Computer Assisted Proofs in Dissipative Partial Differential Equations

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The goal of this paper is to present a brief survey of our research that has focused on studying the dynamics of dissipative partial differential equations by performing computer assisted proofs. We provide a description of the main ideas behind the computer assisted proofs that we have performed, along with related topics. The emphasis is given to the case of the viscous Burgers equation with constant forcing, for which the existence of globally attracting fixed points has been established. To achieve this goal, we used a combination of analytical results with computer assistance.

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Introduction

The goal of this paper is to present a brief survey of our PhD dissertation that will include a report from a study of the dynamics of dissipative partial differential equations (dPDEs) by performing computer assisted proofs. We call a dPDE a partial differential equation (PDE) of the following type

$$\frac{du}{dt} = Lu + N(u, Du, \dots, D^r u). \tag{1}$$

$u\colon \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$, $x \in \mathbb{T}^d$ d-dimensional torus, L is a linear operator, N is a polynomial, D^s- a collection of partial derivatives of the order s. Moreover, L is diagonal in the Fourier basis $\{e^{ikx}\}_{k\in\mathbb{Z}^d}$

$$Le^{ikx} = \lambda_k e^{ikx}, \text{ the eigenvalues satisfy}$$

$$\lambda_k = -\nu(|k|)|k|^p, \quad 0 < \nu_0 \le \nu(|k|) \le \nu_1, \quad p > r.$$

The class of dPDE includes many of the most relevant PDEs, which have been intensively studied by both physicists and mathematicians, for instance, the viscous Burgers equation, the Ginzburg-Landau equation, the Kuramoto-Shivasinsky equation and the Navier-Stokes equations.

Computer techniques for dPDEs

While the topic of computer assisted proofs in ordinary differential equations (ODEs) seems to be thoroughly analyzed and established, the topic of computer assisted proofs in partial differential equations is at the infancy stage. Now, let us give a brief summary of results that exist in literature. There exists a method of proving the existence and stability of steady-states for nonlinear PDEs [1, 2], which has been successfully applied to the two dimensional Navier-Stokes equations, among other equations. This method concerns only the stationary problem. To our knowledge, there exist only two methods that concern the nonstationary (evolution in time) problem for PDEs: the method proposed by G. Arioli and H. Koch in [3] and the method of self-consistent bounds, proposed by P. Zgliczyński, see e.g. [4] and [5]. Both methods have been applied to the Kuramoto-Shivasinsky equation on the real line with periodic boundary conditions. Our approach is based on the method of self-consistent bounds.

The viscous Burgers equation with constant forcing

As the viscous Burgers equation we consider the following PDE

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} - \nu \triangle u = 0 \quad \text{in } \Omega, \quad t > 0,$$

where ν is a positive viscosity constant. The equation was proposed by Burgers (1948) as a mathematical model of turbulence. Later on it was successfully shown that the Burgers equation models certain gas dynamics (Lighthill (1956)) and acoustic (Blackstock (1966)) phenomena. For our purposes we define this equation on the real line $\Omega := \mathbb{R}$, add a constant forcing f to the right-hand side and consider the initial value problem with periodic boundary conditions

$$u \colon \mathbb{R} \times [0, T) \to \mathbb{R},$$

$$f \colon \mathbb{R} \to \mathbb{R},$$

$$u_t + u \cdot u_x - \nu u_{xx} = f(x), \tag{2a}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{2b}$$

$$u(x, t) = u(x + 2k\pi, t), \quad x \in \mathbb{R},\ t \in [0, T),\ k \in \mathbb{Z}, \tag{2c}$$

$$f(x) = f(x + 2k\pi), \quad x \in \mathbb{R},\ k \in \mathbb{Z}. \tag{2d}$$

In the Fourier domain, (2) takes the following form

$$\frac{da_k}{dt} = -i\frac{k}{2}\sum_{k_1 \in \mathbb{Z}} a_{k_1} \cdot a_{k-k_1} - \nu k^2 a_k + f_k, \quad k \in \mathbb{Z}, \tag{3a}$$

$$a_k(0) = \frac{1}{2\pi}\int_0^{2\pi} u_0(x) e^{-ikx}\, dx, \quad k \in \mathbb{Z}, \tag{3b}$$

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\, dx, \quad k \in \mathbb{Z}, \tag{3c}$$

where u_0 is a sufficiently regular initial condition and f is a constant forcing function. In the actual algorithm we require f to be defined by a finite number of modes. We associate a_k with the coefficient corresponding to the Fourier basis function e^{ikx}. We proved the existence of steady-states of (2) by proving the existence of fixed points of (3). For a class of sufficiently regular initial conditions both approaches are equivalent.

Lemma 1 Let $u_0 \in C^4$ be an initial condition for (2), and $\{a_k(0)\}_{k\in\mathbb{Z}}$ the Fourier coefficients of u_0. It is equivalent to solve either (2) with u_0 as the initial condition or (3) with $\{a_k(0)\}_{k\in\mathbb{Z}}$ as the initial condition. Meaning that $\{a_k(t)\}_{k\in\mathbb{Z}}$ are the Fourier coefficients of $u(t)$ for all $t \geq 0$.

The expression $u_0 \in C^4$ means that the fourth derivative of u_0 exists and is continuous. For the proof and the motivation of this result we refer the reader to [6].

Let H be a Hilbert space, actually L_2 or one of its subspaces in the context of dPDEs. We assume that there is a sequence of subspaces $H_k \subset H$ such that $\dim H_k = d_1 < \infty$, H_k and $H_{k'}$ are mutually orthogonal for $k \neq k'$ and $H = \overline{\oplus_{k\in I} H_k}$. For $n > 0$ we set $X_n = \oplus_{|k|\leq n, k\in I} H_k$. By $P_n \colon H \to X_n$ we denote a projection onto X_n.

Definition 2 For any given number m > 0 the m-th Galerkin projection of (3a) is

$$\frac{da_k}{dt} = -i\frac{k}{2}\sum_{\substack{|k-k_1|\leq m \\ |k_1|\leq m}} a_{k_1} \cdot a_{k-k_1} - \nu k^2 a_k + f_k, \quad |k| \leq m. \tag{4}$$

Let $l > 0$, by $\varphi^l(t, x)$ we denote the solution of the l-th Galerkin projection of (3a) at a time $t > 0$ with an initial value $x \in P_l(H)$. The solution $\varphi^l(t, x)$ is well defined, because solutions for each Galerkin projection of (3a) at any time $t > 0$ exist and are unique due to the fact that (4) is a finite system of ODEs with a locally Lipschitz right-hand side. For the purpose of providing a rough explanation of the computer assisted proof in the following section we define two kinds of sets that we used.

For the purpose of providing a rough explanation of the computer assisted proof in the following section we define two kinds of sets that we used.

Definition 3 Let $R \subset H$, R is convex, $l > 0$, $x \in X_l$, $\varphi^l(t, x)$ be the local flow inducted by (4). We call $P_l(R)$ a trapping region for the l-th Galerkin projection (4) if $\varphi^l(t, P_l(R)) \subset P_l(R)$ for all $t > 0$ or equivalently the vector field on the boundary of $P_l(R)$ is pointing inwards.

Definition 4 Let $M_1 > 0$. The set $A \subset H$ is called the absorbing set for any Galerkin projection of (3a), if for any initial condition $\{a_k\}_{k\in\mathbb{Z}} \in H$ there exists a finite time $t_1 \geq 0$, such that $\varphi^l(t, P_l(\{a_k\}_{k\in\mathbb{Z}})) \in P_l(A)$ for all $l > M_1$ and $t \geq t_1$.

The main theorem. A computer assisted proof

Below, we present an examplary theorem that we proved using computer assistance, along with a sketch of the proof.

Theorem 5 Let $\nu = 2$. For any $f \in 1.8(\sin 2x - \cos 3x) + \sum_{k=1}^{3} [-0.025, 0.025](\sin kx + \cos kx)$ there exists a fixed point - a steady state solution of (2), which is unique and globally attracting any initial data u_0 satisfying $u_0 \in C^4$ and $\int_0^{2\pi} u_0(x)\, dx = 2\pi$.

For the proof and a detailed description of the algorithm we refer the reader to [6]. The computer software package is available along with all the data from the proofs [7]. For the purpose of the paper we present only the main ideas in a sketch of the proof.

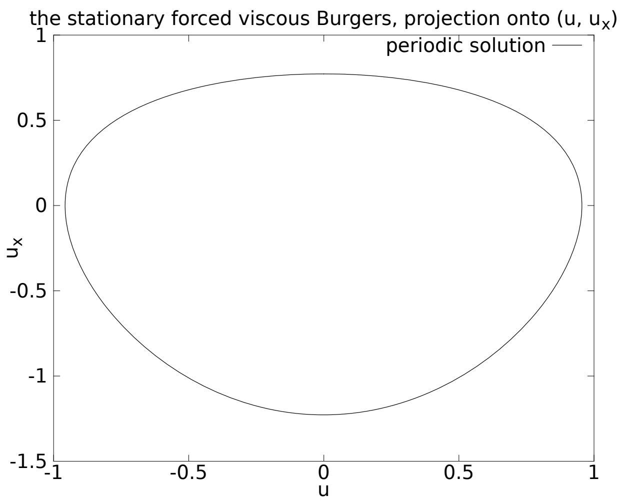
To establish the existence of a locally attracting fixed point for (3) we used a computer technique presented in [5], originally applied to the Kuramoto-Sivashinsky equation. The technique, based on the method of self-consistent bounds, comprises several steps. The first step is to find $\bar{x}$, a candidate for a fixed point of (3) by the Newton method iterations applied to the m-th Galerkin projection of (3a), with $m > 0$. Secondly, new coordinates in which the Jacobian of (4) at $\bar{x}$ has almost diagonal form, are found. Thirdly, $W \subset H$, an isolating box, such that $P_l(W)$ is a trapping region for each l-th Galerkin projection of (3a), is constructed. At this point, we have proved the existence of a fixed point for each Galerkin projection of (3a). Then we pass to the limit with Galerkin projections by estimating the logarithmic norm on $W \subset H$. Furthermore, if we manage to find an upper bound for the logarithmic norm which is negative (logarithmic norm is not a norm in the classical sense, but rather should be thought of as a directional derivative and, thus, can have a negative value), we claim that the found fixed point is, in fact, attracting in $W \subset H$.

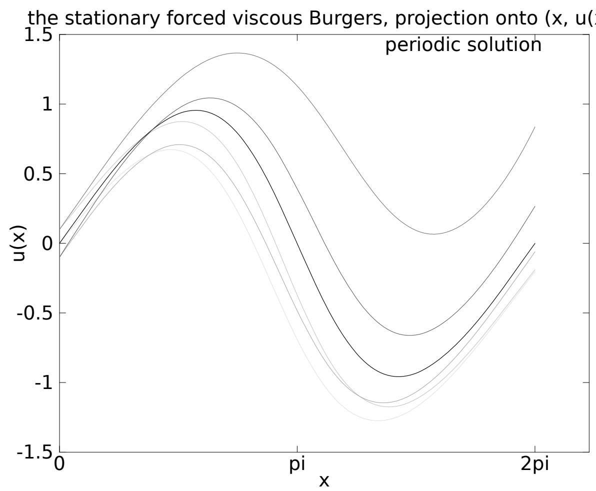
To extend the property of attractiveness of the obtained fixed point from local to global we construct $A \subset H$ - an absorbing set, capturing after a finite time any sufficiently regular solution of any Galerkin projection of (3a). Then, in order to rigorously integrate $P_l(A)$ for all $l > m$ we simultaneously use the algorithm from [4] for solving a differential inclusion. If we manage to verify that there exists a $t_1 > 0$ such that $\varphi^l(t, A) \subset W$ for all $t > t_1$ and for all $l > m$, having in mind that A and W are expressed in different coordinates, we claim that the fixed point within W attracts any sufficiently regular initial condition.

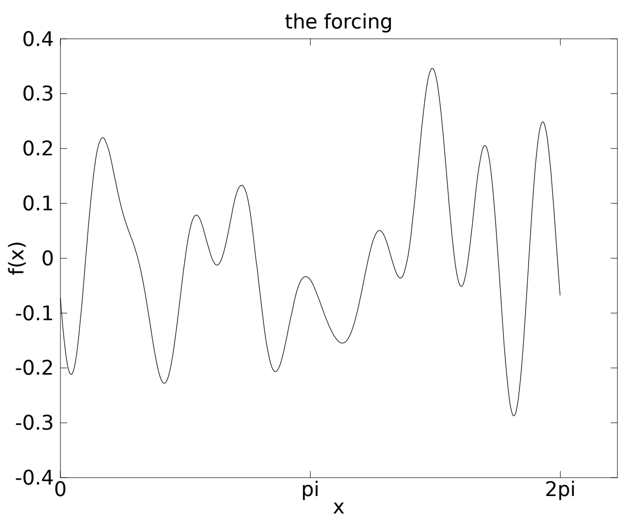
All the calculations performed during the execution of the algorithm are rigorous and the rounding errors are handled by the interval arithmetic. Some elements were realized by using the CAPD library [8], which we recommend for this kind of application.

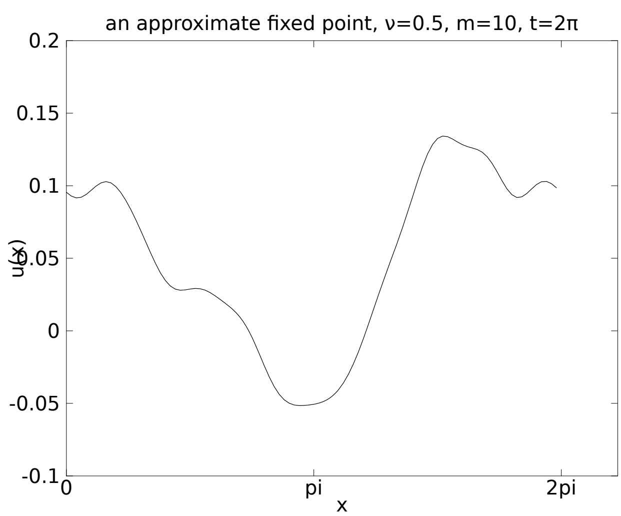
Some figures

To present the results in an intuitive way we provide some figures describing the behavior of solutions.









The upper-left and upper-right figures show projections onto a two-dimensional plane of the solutions of the stationary viscous Burgers equation with a sinusoidal forcing

$$u_{xx} = u \cdot u_x - f(x),$$

which is a system of three non-autonomous ODEs. On the upper-left corner figure we present the periodic solution separately, whereas on the upper-right corner figure the periodic solution is marked with the black color and is plotted along with some nearby non-periodic solutions.

The bottom-left figure shows an example of the forcing function $f(x)$ for which we have managed to prove the existence of a globally attracting fixed point. Other parameter values were as follows: $\nu = 0.5$, $\int_0^{2\pi} u_0(x)\,dx = 0.05\pi$. The approximate fixed point, showed on the bottom-right corner figure, was obtained using a non-rigorous integration of the 10-th Galerkin projection of (3a), which was terminated at $t = 2\pi$. This is provided in order to highlight that the fixed point is not necessarily similar to the forcing function.

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