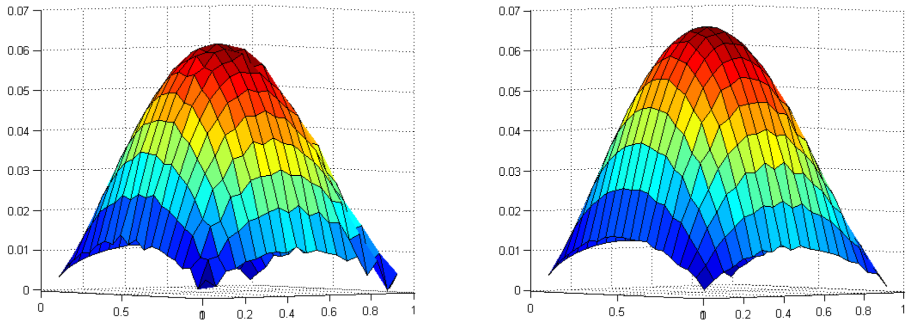

Fig. 3. Initial temperature with $err = 0$, this figure shows that we can rebuild the initial state. We have $\|u_0 - u_0^{\text{exact}}\|_2 = 0.0459$ (left). And with $err = 0.1\|u_0^{\text{exact}}\|_2$, the reconstructed initial condition begins to move away from the true state. We have $\|u_0 - u_0^{\text{exact}}\|_2 = 0.0693$ (right)

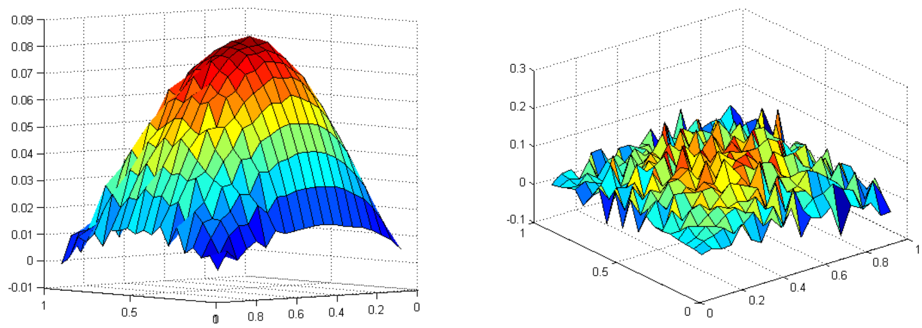

Fig. 4. Initial temperature with $err = \frac{1}{2}\|u_0^{\text{exact}}\|_2$, we have $\|u_0 - u_0^{\text{exact}}\|_2 = 0.2359$ (left). And with $err = \|u_0^{\text{exact}}\|_2$, we have $\|u_0 - u_0^{\text{exact}}\|_2 = 0.5336$ (right). This figure shows that we cannot rebuild the initial state

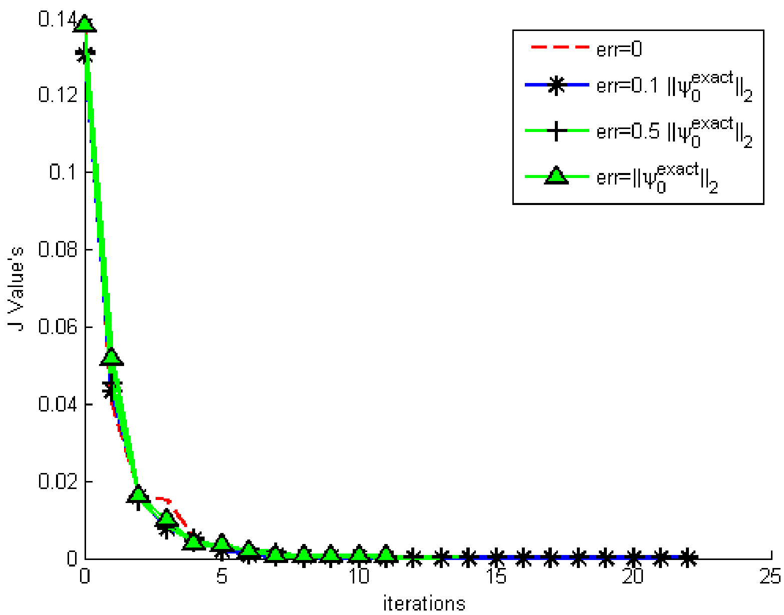

Fig. 5. Graph of J

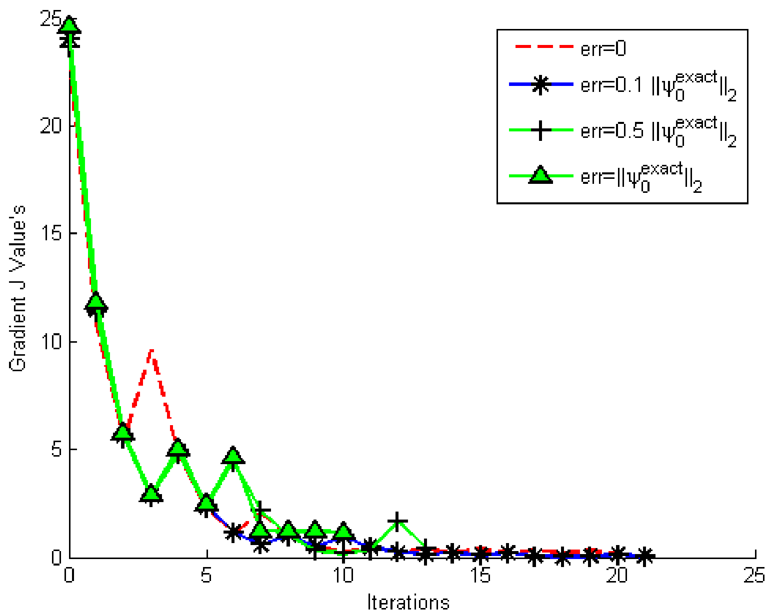

Fig. 6. Graph of $\|\nabla J\|_2$

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