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Immunization and convex interest rate shifts*

by

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Abstract: An important issue in immunization theory is the form of the interest rate process under which immunization is feasible. This paper generalizes Fisher and Weil immunization result to convex interest rate shifts, and examines the practical significance of this generalization. We examine the features of a linear factor model that are consistent with a convex shift. In particular, we show that a specific two factor linear model is sufficient and necessary for a convex shift. This two factor specification allows parallel and damped yield curve shifts, which in combination can twist the yield curve

Keywords: immunization; yield curve; linear factor model; convexity.

1. Introduction

Immunization is widely used by insurance companies and pension funds to build dedicated portfolios.[1] Redington (1952) first examined the problem of immunizing a balance sheet of assets and liabilities against interest rate movements. He showed that the surplus value is immunized against a small parallel change in rates if the asset and liability durations are equal and the asset cash flows are more spread out than liability flows. Fisher and Weil (1971) showed that for a single liability, duration matching assures that the surplus value is immunized against a parallel shift of any magnitude.[2]

It is widely believed that immunization is only valid when the yield curve shifts in a parallel fashion (Lacey and Nawalkha, 1993, and Nawalkha and Latif, 2004). This paper extends the Fisher and Weil result to the class of convex interest rate shifts, which includes a parallel shift, and examines the practical significance of this generalization.

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[1] A dedicated portfolio is one where a pool of assets are invested to fund a future liability stream.

[2] Immunization strategies have also been developed for non-level interest rate shifts in a single-factor model (see Barber (1999), Rzadkowski and Zaremba (2000, 2010).

The classical result by Fisher and Weil is based upon a simple parallel-shift model that has many desirable properties. In particular, the model is parameterized in terms of a random factor, has positive and negative symmetric shifts, and can be empirically estimated and tested. Imposing the above properties on an interest rate shift model, naturally leads to a linear factor model, similar to the form used by Ross (1976) in the development of the Arbitrage Pricing Theory. The linear factor model has been employed by a number of researchers to examine the interest rate changes and yield curve movements (see, for example, Litterman and Scheinkman, 1991; Barber and Copper, 1996; Falkenstein and Hanweck, 1997; Geyer and Pichler, 1999; Golub and Tillman, 2000; Dungey, Martin and Pagan, 2000; Lekkos, 2001; Brummelhuis et al., 2002; Soto, 2003). Jarrow (1996) showed that principal components analysis could be used to estimate a linear factor model specification of forward rate movements in the Heath-Jarrow-Morton (1992) model.

We examine the features of a linear factor model that is consistent with a convex shift. In particular, we show that a specific two factor model is necessary and sufficient for a linear interest rate model to be convex. This two factor specification allows parallel and damped yield curve shifts, which in combination can twist the yield curve.

2. Extension of Fisher-Weil Theorem

An amount M is due in year q. The amount is funded by a stream of positive cash flows represented by a monotonic increasing cumulative cash flow function $C(t)$ for $0 < t \leq T$.[3] The initial spot yield curve is given by $y_0(s)$ defined on a set of maturities s contained in the interval (0, T]. The present value of the cash flow stream is given by the following Stieltjes integral:

$$V(0) = \int_0^T \exp\{-y_0(s)s\}dC(s).$$

Under the influence of a vector of K random factors $f \in \mathrm{R}^K$, the yield curve shifts to $y_0(s) + H(s,f)$, where $H(s,\mathbf{0}) = 0$. Hence, the present value of one dollar promised at date s changes from $\exp\{-y_0(s)s\}$ to $\exp\{-y_0(s)s\}$ $\exp\{-H(s,f)s\}$. After an interest rate shock, the value of the cash flow stream is given by

$$\begin{aligned} V(\mathbf{f}) &= \int_0^T \exp\{-y_0(s)s\}\exp\{-H(s,\mathbf{f})s\}dC(s) \\ &= V(\mathbf{0})\int_0^T \exp\{-H(s,\mathbf{f})s\}dW(s) \end{aligned}$$

where

$$dW(s) = dC(s)\exp\{-y_0(s)s\}/V(\mathbf{0}).$$

[3]The cumulative cash flow at t equals the sum of the cash flows from time 0 up to an including time t.

Observe that $W(s)$ has the properties of a probability distribution defined over the interval $[0,T]$. In particular, W is nondecreasing, $W(0)=0$, and $W(T)=1$. The value of the liability after the shock is given by

$$L(\mathbf{f})=M\exp\{-y_0(s)s\}\exp\{-H(q,\mathbf{f})q\}$$
$$=L(\mathbf{0})\exp\{-H(q,\mathbf{f})q\}$$

where $L(\mathbf{0})$ is the initial value of the liability promised at date q.

The usual immunization condition that duration of assets equals the liability due date is given by

$$\int_0^T sdW(s)=q. \tag{1}$$

Definition 1 *An interest rate shock is convex if* $\exp\{-H(s,f)s\}$ *is a convex function of* $s>0$ *for all* $f\in\mathrm{R}^K$.

Fisher and Weil (1971) showed that a single liability can be immunized for the special case $H(s,f)=f$, where f is any real number. The proposition below generalizes the Fisher and Weil result to convex interest rate shocks. It also relaxes the assumption that the initial asset and liability values are equal, and allows for a mixture of continuous and discrete cash flow streams.

Proposition 1 *For convex interest rate shocks, a portfolio with positive cash flow stream is immunized if condition (1) holds and the initial asset value* $V(0)$ *equals or exceeds the initial liability value* $L(0)$.

Proof. By Definition 1, $\exp\{-H(s,f)s\}$ is a convex function of s. Since W has properties of distribution function, Jensen's inequality along with the conditions $\int_0^T sdW(s)=q$ and $V(0)\geq L(0)$ implies:

$$V(\mathbf{f})=V(\mathbf{0})\int_0^T \exp\{-H(s,\mathbf{f})s\}dW(s)\geq V(\mathbf{0})\exp\{-H(q,\mathbf{f})q\}$$
$$\geq L(0)\exp\{-H(q,\mathbf{f})q\}=L(\mathbf{f})$$

for all $f\in\mathrm{R}^K$. ■

3. Linear factor model and convex shift

Ideally, an interest rate shift model is specified in terms of a set of random factors and generates positive and negative symmetric shifts associated with each random factor. Further, it should be possible to empirically estimate and test the parameters of the model. For example, the Fisher-Weil immunization result is based upon a simple model in which a change in the yield curve equals a random variable f:

$$y(s)-y_0(s)=f \quad \text{for all } f\in\mathrm{R}.$$

This simple model generates positive and negative symmetrical shifts. On the other hand, the following convex-shift model does not generate negative shifts:

$$y(s) - y_0(s) = h(s)f^2 \text{ for all } f \in \mathrm{R}$$

where $-h(s)s$ is a convex function. Now, suppose h is also a function of f such that when f is positive $-h(s,f)s$ is convex in s, and when f is negative $h(s,f)s$ is convex in s. Although the model

$$y(s) - y_0(s) = h(s)f \text{ for all } f \in \mathrm{R}$$

is a convex-shift model and generates positive and negative shifts, the shape of the shift is different for positive and negative shifts. For example, let

$$h(s,f) = \begin{cases} 1 & \text{if} \quad f \geq 0 \\ \log s & \text{if} \quad f < 0. \end{cases}$$

Then a positive shift is parallel and a negative shift is increasing in maturity. It seems reasonable to require that the shape of the shift, indicated h, is independent of the random factor f. Therefore, we explicitly assume that the shape of the shift is independent of the random factor[4]

$$h(s,f) = h(s)f. \tag{2}$$

A linear factor model has the desired properties and can be estimated using principal component analysis (see Jarrow, 1996; Barber and Copper, 1996). Our aim is to determine the necessary and sufficient conditions for a convex shift to obtain under the linear factor model.

Definition 2 *Linear Factor Model for yield curve shift:*

$$H(s,\mathbf{f}) \equiv y(s,\mathbf{f}) - y_0(s) = \sum_{k=1}^{K} h_k(s) f_k \tag{3}$$

where $f_1, ..., f_K$ are random variables that can assume any real number and $h_1, ..., h_K$ are twice differentiable for $s > 0$.

We are not requiring that the yield curve be defined at every maturity. If the yield curve is defined on a discrete set of dates, say at half-year intervals, then $y - y_0$ is a twice differentiable curve that fits the change in the observed yields at half-year intervals. For example, based upon principal component analysis of monthly yield curve changes from 1992 to 2001,[5] we find that a linear two-factor model explains roughly 93% of the variation of monthly yield curve shifts. Fig. 1

[4]There is no point in transforming the random factor, because the model can always be specified in terms of the transformed factor.

[5]Yield curves are estimated from CRSP Treasury bond data for each month from 1992 to 2001 using McCulloch's (1975) methodology .

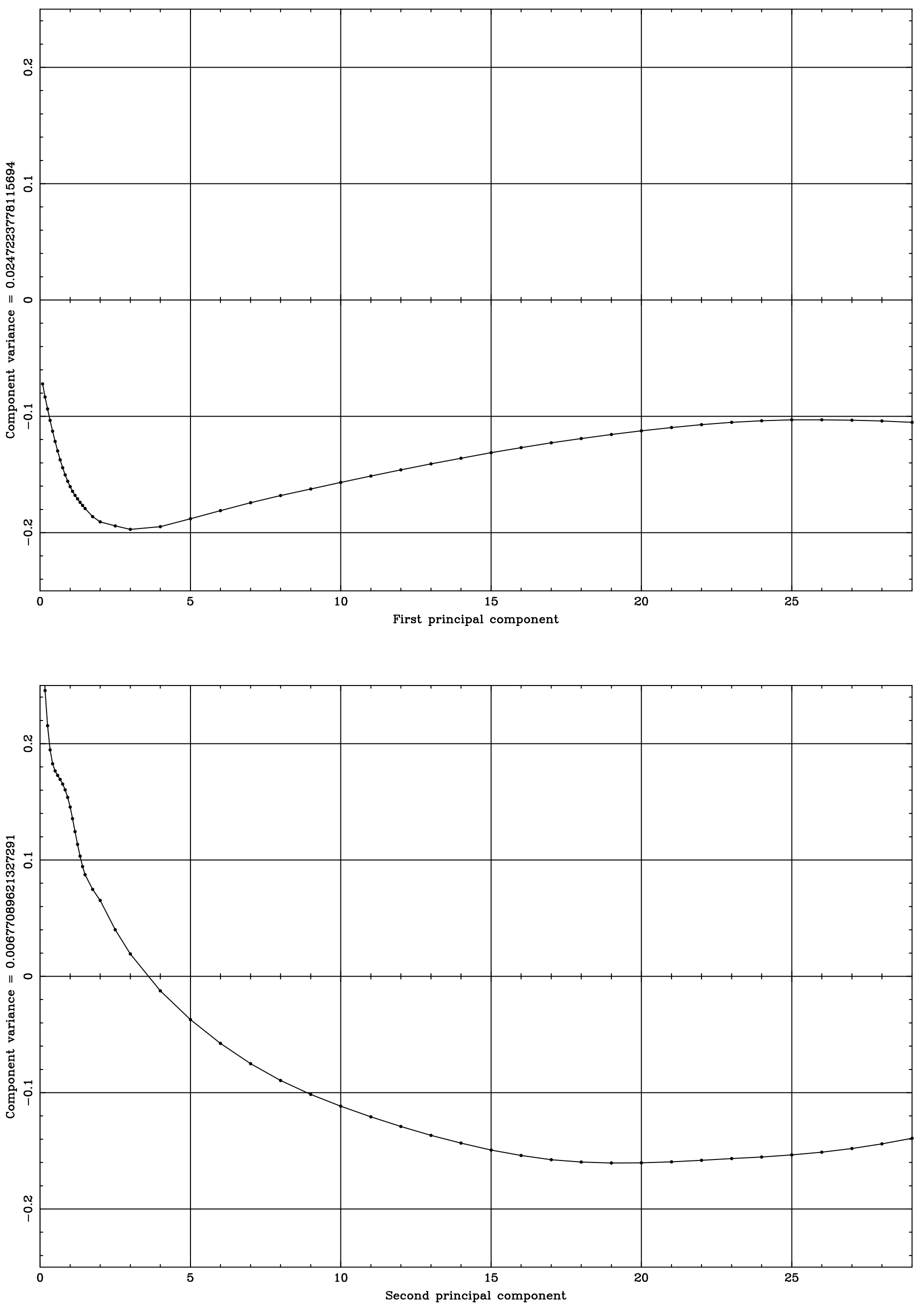

Figure 1. Principal components estimated from monthly yield curve changes from 1992 to 2001

shows the graphs of h_1 and h_2 versus the time to maturity. The first component is roughly a parallel shift, and the second component is a twist. Given that the actual treasury bond data is discrete, the functions h_1, h_2 are twice differentiable curves fitted to the estimated values at discrete points. The assumption that the fitted curves are twice differentiable simplifies the convexity analysis, but does not require that the actual yield curves be continuous functions of maturity.

The linear factor model is parsimonious if it cannot be reduced to linear model with fewer factors. A parallel shift is a single factor model wherein h equals a constant. Obviously, a multifactor parallel shift specification is not parsimonious, because it can be expressed in terms of a single factor not dependent upon maturity:

$$y(s, \mathbf{f}) - y_0(s) = af_1 + bf_2 = u.$$

Proposition 2 *Under a parsimonious linear factor model, the interest rate shock is convex if and only if* $y(s,f) - y_0(s)$ *can be expressed as* $\frac{1}{s}f_1 + f_2$*, where* f_1, f_2 *are random factors that can assume any real number.*

Proof. Sufficiency. If $y(s,f) - y_0(s) = \frac{1}{s}f_1 + f_2$, then

$$\frac{\partial^2}{\partial s^2}\exp\{-f_1 - f_2 s\} = f_2^2 e^{-f_1 - f_2 s} \geq 0$$

for all $f_1, f_2 \in$ R. Therefore, interest rate shock is convex.

Necessity. First, we establish the form of a two-factor model, and then, second, show that additional factors are redundant. Suppose that y depends upon two random factors $u_1, u_2 \in$ R: $y(s, \mathbf{u}) = y_0(s) + h_1(s)u_1 + h_2(s)u_2$, where h_1, h_2 are twice differentiable. For simplicity, let $g_1 = sh_2$ and $g_2 = sh_2$.Then the interest rate model is convex if:

$$\frac{\partial^2}{\partial s^2}\exp\{-g_1 u_1 - g_2 u_2\} \geq 0 \text{ for all } u_1, u_2 \in \text{R}$$

or

$$-\frac{\partial^2 g_1}{\partial s^2}u_1 - \frac{\partial^2 g_2}{\partial s^2}u_2 + \left(\frac{\partial g_1}{\partial s}\right)^2 u_1^2 + 2\frac{\partial g_1}{\partial s}u_1\frac{\partial g_2}{\partial s}u_2 + \left(\frac{\partial g_2}{\partial s}\right)^2 u_2^2 \geq 0 \quad (4)$$

for all $u_1, u_2 \in$ R. Notice if $\frac{\partial^2 g_1}{\partial s^2} = \frac{\partial^2 g_2}{\partial s^2} = 0$, the condition is satisfied. On the other hand, if $u_2 = 0$, then (4) becomes

$$-\frac{\partial^2 g_1}{\partial s^2}u_1 + \left(\frac{\partial g_1}{\partial s}\right)^2 u_1^2 \geq 0 \text{ for all } u_1 \in \text{R} \quad (5)$$

which is only satisfied if $\frac{\partial^2 g_1}{\partial s^2} = 0$. Otherwise, the right-hand side of (5) is a quadratic in u_1 whose minimum value (coefficient on u_1^2 is positive) is negative. Therefore, the condition$\frac{\partial^2 g_1}{\partial s^2} = 0$ is necessary for the shock to be convex. This condition implies that $g_1(s) = a + bs$, or $h_1(s) = \frac{a}{s} + b$, where a, b are positive constants. By the same argument, we conclude that $h_2(s) = \frac{c}{s} + d$, where c, d are constants. Therefore, the only interest rate models that satisfy (4) have the form:

$$\begin{aligned} y(s) - y_0(s) &= \left(\frac{a}{s} + b\right) u_1 + (\frac{c}{s} + d)u_2) \\ &= \frac{1}{s}(au_1 + u_2c) + (bu_1 + du_2) \\ &= \frac{1}{s}f_1 + f_2 \end{aligned}$$

for all $f_1 = au_1 + u_2c \in \mathrm{R}$ and $f_2 = bu_1 + du_2 \in \mathrm{R}$. Obviously, if we add additional factors, the interest-rate model will always reduce to the two-factor form shown above. ■

Proposition 3 *If $y(s,f) - y_0(s) = \frac{1}{s}f_1 + f_2$, a single liability due in T years can be immunized if the duration of asset cash flows equals T and initial asset value equals or exceeds initial liability value.*

Multiple liabilities can be handled as an extension of the single liability case by separately immunizing each liability cash flow (see Bierwag, Kaufman, and Toevs, 1983).

Immunization is feasible under a linear factor model that generates parallel and damped parallel shifts. A combination of a parallel and damped shift can give rise to a twist, wherein short-term rates increase (decrease) and long-term rates decrease (increase). A damped shift fits the empirical fact that long-term rates are less volatile than short-term rates. In fact, the volatility function is given by decreasing function of maturity s:

$$\frac{\mathrm{var}(f)}{s}.$$

In the limit, as maturity becomes large, the volatility approaches zero. This feature is consistent with Dybvig, Ingersoll, and Ross's (1996) proof that absence of arbitrage implies the rate on a long (infinite in the limit) term zero coupon bond can never rise. Another desirable feature of the two-factor model is that a combination of a parallel and damped shift can give rise to a twist, wherein short-term rates increase (decrease) and long-term rates decrease (increase).

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