BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 62, No. 3, 2014

PARTIAL DIFFERENTIAL EQUATIONS

Finite-dimensional Pullback Attractors for Non-autonomous Newton–Boussinesq Equations in Some Two-dimensional Unbounded Domains

by

Cung The ANH and Dang Thanh SON

Presented by Jerzy ZABCZYK

Summary. We study the existence and long-time behavior of weak solutions to Newton–Boussinesq equations in two-dimensional domains satisfying the Poincaré inequality. We prove the existence of a unique minimal finite-dimensional pullback D_σ-attractor for the process associated to the problem with respect to a large class of non-autonomous forcing terms.

1. Introduction. Let Ω be an arbitrary (bounded or unbounded) domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. In this paper we study the long-time behavior of solutions to the following non-autonomous Newton–Boussinesq equations in Ω:

$$(1.1)\qquad \begin{cases} \partial_t\omega + u\partial_{x_1}\omega + v\partial_{x_2}\omega = \Delta\omega - \dfrac{R_a}{P_r}\partial_{x_1}\theta + f(x_1,x_2,t), \\ \Delta\Psi = \omega, \quad u = \Psi_{x_2}, \quad v = -\Psi_{x_1}, \\ \partial_t\theta + u\partial_{x_1}\theta + v\partial_{x_2}\theta = \dfrac{1}{P_r}\Delta\theta + g(x_1,x_2,t), \end{cases}$$

where $\vec{u} = (u,v)$ is the unknown velocity vector, θ is the flow temperature of the fluid at the point $(x_1,x_2)\in\Omega$ and at time $t \geq \tau$, Ψ is the flow function, ω is the vortex; the positive constants P_r and R_a are the Prandtl number and the Rayleigh number, respectively; $f(x_1,x_2,t)$ is the external body force, and $g(x_1,x_2,t)$ is the external heat source.

2010 *Mathematics Subject Classification*: 35B41, 37L30, 76D99.
Key words and phrases: Newton–Boussinesq equations, unbounded domain, pulback attractor, fractal dimension, energy equation method.

DOI: 10.4064/ba62-3-7

Notice that system (1.1) can be written as follows: for every $x = (x_1, x_2) \in \Omega$ and $t > \tau$,

$$(1.2)\qquad \begin{cases} \dfrac{\partial \omega}{\partial t} - \Delta\omega + J(\Psi, \omega) + \dfrac{R_a}{P_r}\dfrac{\partial \theta}{\partial x_1} = f(x,t), \\ \Delta\Psi = \omega, \\ \dfrac{\partial \theta}{\partial t} - \dfrac{1}{P_r}\Delta\theta + J(\Psi, \theta) = g(x,t), \end{cases}$$

where the function J is given by

$$J(u, v) = u_{x_2} v_{x_1} - u_{x_1} v_{x_2}.$$

We consider system (1.2) with the following boundary conditions:

$$(1.3)\qquad \begin{cases} \omega(x,t) = 0, & \forall (x,t) \in \partial\Omega \times (\tau, \infty), \\ \theta(x,t) = 0, & \forall (x,t) \in \partial\Omega \times (\tau, \infty), \\ \Psi(x,t) = 0, & \forall (x,t) \in \partial\Omega \times (\tau, \infty), \end{cases}$$

and the initial conditions

$$(1.4)\qquad \begin{cases} \omega(x,\tau) = \omega_0(x), & x \in \Omega, \\ \theta(x,\tau) = \theta_0(x), & x \in \Omega. \end{cases}$$

The Newton–Boussinesq equations describe many physical phenomena such as Bénard flow (see [4, 6]). If the domain under consideration is bounded, the existence, uniqueness and the long-time behavior of solutions to system (1.1) have been studied by several authors, in both the autonomous case [5, 8, 9, 10, 11] and the non-autonomous case [16]. Note that in these works, the compactness of the Sobolev embeddings, due to the boundedness of the domain, plays an essential role. However, as fas as we know, there are no results for problem (1.1) in unbounded domains, a more complicated case due to the lack of compactness of the Sobolev embeddings.

The aim of this paper is to continue the study of the long-time behavior of weak solutions to problem (1.1) in a two-dimensional channel-like domain Ω. More precisely, Ω can be an arbitrary (bounded and unbounded) open set in $\mathbb{R}^2$ without any regularity assumption on its boundary. The only assumption is that the Poincaré inequality holds on it, i.e., there exists $\lambda_1 > 0$ such that

$$(1.5)\qquad \int_\Omega |\phi|^2 dx \le \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2\, dx, \quad \forall \phi \in H_0^1(\Omega).$$

Because the external forces $f(x,t)$ and $g(x,t)$ are time-dependent, to study the long-time behavior of solutions we will use the theory of pullback attractors. This is a natural generalization of the theory of global attractors for autonomous dynamical systems and allows considering a number of different problems for non-autonomous dynamical systems with a large class of non-autonomous forcing terms (see the recent monograph [3]). Note that the unboundedness of Ω introduces a major difficulty for proving the existence of a pullback attractor because the Sobolev embeddings are no longer compact, and hence the pulback asymptotic compactness of the process cannot be obtained by a standard method as in [16]. To overcome this difficulty, we exploit the energy equation method introduced by Ball [1] to prove the pullback asymptotic compactness of the process, and as a result, the existence of a pullback attractor. Such an approach has been used to prove the existence of pullback attractors for non-autonomous 2D Navier–Stokes equations in some unbounded domains [2]. Finally, following the general lines of the approach in [13], we show that the pullback attractor has a finite fractal dimension under some additional conditions. The results obtained, in particular, improve and extend all known results about attractors for the 2D Newton–Boussinesq equations. Notice that in the autonomous case, to overcome the difficulty due to the unboundedness of the domain, the authors of [7] used another approach based on the tail estimates method introduced by Wang [18]. It is worth noticing that the estimate of fractal dimension of the attractor obtained here is new even in the autonomous case.

The paper is organized as follows. In Section 2, for the convenience of the reader, we recall the mathematical framework which is necessary to set up the problem, and abstract results on the existence and fractal dimension of pullback attractors. In Section 3, we give the proof of existence and uniqueness of weak solutions. In Section 4, we prove the existence of a pullback $\mathcal{D}_\sigma$-attractor for the associated process by using the energy equation method. In the last section, we estimate the fractal dimension of the pullback $\mathcal{D}_\sigma$-attractor.

2. Preliminary results

2.1. Function spaces and operators. We now recall several function spaces and operators necessary to write the problem (1.1) in its variational formulation. Let

$$V := H_0^1(\Omega) \times H_0^1(\Omega), \quad H = L^2(\Omega) \times L^2(\Omega).$$

Due to (1.5), the inner product and norm in $H_0^1(\Omega)$ are given by

$$((\omega, \tilde{\omega})) = \int_\Omega \nabla\omega \cdot \nabla\tilde{\omega}\, dx, \quad \forall \omega, \tilde{\omega} \in H_0^1(\Omega),$$
$$\|\omega\| = ((\omega, \omega))^{1/2}, \qquad \forall \omega \in H_0^1(\Omega).$$

Abusing notation for simplicity, we define the inner product and norm in V by

$$((z, \tilde{z})) = ((\omega, \tilde{\omega})) + \gamma((\theta, \tilde{\theta})), \quad \forall z = (\omega, \theta), \tilde{z} = (\tilde{\omega}, \tilde{\theta}) \in V,$$
$$\|z\| = ((z, z))^{1/2}, \qquad \forall z \in V,$$

where

$$\gamma \geq \frac{R_a^2}{P_r \lambda_1}. \tag{2.1}$$

This constant γ is chosen so that an operator to be defined later (related to the linear part of the system) is coercive under the norm defined. Moreover, the choice of γ makes the two terms in the definition of the inner product in V dimensionally consistent in physical units.

We define the inner product and norm in H by

$$(z, \tilde{z}) = (\omega, \tilde{\omega}) + \gamma(\theta, \tilde{\theta}), \quad \forall z = (\omega, \theta), \tilde{z} = (\tilde{\omega}, \tilde{\theta}) \in H,$$
$$|z| = (z, z)^{1/2}, \quad \forall z \in H.$$

It follows from (1.5) that for all $\omega, \theta \in H_0^1$ we have

$$|\omega|^2 \leq \frac{1}{\lambda_1}\|\omega\|^2, \quad |\theta|^2 \leq \frac{1}{\lambda_1}\|\theta\|^2. \tag{2.2}$$

Applying Riesz's representation theorem, we can identify the dual space H' with H, obtaining $V \subset H = H' \subset V'$, where the injections are continuous and each space is dense in the following ones.

Let us define a bilinear form $a : V \times V \to \mathbb{R}$, and the corresponding linear operator $A : V \to V'$, by

$$a(z, \tilde{z}) = \langle Az, \tilde{z}\rangle_{V',V} = \int_\Omega \nabla\omega \cdot \nabla\tilde{\omega}\, dx + \gamma\frac{1}{P_r}\int_\Omega \nabla\theta \cdot \nabla\tilde{\theta}\, dx.$$

The operator A is clearly linear from V into V', and the bilinear form a is coercive since

$$\min\left(1, \frac{1}{P_r}\right)\|z\|^2 \leq a(z, z) = \langle Az, z\rangle_{V',V} \leq \max\left(1, \frac{1}{P_r}\right)\|z\|^2. \tag{2.3}$$

Let us define $b : (H^2 \cap H_0^1) \times V \times V \to \mathbb{R}$, and $B(z) = B(\Psi^z, z)$, the associated bilinear operator $B : (H^2 \cap H_0^1) \times V \to V'$, by

$$b(\Psi^z, z, \tilde{z}) = \langle B(\Psi^z, z), \tilde{z}\rangle = \int_\Omega J(\Psi^z, \omega)\tilde{\omega}\, dx + \gamma\int_\Omega J(\Psi^z, \theta)\tilde{\theta}\, dx,$$

where $\Delta\Psi^z = \omega$. It is easy to check that if $\Psi^z \in H^2 \cap H_0^1, z, \tilde{z} \in V$, then

$$b(\Psi^z, z, \tilde{z}) = -b(\Psi^z, \tilde{z}, z). \tag{2.4}$$

Hence

$$b(\Psi^z, z, z) = 0. \tag{2.5}$$

The following result is well-known.

Lemma 2.1 (Ladyzhenskaya's inequality). *For any open set $\Omega \subset \mathbb{R}^2$, we have*

$$\|\phi\|_{L^4(\Omega)} \leq \frac{1}{2^{1/4}}\|\phi\|_{L^2(\Omega)}^{1/2}\|\nabla\phi\|_{L^2(\Omega)}^{1/2}, \quad \forall\phi \in H_0^1(\Omega). \tag{2.6}$$

Using Lemma 2.1 and the Poincaré inequality (1.5), we deduce from (2.6) that

$$\|\phi\|_{L^4(\Omega)} \le \left(\frac{1}{2\lambda_1}\right)^{1/4} \|\nabla\phi\|_{L^2(\Omega)}, \quad \forall \phi \in H_0^1(\Omega). \tag{2.7}$$

LEMMA 2.2. *For any open set $\Omega \subset \mathbb{R}^2$ and $\Psi^z \in H^2 \cap H_0^1$, $z, \tilde{z} \in V$ we have*

$$|b(\Psi^z, z, \tilde{z})| \le C|z|\,\|z\|\,\|\tilde{z}\|,$$

where C is a positive constant.

Proof. It is obvious that

$$\frac{1}{\sqrt{2}}(\|\tilde{\omega}\| + \gamma^{1/2}\|\tilde{\theta}\|) \le \sqrt{\|\tilde{\omega}\|^2 + \gamma\|\tilde{\theta}\|^2} = \|\tilde{z}\|. \tag{2.8}$$

By using the Hölder inequality and (2.7) we obtain

$$\begin{aligned}\Big|\int_\Omega \Psi_y^z \omega_x \tilde{\omega}\,dx\Big| &\le \|\Psi_y^z\|_{L^4}\|\omega_x\|_{L^2}\|\tilde{\omega}\|_{L^4} \\ &\le C\|\Delta\Psi^z\|_{L^2}\|\omega\|\,\|\tilde{\omega}\| \le C|\omega|\,\|\omega\|\,\|\tilde{\omega}\|.\end{aligned}$$

Thus

$$\Big|\int_\Omega J(\Psi^z, \omega)\tilde{\omega}\,dx\Big| \le \frac{1}{\sqrt{2}}C|z|\,\|z\|\,\|\tilde{\omega}\|.$$

Similarly,

$$\begin{aligned}\Big|\int_\Omega J(\Psi^z, \theta)\tilde{\theta}\,dx\Big| &\le \frac{1}{\sqrt{2}}C\|\Psi^z\|_{H^2}\|\theta\|\,\|\tilde{\theta}\| \le \frac{1}{\sqrt{2}}C\gamma^{-1/2}|\Delta\Psi^z|\,\|z\|\,\|\tilde{\theta}\| \\ &\le \frac{1}{\sqrt{2}}C\gamma^{-1/2}|\omega|\,\|z\|\,\|\tilde{\theta}\| \le \frac{1}{\sqrt{2}}C\gamma^{-1/2}|z|\,\|z\|\,\|\tilde{\theta}\|.\end{aligned}$$

Hence

$$|b(\Psi^z, z, \tilde{z})| \le \frac{1}{\sqrt{2}}C|z|\,\|z\|(\|\tilde{\omega}\| + \gamma^{1/2}\|\tilde{\theta}\|).$$

Using (2.8) we get the desired result. ■

Applying (2.5) and Lemma 2.2 we see that

$$\langle B(\Psi^z, z), z\rangle_{V',V} = 0, \tag{2.9}$$

$$\|B(z)\|_{V'} \le C|z|\,\|z\|, \quad \forall z \in V. \tag{2.10}$$

2.2. Pullback attractors. Let (X, d) be a metric space. For $A, B \subset X$, we define the Hausdorff semidistance between A and B by

$$\operatorname{dist}(A, B) := \sup_{a\in A}\inf_{b\in B} d(a, b).$$

A *process* on X is a two-parameter family $\{Z(t,\tau)\}$ of mappings in X with the following properties:

$$Z(t,r)Z(r,\tau) = Z(t,\tau) \quad \text{for all } t \geq r \geq \tau,$$
$$Z(\tau,\tau) = \mathrm{Id} \quad \text{for all } \tau \in \mathbb{R}.$$

Suppose that $\mathcal{B}(X)$ is the family of all non-empty bounded subsets of X, and $\mathcal{D}$ is a non-empty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Definition 2.1. The process $\{Z(t,\tau)\}$ is said to be *pullback $\mathcal{D}$-asymptotically compact* if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{Z(t,\tau_n)x_n\}_n$ is relatively compact in X.

Definition 2.2. A family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ of bounded sets is called *pullback $\mathcal{D}$-absorbing* for the process $Z(t,\tau)$ if for any $t \in \mathbb{R}$ and $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \leq t$ such that

$$\bigcup_{\tau \leq \tau_0} Z(t,\tau)D(\tau) \subset B(t).$$

Definition 2.3. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a *pullback $\mathcal{D}$-attractor* for $\{Z(t,\tau)\}$ if:

(i) $A(t)$ is compact for all $t \in \mathbb{R}$;
(ii) $\hat{\mathcal{A}}$ is invariant, i.e., $Z(t,\tau)A(\tau) = A(t)$ for all $t \geq \tau$;
(iii) $\hat{\mathcal{A}}$ is *pullback $\mathcal{D}$-attracting*, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}(Z(t,\tau)D(\tau), A(t)) = 0 \quad \text{for all } \hat{\mathcal{D}} \in \mathcal{D} \text{ and } t \in \mathbb{R};$$

(iv) if $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 2.1 ([2]). *Let $\{Z(t,\tau)\}$ be a continuous process such that $\{Z(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a pullback $\mathcal{D}$-absorbing family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{Z(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} Z(s,\tau)B(\tau)}.$$

We now recall from [13] some estimates of the fractal dimension of pullback attractors.

Let H be a separable real Hilbert space. Given a compact set $K \subset H$, and $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimum number of open balls in H with radius ε that are necessary to cover K.

DEFINITION 2.4. For any non-empty compact set $K \subset H$, the *fractal dimension* of K is the number

$$d_{\mathrm{F}}(K) = \limsup_{\varepsilon \downarrow 0} \frac{\log(N_\varepsilon(K))}{\log(1/\varepsilon)}.$$

Consider a separable real Hilbert space $V \subset H$ such that the injection of V in H is continuous, and V is dense in H. We identify H with its topological dual H', and we consider V as a subspace of H', identifying $\eta \in V$ with the element $f_\eta \in H'$ defined by

$$f_\eta(h) = (\eta, h), \quad h \in H.$$

Let $F : V \times \mathbb{R} \to V'$ be a given family of non-linear operators such that, for all $\tau \in \mathbb{R}$ and any $z_0 \in H$, there exists a unique function $z(t) = z(t; \tau, z_0)$ satisfying

$$\begin{cases} z \in L^2(\tau, T; V) \cap C([\tau, T]; H), \\ F(z(\cdot), \cdot) \in L^1(\tau, T; V') \quad \text{for all } T > \tau, \\ \dfrac{dz}{dt} = F(z(t), t), \quad t > \tau, \\ z(\tau) = z_0. \end{cases} \tag{2.11}$$

Let us define

$$Z(t, \tau) z_0 = z(t; \tau, z_0), \quad \tau \le t,\ z_0 \in H.$$

Fix $T^* \in \mathbb{R}$. We assume that there exists a family $\{A(t) : t \le T^*\}$ of non-empty compact subsets of H with the invariance property

$$Z(t, \tau) A(\tau) = A(t) \quad \text{for all } \tau \le t \le T^*,$$

and such that, for all $\tau \le t \le T^*$ and any $z_0 \in A(\tau)$, there exists a continuous linear operator $L(t; \tau, z_0) \in \mathcal{L}(H)$ such that

$$|Z(t,\tau)\bar{z}_0 - Z(t,\tau)z_0 - L(t;\tau,z_0)(\bar{z}_0 - z_0)| \le \chi(t-\tau, |\bar{z}_0 - z_0|)|\bar{z}_0 - z_0| \tag{2.12}$$

for all $\bar{z}_0 \in A(\tau)$, where $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\chi(s, \cdot)$ is non-decreasing for all $s \ge 0$, and

$$\lim_{r \to 0} \chi(s, r) = 0 \quad \text{for any } s \ge 0. \tag{2.13}$$

We assume that, for all $t \le T^*$, the mapping $F(\cdot, t)$ is Gateaux differentiable in V, i.e., for any $z \in V$ there exists a continuous linear operator $F'(z, t) \in \mathcal{L}(V; V')$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(z + \epsilon\eta, t) - F(z, t) - \epsilon F'(z, t)\eta] = 0 \quad \text{in } V'.$$

Moreover, we suppose that the mapping

$$F' : V \times (-\infty, T^*] \ni (z, t) \mapsto F'(z, t) \in \mathcal{L}(V; V')$$

is continuous (thus, in particular, for each $t \le T^*$, the mapping $F(\cdot, t)$ is continuously Fréchet differentiable in V).

Then, for all $\tau \le T^*$ and $z_0, \eta_0 \in H$, there exists a unique $\eta(t) = \eta(t;\tau, z_0, \eta_0)$ which is a solution of

$$\begin{cases} \eta \in L^2(\tau, T; V) \cap C([\tau, T]; H) \quad \text{for all } \tau < T \le T^*, \\ \dfrac{d\eta}{dt} = F'(Z(t,\tau)z_0, t)\eta, \quad \tau < t < T^*, \\ \eta(\tau) = \eta_0. \end{cases}$$

We make the assumption that

$$\eta(t;\tau, z_0, \eta_0) = L(t;\tau, z_0)\eta_0 \quad \text{for all } \tau \le t \le T^*,\ z_0, \eta_0 \in A(\tau). \tag{2.14}$$

Let us write, for $m = 1, 2, \dots,$

$$\tilde{q}_m = \limsup_{T\to\infty} \sup_{z_0 \in A(\tau-T)} \frac{1}{T} \int_{\tau-T}^{\tau} \operatorname{Tr}_m\big(F'(Z(s,\tau-T)z_0, s)\big)\, ds,$$

where

$$\operatorname{Tr}_m\big(F'(Z(s,\tau-T)z_0, s)\big) = \sup_{\eta_0^i \in H,\, |\eta_0^i| \le 1,\, i \le m} \sum_{i=1}^{m} \langle F'(Z(s,\tau-T)z_0, s)\varphi_i, \varphi_i\rangle,$$

$\{\varphi_i\}_{i=1,\dots,m}$ being an orthonormal basis of the subspace in H spanned by

$$\eta(s;\tau, z_0, \eta_0^1), \dots, \eta(s;\tau, z_0, \eta_0^m).$$

THEOREM 2.2 ([13, Theorem 2.2]). *Under the assumptions above, suppose that*

$$\bigcup_{\tau \le T^*} A(\tau) \text{ is relatively compact in } H,$$

and there exist q_m, $m = 1, 2, \dots,$ such that

$$\begin{aligned} &\tilde{q}_m \le q_m \quad \text{for any } m \ge 1, \\ &q_{n_0} \ge 0,\ q_{n_0+1} < 0 \quad \text{for some } n_0 \ge 1, \\ &q_m \le q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - m) \quad \text{for all } m = 1, 2, \dots. \end{aligned}$$

Then

$$d_{\mathrm{F}}(A(\tau)) \le d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}} \quad \text{for all } \tau \le T^*.$$

3. Existence and uniqueness of weak solutions. We define $r : V \times V \to \mathbb{R}$ and the associated linear operator $R : V \to V'$ by

$$r(z, \tilde{z}) = \langle Rz, \tilde{z}\rangle = \frac{R_a}{P_r}(\theta_{x_1}, \tilde{\omega}).$$

By (2.1), and the Hölder and Poincaré inequalities, we have

$$\left|\frac{R_a}{P_r}(\theta_{x_1}, \tilde{\omega})\right| \le \frac{R_a}{P_r}|\theta_{x_1}|\,|\tilde{\omega}| \le \frac{R_a}{\sqrt{\lambda_1}\,P_r}\|\theta\|\,\|\tilde{\omega}\| \le \frac{1}{\sqrt{P_r}}\|z\|\,\|\tilde{z}\|.$$

Hence we obtain

$$|r(z,\tilde{z})| \le \frac{1}{\sqrt{P_r}}\|z\|\,\|\tilde{z}\| \quad \text{and} \quad \|Rz\|_{V'} \le \frac{1}{\sqrt{P_r}}\|z\|. \tag{3.1}$$

We assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\Omega))$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\Omega))$. Then it is easy to see that $\Phi = (f,g) \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, and

$$\langle \Phi, z\rangle_{V',V} = \langle f,\omega\rangle_{H^{-1},H^1_0} + \gamma\langle g,\theta\rangle_{H^{-1},H^1_0} \quad \text{for a.e. } t \in \mathbb{R}. \tag{3.2}$$

We define $e : V \to \mathbb{R}$ by $e(z) = \langle \Phi, z\rangle_{V',V}$. It is obvious that

$$|e(z)| = |\langle \Phi, z\rangle| \le \|\Phi\|_{V'}\|z\|.$$

We now consider the following weak formulation of problem (1.2)–(1.4).

Problem. For $z_0 \in H$ given, find $z = (\omega,\theta)$ such that

$$\begin{cases} z \in L^2(\tau,T;V) \cap L^\infty(\tau,T;H), \\ \dfrac{d}{dt}(z,\tilde{z}) + a(z,\tilde{z}) + r(z,\tilde{z}) + b(\Psi^z, z, \tilde{z}) = e(\tilde{z}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \tilde{z} \in V, \ \text{for a.e. } t, \\ z(\tau) = z_0. \end{cases} \tag{3.3}$$

Equation (3.3) is equivalent to the functional equation in V',

$$\begin{cases} z \in L^2(\tau,T;V) \cap L^\infty(\tau,T;H), \\ z' + (A+R)z + B(z) = \Phi \quad \text{in } V', \ \text{for a.e. } t, \\ z(\tau) = z_0, \end{cases} \tag{3.4}$$

where $z' = (d\omega/dt, d\theta/dt)$.

In order to prove the existence of global solutions, we first show that $A+R$ is V-elliptic. More precisely, we will prove that there exists $\delta > 0$ such that

$$\langle (A+R)z, z\rangle \ge \delta\Big(\|\omega\|^2 + \gamma\frac{1}{P_r}\|\theta\|^2\Big). \tag{3.5}$$

This implies that there exists $\delta' > 0$ such that

$$\langle (A+R)z, z\rangle \ge \delta'\|z\|^2.$$

Lemma 3.1. *The operator $A+R$ satisfies* (3.5) *with some positive number δ.*

Proof. From (2.1), $R_a/(\sqrt{\lambda_1}\,P_r) \le (\gamma P_r^{-1})^{1/2}$. Using this and the Poincaré inequality (1.5) we obtain

$$\begin{aligned} \left|\frac{R_a}{P_r}(\theta_{x_1},\omega)\right| &\le \frac{R_a}{P_r}|\theta_{x_1}|\,|\omega| \le \frac{R_a}{\sqrt{\lambda_1}\,P_r}\|\theta\|\,\|\omega\| \\ &\le \frac{\|\omega\|^2 + \gamma P_r^{-1}\|\theta\|^2}{2}. \end{aligned}$$

From the definition of R and the inequality above we have

$$|\langle Rz, z\rangle| = |r(z,z)| \le \frac{1}{2}\bigg(\|\omega\|^2 + \frac{\gamma}{P_r}\|\theta\|^2\bigg).$$

Using the definition of A, the definition of $\|z\|$ and the inequality above we obtain

$$\langle (A+R)z, z\rangle \ge \|\omega\|^2 + \frac{\gamma}{P_r}\|\theta\|^2 - |\langle Rz, z\rangle| \ge \frac{1}{2}\bigg(\|\omega\|^2 + \frac{\gamma}{P_r}\|\theta\|^2\bigg).$$

Hence we can choose, for instance, $\delta = 1/2$. ∎

We are now ready to prove the existence of a weak solution to problem (3.3).

THEOREM 3.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\Omega))$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\Omega))$. Then for any $z_0 \in H$, $\tau \in \mathbb{R}$, and $T > \tau$, there exists a unique solution $z \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ of problem (3.4). Moreover, $z \in C([\tau, T]; H)$ and*

$$|z(t)|^2 \le e^{-\sigma(t-\tau)}|z_0|^2 + \frac{e^{-\sigma t}}{\zeta}\int_{-\infty}^{t} e^{\sigma s}\|\Phi(s)\|^2_{V'}\,ds, \tag{3.6}$$

where $\sigma = \zeta\lambda_1$ and $\zeta = \delta\min(1, 1/P_r)$.

Proof. (i) *Existence.* From (3.5) we see that $A + R$ is V-elliptic. The existence of a weak solution on (τ, T) is based on Galerkin approximations, *a priori* estimates, and the compactness method. As it is standard and similar to the case of the Navier–Stokes equations [17], we only provide some basic *a priori* estimates used frequently later.

Now we determine an energy equation for the solution. We define a symmetric bilinear form $[\cdot,\cdot] : V \times V \to \mathbb{R}$ by

$$[z, \tilde z] = \langle (A+R)z, \tilde z\rangle - \frac{\zeta\lambda_1}{2}(z, \tilde z), \qquad \forall z, \tilde z \in V, \tag{3.7}$$

where ζ is defined as

$$\zeta = \delta \min\bigg(1, \frac{1}{P_r}\bigg), \tag{3.8}$$

and where δ is given by (3.5), the V-ellipticity condition.

From (3.1) and the definition of A we have

$$[z, z] + \frac{\zeta\lambda_1}{2}|z|^2 = \langle (A+R)z, z\rangle \le \bigg(\max\bigg(1, \frac{1}{P_r}\bigg) + \frac{1}{\sqrt{P_r}}\bigg)\|z\|^2.$$

Thus,

$$[z]^2 \equiv [z, z] \le \bigg(\max\bigg(1, \frac{1}{P_r}\bigg) + \frac{1}{\sqrt{P_r}}\bigg)\|z\|^2. \tag{3.9}$$

Let $z = (\omega, \theta)$. From the definition of $|z|$ and (2.2) we have

$$\frac{\zeta\lambda_1}{2}|z|^2 = \frac{\zeta\lambda_1}{2}(|\omega|^2 + \gamma|\theta|^2) \leq \frac{\zeta}{2}(\|\omega\|^2 + \gamma\|\theta\|^2) = \frac{\zeta}{2}\|z\|^2.$$

Using this, (3.5) and (3.8) we obtain

$$[z]^2 \geq \zeta\|z\|^2 - \frac{\zeta\lambda_1}{2}|z|^2 \geq \frac{\zeta}{2}\|z\|^2. \tag{3.10}$$

Putting together (3.9) and (3.10) we obtain

$$\frac{\zeta}{2}\|z\|^2 \leq [z]^2 \leq \left(\max\left(1, \frac{1}{P_r}\right) + \frac{1}{\sqrt{P_r}}\right)\|z\|^2, \qquad \forall z \in V. \tag{3.11}$$

Thus, $[\cdot,\cdot]$ defines an inner product in V with norm $[\cdot] = [\cdot,\cdot]^{1/2}$ equivalent to $\|\cdot\|$.

Now let $z(t) = (\omega(t), \theta(t))$ be a solution given by Theorem 3.1. Since $z = (\omega, \theta) \in L^2(\tau, T; V)$ and $z' = (\omega', \theta') \in L^2(\tau, T; V')$, we have

$$\frac{1}{2}\frac{d}{dt}|\omega|^2 = \langle\omega', \omega\rangle_{H^{-1},H_0^1} \quad \text{and} \quad \frac{1}{2}\frac{d}{dt}|\theta|^2 = \langle\theta', \theta\rangle_{H^{-1},H_0^1}.$$

Using (3.2) we have

$$\frac{1}{2}\frac{d}{dt}|z|^2 = \frac{1}{2}\frac{d}{dt}(|\omega|^2 + \gamma|\theta|^2) = \langle\omega', \omega\rangle_{H^{-1},H_0^1} + \gamma\langle\theta', \theta\rangle_{H^{-1},H_0^1} = \langle z', z\rangle_{V',V}.$$

So from (3.4) and (2.9) we obtain

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \langle(A+R)z, z\rangle = \langle\Phi, z\rangle.$$

From the definition of the norm $[\cdot]$ given by (3.7) we deduce that

$$\frac{d}{dt}|z|^2 + \zeta\lambda_1|z|^2 + 2[z]^2 = 2\langle\Phi, z\rangle. \tag{3.12}$$

Using the equivalence of norms given by (3.11) and the Cauchy inequality, we obtain

$$\frac{d}{dt}|z|^2 + \zeta\lambda_1|z|^2 + \zeta\|z\|^2 \leq \frac{2}{\zeta}\|\Phi\|_{V'}^2 + \frac{\zeta}{2}\|z\|^2,$$

and hence

$$\frac{d}{dt}|z|^2 + \frac{\zeta}{2}\|z\|^2 \leq \frac{2}{\zeta}\|\Phi\|_{V'}^2.$$

Let $T > \tau$ be arbitrary. Integrating both sides of the above inequality from τ to T, we get

$$|z(T)|^2 + \frac{\zeta}{2}\int_\tau^T \|z(s)\|^2\,ds \leq |z_0|^2 + \frac{2}{\zeta}\|\Phi\|_{L^2(\tau,T;V')}^2.$$

This inequality implies estimates of z in the function space $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$.

Since $z \in L^2(\tau, T; V)$, (3.4) implies that $z' \in L^2(\tau, T; V')$. Hence, $z \in C([\tau, T]; H)$.

(ii) *Uniqueness and continuous dependence.* Assume that z^1 and z^2 are two weak solutions of (3.3) with initial data z_0^1, z_0^2. Set $w = z^1 - z^2$ and $\Psi^w = \Psi^{z^1} - \Psi^{z^2}$, where $\Delta\Psi^{z^1} = \omega^1$ and $\Delta\Psi^{z^2} = \omega^2$. Then $w \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$, and w satisfies

$$\begin{aligned} &\frac{d}{dt}w + (A + R)w = B(\Psi^{z^2}, z^2) - B(\Psi^{z^1}, z^1), \\ &w(\tau) = z_0^1 - z_0^2. \end{aligned}$$

We deduce that

$$\begin{aligned} \frac{d}{dt}|w|^2 + 2[w]^2 &= -\zeta\lambda_1|w|^2 + 2b(\Psi^{z^2}, z^2, w) - 2b(\Psi^{z^1}, z^1, w) \\ &= -\zeta\lambda_1|w|^2 - 2b(\Psi^w, z^1, w). \end{aligned}$$

By Lemma 2.2, we have

$$|-2b(\Psi^w, z^1, w)| \le 2C|w|\,\|z^1\|\,\|w\| \le \zeta\|w\|^2 + \frac{C^2}{\zeta}|w|^2\|z^1\|^2.$$

Hence

$$\frac{d}{dt}|w|^2 \le \left(\zeta\lambda_1 + \frac{C^2}{\zeta}\|z^1\|^2\right)|w|^2.$$

Applying the Gronwall inequality, we obtain

$$|w(t)|^2 \le |w(\tau)|^2 \exp\left(\int_\tau^t \left(\zeta\lambda_1 + \frac{C^2}{\zeta}\|z^1(s)\|^2\right) ds\right).$$

The last estimate implies the uniqueness (if $z_0^1 = z_0^2$) and the continuous dependence of solutions on the intial data.

(iii) *The a priori estimate* (3.6). Applying the Cauchy inequality in (3.12) we get

$$\frac{d}{dt}|z|^2 + \zeta\lambda_1|z|^2 + \zeta\|z\|^2 \le \frac{1}{\zeta}\|\Phi\|_{V'}^2 + \zeta\|z\|^2.$$

By the Gronwall inequality, we obtain (3.6). Hence it follows that the solution z can be extended to $[\tau, \infty)$. ∎

4. Existence of a pullback $\mathcal{D}_\sigma$-attractor. Thanks to Theorem 3.1, we can define a continuous process $Z(t, \tau)$ in H by

$$Z(t, \tau)z_0 = z(t; \tau, z_0), \qquad \tau \le t,\ z_0 \in H,$$

where $z(t) = z(t; \tau, z_0)$ is the unique weak solution to problem (3.4) with the initial datum $z(\tau) = z_0$.

The following lemma shows the weak continuity of the process $Z(t, \tau)$.

LEMMA 4.1. *Let $\{z_{0_n}\}_n$ be a sequence in H converging weakly in H to an element $z_0 \in H$. Then*

$$Z(t,\tau)z_{0_n} \rightharpoonup Z(t,\tau)z_0 \quad \text{weakly in } H \text{ for all } t \geq \tau, \tag{4.1}$$

$$Z(\cdot,\tau)z_{0_n} \rightharpoonup Z(\cdot,\tau)z_0 \quad \text{weakly in } L^2(\tau,T;V) \text{ for all } T > \tau. \tag{4.2}$$

Proof. The proof is similar to that of Lemma 2.1 in [15], so it is omitted here. ∎

Let $\mathcal{R}_\sigma$ be the set of all functions $r : \mathbb{R} \to (0,\infty)$ such that

$$\lim_{t\to-\infty} e^{\sigma t} r^2(t) = 0,$$

and denote by $\mathcal{D}_\sigma$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(H)$ such that $D(t) \subset B(0,\hat{r}(t))$ for some $\hat{r}(t) \in \mathcal{R}_\sigma$, where $B(0,r)$ denotes the closed ball in H, centered at zero with radius r.

Now, in order to prove the existence of a pullback $\mathcal{D}_\sigma$-attractor for the process $\{Z(t,\tau)\}$ we assume that $\Phi = (f,g) \in L^2_{\mathrm{loc}}(\mathbb{R};V')$ and

$$\int_{-\infty}^{t} e^{\sigma s} \|\Phi(s)\|^2_{V'}\, ds < \infty \quad \text{for all } t \in \mathbb{R}, \tag{4.3}$$

where $\sigma = \zeta\lambda_1$ and $\zeta = \delta \min(1, 1/P_r)$.

THEOREM 4.1. *Under the conditions of Theorem* 3.1 *and* (4.3), *there exists a unique pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ for the process $\{Z(t,\tau)\}$ associated to problem* (3.4).

Proof. Let $\tau \in \mathbb{R}$ and $z_0 \in H$ be fixed, and denote

$$z(t) = z(t;\tau,z_0) = Z(t,\tau)z_0 \quad \text{for all } t \geq \tau.$$

In order to apply Theorem 2.1, we will check the conditions in the abstract theorem.

(i) *The process $Z(t,\tau)$ has a pullback $\mathcal{D}_\sigma$-absorbing family of sets.* Let $\hat{\mathcal{D}} \in \mathcal{D}_\sigma$. From (3.6) we have

$$|Z(t,\tau)z_0|^2 \leq e^{-\sigma(t-\tau)}\hat{r}^2(\tau) + \frac{e^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s} \|\Phi(s)\|^2_{V'}\, ds \tag{4.4}$$

for all $z_0 \in D(\tau)$ and all $t \geq \tau$. Define $R_\sigma(t) \in \mathcal{R}_\sigma$ by

$$R^2_\sigma(t) = \frac{2e^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s} \|\Phi(s)\|^2_{V'}\, ds, \tag{4.5}$$

and consider the family $\hat{\mathcal{B}}_\sigma$ of closed balls in H defined by $B_\sigma(t) = B(0,R_\sigma(t))$. It is straightforward to check that $\hat{\mathcal{B}}_\sigma \in \mathcal{D}_\sigma$, and moreover, by (4.4) and (4.5), the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing for the process $Z(t,\tau)$.

(ii) $Z(t,\tau)$ *is pullback $\mathcal{D}_\sigma$-asymptotically compact.* Fix $\hat{\mathcal{D}} \in \mathcal{D}_\sigma$, a sequence $\tau_n \to -\infty$, a sequence $z_{0_n} \in D(\tau_n)$ and $t \in \mathbb{R}$. We must prove that from the sequence $\{Z(t,\tau_n)z_{0_n}\}_n$ we can extract a subsequence that converges in H.

As the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing, for each integer $k \geq 0$, there exists a $\tau_{\hat{D}}(k) \leq t-k$ such that

$$Z(t-k,\tau)D(\tau) \subset B_\sigma(t-k) \quad \text{for all } \tau \leq \tau_{\hat{D}}(k), \tag{4.6}$$

so that for $\tau_n \leq \tau_{\hat{D}}(k)$,

$$Z(t-k,\tau_n)z_{0_n} \subset B_\sigma(t-k).$$

Thus, $\{Z(t-k,\tau_n)z_{0_n}\}_n$ is weakly precompact in H and since $B_\sigma(t-k)$ is closed and convex, there exists a subsequence $\{\tau_{n'}, z_{0_{n'}}\}_{n'} \subset \{\tau_n, z_{0_n}\}_n$ and a sequence $\{w_k : k \geq 0\} \subset H$ such that for all $k \geq 0, w_k \in B_\sigma(t-k)$, and

$$Z(t-k,\tau_{n'})z_{0_{n'}} \rightharpoonup w_k \quad \text{weakly in } H. \tag{4.7}$$

Note that from the weak continuity of $Z(t,\tau)$ established in Lemma 4.1, we get

$$\begin{aligned} w_0 &= \lim_{n'\to\infty}{}^{H_w} Z(t,\tau_{n'})z_{0_{n'}} = \lim_{n'\to\infty}{}^{H_w} Z(t,t-k)Z(t-k,\tau_{n'})z_{0_{n'}} \\ &= Z(t,t-k)\lim_{n'\to\infty}{}^{H_w} Z(t-k,\tau_{n'})z_{0_{n'}} = Z(t,t-k)w_k, \end{aligned}$$

where $\lim^{H_w}$ denotes the limit taken in the weak topology of H. Thus,

$$Z(t,t-k)w_k = w_0 \quad \text{for all } k \geq 0. \tag{4.8}$$

Now, from (4.7), by the lower semicontinuity of the norm, we have

$$|w_0| \leq \liminf_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}|.$$

If we now prove that also

$$\limsup_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}| \leq |w_0|, \tag{4.9}$$

then we will have

$$\lim_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}| = |w_0|,$$

and this, together with the weak convergence, will imply the strong convergence in H of $Z(t,\tau_{n'})z_{0_{n'}}$ to w_0.

Now, from (3.12) we get

$$|z(t)|^2 \leq e^{-\sigma(t-\tau)}|z_0|^2 + 2\int_\tau^t e^{-\sigma(t-s)}\big(\langle \Phi(s), z(s)\rangle - [z(s)]^2\big)\,ds,$$

which can be written as

$$(4.10)\qquad |Z(t,\tau)z_0|^2 \le e^{\sigma(\tau-t)}|z_0|^2 + 2\int_\tau^t e^{\sigma(s-t)}\big(\langle\Phi(s),z(s)\rangle - [z(s)]^2\big)\,ds$$

for all $\tau \le t$ and all $z_0 \in H$. Thus, for all $k \ge 0$ and all $\tau_{n'} \le t-k$,

$$\begin{aligned}(4.11)\qquad |Z(t,\tau_{n'})z_{0_{n'}}|^2 &= |Z(t,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}|^2\\ &\le e^{-\sigma k}|Z(t-k,\tau_{n'})z_{0_{n'}}|^2\\ &\quad + 2\int_{t-k}^t e^{\sigma(s-t)}\langle\Phi(s),Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}\rangle\,ds\\ &\quad - 2\int_{t-k}^t e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\,ds.\end{aligned}$$

We now estimate each of the three terms above.

By (4.6), $Z(t-k,\tau_{n'})z_{0_{n'}} \in B_\sigma(t-k)$ for all $\tau_{n'} \le \tau_{\hat D}(k)$, $k \ge 0$, and we have

$$(4.12)\qquad \limsup_{n'\to\infty} e^{-\sigma k}|Z(t,\tau_{n'})z_{0_{n'}}|^2 \le e^{-\sigma k}R_\sigma^2(t-k),\quad k\ge 0.$$

This takes care of the first term in (4.11).

As $Z(t-k,\tau_{n'})z_{0_{n'}} \rightharpoonup w_k$ weakly in H, from Lemma 4.1 we obtain

$$(4.13)\quad Z(\cdot,t-k)Z(t-k,\tau_{n'})z_{0_{n'}} \rightharpoonup Z(\cdot,t-k)w_k \text{ weakly in } L^2(t-k,t;V).$$

Taking into account that, in particular, $e^{\sigma(s-t)}\Phi(s) \in L^2(t-k,t;V')$, from (4.13) we get

$$\begin{aligned}(4.14)\qquad \lim_{n'\to\infty}\int_{t-k}^t e^{\sigma(s-t)}\langle\Phi(s),Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}\rangle\,ds\\ = \int_{t-k}^t e^{\sigma(s-t)}\langle\Phi(s),Z(s,t-k)w_k\rangle\,ds.\end{aligned}$$

This takes care of the second term in (4.11).

From (3.11) the norm $[\cdot]$ is equivalent to $\|\cdot\|$ in V. Also

$$0 < e^{-\sigma k} \le e^{\sigma(s-t)} \le 1,\qquad \forall s\in[t-k,t],$$

and therefore

$$\Big(\int_{t-k}^t e^{-\sigma(t-s)}[\cdot]^2\,ds\Big)^{1/2}$$

is a norm in $L^2(t-k,t;V)$ equivalent to the usual norm. Hence from (4.13) we deduce that

$$\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)w_k]^2\,ds \le \liminf_{n'\to\infty} \int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\,ds.$$

Hence

$$\limsup_{n'\to\infty} -2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\,ds = -\liminf_{n'\to\infty} 2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\,ds \le -2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)w_k]^2\,ds. \tag{4.15}$$

This takes care of the last term in (4.11).

We can now pass to the lim sup as $n'\to\infty$ in (4.11), and take (4.12), (4.14) and (4.15) into account to obtain

$$\limsup_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}|^2 \le e^{-\sigma k}R_\sigma^2(t-k) + 2\int_{t-k}^{t} e^{\sigma(s-t)}\big(\langle \Phi(s), Z(s,t-k)w_k\rangle - [Z(s,t-k)w_k]^2\big)\,ds. \tag{4.16}$$

On the other hand, from (4.10) applied to (4.8) we find that

$$|w_0| = |Z(t,t-k)w_k|^2 = |w_k|^2 e^{-\sigma k} + 2\int_{t-k}^{t} e^{\sigma(s-t)}\big(\langle \Phi(s), Z(s,t-k)w_k\rangle - [Z(s,t-k)w_k]^2\big)\,ds.$$

From (4.15) and (4.16), we have

$$\limsup_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}|^2 \le e^{-\sigma k}R_\sigma^2(t-k) + |w_0|^2 - |w_k|^2 e^{-\sigma k} \le e^{-\sigma k}R_\sigma^2(t-k) + |w_0|^2,$$

and thus, taking into account that

$$e^{-\sigma k}R_\sigma^2(t-k) = \frac{2e^{-\sigma t}}{\zeta}\int_{-\infty}^{t-k} e^{\sigma s}\|\Phi(s)\|_{V'}^2\,ds \to 0 \quad \text{as } k\to\infty,$$

we easily obtain (4.9) from the last inequality. ∎

5. Fractal dimension estimates of the pullback $\mathcal{D}_\sigma$-attractor. Observe that problem (3.4) can be written in the form (2.11) by taking

$$F(z,t) = -Az(t) - Rz(t) - Bz(t) + \Phi(t).$$

Then it follows immediately that for all $t \in \mathbb{R}$, the mapping $F(\cdot,t)$ is Gateaux differentiable in V with

$$F'(z,t)\eta = -A\eta - R\eta - B(\Psi^z,\eta) - B(\Psi^\eta,z), \qquad z,\eta \in V,$$

and the mapping $F' : V \times \mathbb{R} \ni (z,t) \mapsto F'(z,t) \in \mathcal{L}(V;V')$ is continuous.

Evidently, for any $\tau \in \mathbb{R}$ and $z_0, \eta_0 \in H$, there exists a unique solution $\eta(t) = \eta(t;\tau,z_0,\eta_0)$ of the problem

$$\text{(5.1)} \qquad \begin{cases} \eta \in L^2(\tau,T;V) \cap L^\infty(\tau;T;H), \\ \dfrac{d\eta}{dt} = -(A+R)\eta - B(\Psi^z,\eta) - B(\Psi^\eta,z), \quad \tau < t, \\ \eta(\tau) = \eta_0. \end{cases}$$

From now on, besides $\Phi = (f,g) \in L^2_{\mathrm{loc}}(\mathbb{R};V')$, we suppose that

$$\text{(5.2)} \qquad f,g \in L^\infty(-\infty,T^*;H^{-1}(\Omega)) \quad \text{for some } T^* \in \mathbb{R}.$$

Thus, $\Phi \in L^2_{\mathrm{loc}}(\mathbb{R};V') \cap L^\infty(-\infty,T^*;V')$.

LEMMA 5.1. *Under the conditions of Theorem* 3.1 *and* (5.2)*, the pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ obtained in Theorem* 4.1 *satisfies:*

$$\text{(5.3)} \qquad \bigcup_{\tau \le T^*} A(\tau) \text{ is relatively compact in } H.$$

Proof. Denoting $M = \|\Phi\|^2_{L^\infty(-\infty,T^*;V')}$, from (4.5) we have

$$R^2_\sigma(t) \le \frac{2Me^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s}\,ds = \frac{2M}{\sigma\zeta},$$

and consequently

$$B^* := \bigcup_{\tau \le T^*} B_\sigma(\tau) \text{ is bounded in } H,$$

where $B_\sigma(\tau) = B(0,R_\sigma(\tau))$.

Denote by $\mathcal{M}$ the set of all $y \in H$ such that there exists a sequence $\{t_n,\tau_n\}_n \subset \mathbb{R}^2$ satisfying

$$\tau_n \le t_n \le T^*, \quad n \ge 1, \qquad \lim_{n\to\infty}(t_n - \tau_n) = \infty,$$

and a sequence $\{z_{0_n}\}_n \subset B^*$ such that $\lim_{n\to\infty} |Z(t,\tau_n)z_{0_n} - y| = 0$.

It is easy to see that $A(t) \subset \mathcal{M}$ for all $t \le T^*$. If we prove that $\mathcal{M}$ is relatively compact in H, then (5.3) follows immediately.

Let $\{y_k\}_k \subset \mathcal{M}$. For each $k \ge 1$, we can take $(t_k,\tau_k) \in \mathbb{R}^2$ and an element $z_{0_k} \in B^*$ such that $t_k \le T^*, t_k - \tau_k \ge k$ and $|Z(t_k,\tau_k)z_{0_k} - y_k| \le 1/k$. Using

(5.2), by arguments as in [13, Proposition 3.4], we can extract from $\{y_k\}_k$ a subsequence that converges in H. ∎

LEMMA 5.2. *Under the conditions of Theorem* 3.1 *and* (5.2), *the process* $Z(t,\tau)$ *associated to problem* (3.4) *has the quasidifferentiability properties* (2.12)–(2.14), *with* $\eta(t) = \eta(t;\tau, z_0, \eta_0)$ *defined by* (5.1).

Proof. By (5.2) and Lemma 5.1 there exists a constant $C > 1$ such that

$$\|\Phi\|^2_{L^\infty(-\infty,T^*;V')} \le \frac{C\zeta}{2}, \quad |z_0|^2 \le C \quad \text{for all } z_0 \in \bigcup_{\tau \le T^*} A(\tau). \tag{5.4}$$

Fix $\tau \le T^*$ and $z_0, \bar{z}_0 \in A(\tau)$, denote $z(t) = Z(t,\tau)z_0$ and $\bar{z}(t) = Z(t,\tau)\bar{z}_0$, and let $\eta(t)$ be the solution of (5.1) with $\eta_0 = \bar{z}_0 - z_0$.

From (3.12) we easily find that

$$|z(t)|^2 + \frac{\zeta}{2}\int_\tau^t \|z(s)\|^2\,ds \le |z_0|^2 + \frac{2}{\zeta}\int_\tau^t \|\Phi(s)\|^2_{V'}\,ds. \tag{5.5}$$

Taking into account (5.4), we easily deduce from (5.5) that

$$\int_\tau^t \|z(s)\|^2\,ds \le \frac{2C}{\zeta}(1 + t - \tau) \quad \text{for all } \tau \le t \le T^*. \tag{5.6}$$

Denoting

$$w(t) = \bar{z}(t) - z(t), \quad \tau \le t,$$

we have

$$\begin{aligned}\frac{d}{dt}|w|^2 + 2[w]^2 &= -\zeta\lambda_1|w|^2 + 2b(\Psi^z, z, w) - 2b(\Psi^{\bar{z}}, \bar{z}, w)\\ &= -\zeta\lambda_1|w|^2 - 2b(\Psi^w, z, w).\end{aligned}$$

By Lemma 2.2, we have

$$|-2b(\Psi^w, z, w)| \le 2C|w|\,\|z\|\,\|w\| \le \frac{\zeta}{2}\|w\|^2 + \frac{2C^2}{\zeta}|w|^2\|z\|^2.$$

Hence

$$\frac{d}{dt}|w|^2 + \frac{\zeta}{2}\|w\|^2 \le \left(\zeta\lambda_1 + \frac{2C^2}{\zeta}\|z\|^2\right)|w|^2. \tag{5.7}$$

In particular

$$|w(t)|^2 \le |w(\tau)|^2 \exp\left(\int_\tau^t \left(\zeta\lambda_1 + \frac{2C^2}{\zeta}\|z(s)\|^2\right) ds\right).$$

Thus, by using (5.6),

$$|w(t)|^2 \le |w(\tau)|^2 \exp(K(1 + t - \tau)) \quad \text{for all } \tau \le t \le T^*, \tag{5.8}$$

where $K = \max(4C^3/\zeta^2 + \zeta\lambda_1, 1)$.

Now from (5.7) and (5.8) we have

$$\begin{aligned}\frac{\zeta}{2}\int_\tau^t \|w(s)\|^2\,ds &\le |w(\tau)|^2 + \int_\tau^t \Big(\zeta\lambda_1 + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)|w(s)|^2\,ds\\ &\le |w(\tau)|^2 + \int_\tau^t \Big(\zeta\lambda_1 + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)|w(\tau)|^2 \exp[K(1+s-\tau)]\,ds\\ &\le |w(\tau)|^2\Big[1+\exp[K(1+t-\tau)]\int_\tau^t\Big(\zeta\lambda_1 + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)\,ds\Big].\end{aligned}$$

Hence

$$\begin{aligned}\frac{\zeta}{2}\int_\tau^t \|w(s)\|^2\,ds &\le |w(\tau)|^2\big[1+K(1+t-\tau)\exp[K(1+t-\tau)]\big]\\ &\le |w(\tau)|^2[1+K(1+t-\tau))]\exp[K(1+t-\tau)]\\ &\le |w(\tau)|^2\exp[2K(1+t-\tau)].\end{aligned} \tag{5.9}$$

Let $\alpha(t)$ be defined by

$$\alpha(t)=\bar z(t)-z(t)-\eta(t)=w(t)-\eta(t),\quad t\ge\tau.$$

Evidently, $\alpha(t)$ satisfies

$$\begin{cases}\alpha\in L^2(\tau,T;V)\cap C([\tau,T];H)\quad\text{for all } T>\tau,\\ \dfrac{d\alpha}{dt}=-(A+R)\alpha - B(\Psi^{\bar z},\bar z)+B(\Psi^z,z)+B(\Psi^z,\eta)+B(\Psi^\eta,z),\quad t>\tau,\\ \alpha(\tau)=0.\end{cases}$$

It is easy to see that

$$\begin{aligned}-B(\Psi^{\bar z},\bar z)+B(\Psi^z,z)+B(\Psi^z,\eta)+B(\Psi^\eta,z)&\\ =-B(\Psi^z,\alpha)-B(\Psi^\alpha,z)-B(\Psi^w,w),&\end{aligned}$$

and consequently, for all $t>\tau$,

$$\begin{aligned}\frac{d}{dt}|\alpha|^2+\zeta\|\alpha\|^2 &= -\zeta\lambda_1|\alpha|^2 - 2b(\Psi^\alpha,z,\alpha)-2b(\Psi^w,w,\alpha)\\ &\le \zeta\lambda_1|\alpha|^2+2C\|\Psi^\alpha\|_{H^2}\|z\|\,\|\alpha\|+2C\|\Psi^w\|_{H^2}\|w\|\,\|\alpha\|\\ &\le \zeta\lambda_1|\alpha|^2+2C|\alpha|\,\|z\|\,\|\alpha\|+2C|w|\,\|w\|\,\|\alpha\|\\ &\le \zeta\lambda_1|\alpha|^2+\frac{\zeta}{2}\|\alpha\|^2+\frac{2C^2}{\zeta}|\alpha|^2\|z\|^2+\frac{\zeta}{2}\|\alpha\|^2+\frac{2C^2}{\zeta}|w|^2\|w\|^2\\ &\le \zeta\|\alpha\|^2+\Big(\zeta\lambda_1+\frac{2C^2}{\zeta}\|z\|^2\Big)|\alpha|^2+\frac{2C^2}{\zeta}|w|^2\|w\|^2.\end{aligned}$$

Using the Gronwall inequality, we have

$$|\alpha(t)|^2 \le \exp\left[\int_\tau^t \left(\zeta\lambda_1 + \frac{2C^2}{\zeta}\|z(s)\|^2\right) ds\right] \int_\tau^t \frac{2C^2}{\zeta}|w(s)|^2\|w(s)\|^2\, ds.$$

From (5.8) we obtain

$$|\alpha(t)|^2 \le \frac{2C^2}{\zeta}|w(\tau)|^2 \exp[2K(1+t-\tau)] \int_\tau^t \|w(s)\|^2\, ds.$$

Plugging (5.9) into the last estimate, we get

$$|\alpha(t)|^2 \le \frac{4C^2}{\zeta^2}|w(\tau)|^4 \exp[4K(1+t-\tau)],$$

i.e., (2.12)–(2.14) hold with

$$\chi(s,r) = \frac{2Cr}{\zeta}\exp[2K(1+s)],$$

where $K > 1$. ∎

We now prove the main result in this section.

THEOREM 5.1. *Assume the conditions of Theorem* 3.1 *and* (5.2) *hold. Then the pullback* $\mathcal{D}_\sigma$*-attractor* $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ *of the process* $Z(t,\tau)$ *associated to problem* (3.4) *satisfies*

$$d_{\mathrm{F}}(A(\tau)) \le \max\left\{1, \frac{8\Lambda P_r}{\delta\lambda_1(1+P_r)}\right\},$$

where Λ *is given in* (5.20) *below.*

Proof. In order to estimate the number $\tilde{q}_m$, let $z_0 \in \hat{\mathcal{A}}$ and $\xi_1, \dots, \xi_m \in H$. Set $z(t) = Z(t,\tau)z_0$ and $\eta_i(t) = L(t;\tau,z_0)\xi_i, t \ge \tau$. Let $\{\tilde{\zeta}_i\}_{i=1,\dots,m}$ be orthonormal in $L^2(\Omega)$, with $\Delta\tilde{\zeta}_i = \tilde{\phi}_i$. Let $\{(\tilde{\phi}_i(t)/\lambda_i, \tilde{\psi}_i(t))\}_{i=1,\dots,m}$, $t \ge \tau$, be a basis for $\mathrm{span}\{\eta_1(t), \dots, \eta_m(t)\}$ and $\{\tilde{\phi}_i(t)/\lambda_i\}_{i=1,\dots,m}$ (λ_i, $i = 1, \dots, m$, are the first m eigenvalues of the operator A) and $\{\tilde{\psi}_i(t)\}_{i=1,\dots,m}$ are orthonormal in $L^2(\Omega)$. Let us define

$$\varphi_i = (\phi_i, \psi_i) = \left(\frac{\tilde{\phi}_i}{\lambda_i\sqrt{2}}, \frac{\tilde{\psi}_i}{\sqrt{2\gamma}}\right).$$

An easy computation shows that $\{\varphi_i\}_{i=1,\dots,m}$ is orthonormal in H. Since $\eta_i(t) \in V$ for a.e. $t \ge \tau$, we can assume that $\varphi_i(t) \in V$ for a.e. $t \ge \tau$ (by the Gram–Schmidt orthogonalization process).

From (5.1), (2.5) and (3.5), we have

$$\begin{aligned}
(5.10)\qquad \mathrm{Tr}_m(F'(Z(s,\tau)z_0,s)) &= \sum_{i=1}^{m}\langle F'(Z(s,\tau)z_0,s)\varphi_i,\varphi_i\rangle_{V',V}\\
&= \sum_{i=1}^{m}\langle -(A+R)\varphi_i - B(\Psi^z,\varphi_i) - B(\Psi^{\varphi_i},z),\varphi_i\rangle_{V',V}\\
&\le \sum_{i=1}^{m} -\delta\Big(\|\phi_i\|^2+\frac{\gamma}{P_r}\|\psi_i\|^2\Big)+|b(\Psi^{\varphi_i},z,\varphi_i)|
\end{aligned}$$

for a.e. $s\ge\tau$. Now let

$$\rho(x)=\sum_{i=1}^{m}\Big(|\phi_i(x)|^2+\frac{\gamma}{\sqrt{P_r}}|\psi_i(x)|^2\Big).$$

From the definition of $\tilde\phi_i$ and $\tilde\psi_i$ we observe that

$$\rho(x)=\frac12\sum_{i=1}^{m}\Big(\frac{|\tilde\phi_i(x)|^2}{\lambda_i^2}+\frac{1}{\sqrt{P_r}}|\tilde\psi_i(x)|^2\Big).$$

The generalized Lieb–Thirring inequality (see [12, Corollary 2.1]) can be applied to the finite orthonormal families $\{\tilde\phi_i/\lambda_i\}_i$ and $\{\tilde\psi_i\}_i$. This guarantees the existence of a constant μ independent of the number m of functions (but depending on the shape of Ω) such that

$$\begin{aligned}
(5.11)\qquad |\rho|_{L^2}^2 &\le \frac12\bigg(\bigg|\sum_{i=1}^{m}\Big(\frac{\tilde\phi_i}{\lambda_i}\Big)^2\bigg|_{L^2}^2+\frac{1}{P_r}\bigg|\sum_{i=1}^{m}(\tilde\psi_i)^2\bigg|_{L^2}^2\bigg)\\
&\le \frac{\mu}{2}\sum_{i=1}^{m}\Big(\frac{1}{\lambda_i^2}\|\tilde\phi_i\|^2+\frac{1}{P_r}\|\tilde\psi_i\|^2\Big)=\mu\sum_{i=1}^{m}\Big(\|\phi_i\|^2+\frac{\gamma}{P_r}\|\psi_i\|^2\Big).
\end{aligned}$$

Moreover, setting $q(x):=\sum_{i=1}^m|\nabla\Psi^{\varphi_i}|^2$, we can also apply the generalized Lieb–Thirring inequality to the orthonormal family $\{\lambda_i\Psi^{\varphi_i}\}_i$, to obtain

$$(5.12)\qquad \int_\Omega q^2(x)\,dx\le\mu\sum_{i=1}^{m}\int_\Omega\frac{|\Delta\tilde\zeta_i(x)|^2}{\lambda_i^4}\,dx=\frac{\mu}{\lambda_1^2}\sum_{i=1}^{m}|\phi_i|^2\le\frac{\mu}{\lambda_1^3}\sum_{i=1}^{m}\|\phi_i\|^2.$$

We have

$$\begin{aligned}
(5.13)\qquad &\sum_{i=1}^{m}|b(\Psi^{\varphi_i},z,\varphi_i)|\\
&\qquad=\sum_{i=1}^{m}\Big|\int_\Omega J(\Psi^{\varphi_i},\omega)\phi_i\,dx\,dy+\gamma\int_\Omega J(\Psi^{\varphi_i},\theta)\psi_i\,dx\,dy\Big|.
\end{aligned}$$

We now estimate the first term on the right hand side of (5.13). By using the Cauchy–Schwarz, Hölder and Cauchy inequalities, (5.12) and (5.13), we obtain

$$
\begin{aligned}
(5.14)\qquad \sum_{i=1}^{m}\Big|\int_\Omega J(\Psi^{\varphi_i},\omega)\phi_i\,dx\Big| &\le \sum_{i=1}^{m}\int_\Omega |\nabla\Psi^{\varphi_i}|\,|\nabla\omega|\,|\phi_i|\,dx\\
&\le \int_\Omega |\nabla\omega| q^{1/2}(x)\rho^{1/2}(x)\,dx \le |\nabla\omega|_{L^2}\Big(\int_\Omega q(x)\rho(x)\,dx\Big)^{1/2}\\
&\le \frac{|\nabla\omega|_{L^2}}{\sqrt{2}}\Big(\int_\Omega q^2(x)\,dx+\int_\Omega \rho^2(x)\,dx\Big)^{1/2}\\
&\le \frac{\mu^{1/2}}{\sqrt{2}}|\nabla\omega|_{L^2}\Big((\lambda_1^{-3}+1)\sum_{i=1}^{m}\|\phi_i\|^2+\frac{\gamma}{P_r}\sum_{i=1}^{m}\|\psi_i\|^2\Big)^{1/2}\\
&\le \frac{1}{2}\mu\delta^{-1}(\lambda_1^{-3}+2)\|\omega\|^2+\frac{\delta}{4}\sum_{i=1}^{m}\Big(\|\phi_i\|^2+\frac{\gamma}{P_r}\|\psi_i\|^2\Big).
\end{aligned}
$$

Similarly for the second term on the right hand side of (5.13) we have

$$
\begin{aligned}
(5.15)\qquad \gamma\Big|\sum_{i=1}^{m}\int_\Omega J(\Psi^{\varphi_i},\theta)\psi_i\,dx\Big| &\le \gamma\sum_{i=1}^{m}\int_\Omega |\nabla\Psi^{\varphi_i}|\,|\nabla\theta|\,|\psi_i|\,dx\\
\le \gamma^{1/2}P_r^{1/4}\int_\Omega |\nabla\theta| q^{1/2}(x)\rho^{1/2}(x)\,dx &\le \gamma^{1/2}P_r^{1/4}|\nabla\theta|_{L^2}\Big(\int_\Omega q(x)\rho(x)\,dx\Big)^{1/2}\\
\le \frac{|\nabla\theta|_{L^2}}{\sqrt{2}}\gamma^{1/2}P_r^{1/4}\Big(\int_\Omega q^2(x)\,dx+\int_\Omega \rho^2(x)\,dx\Big)^{1/2}&\\
\le \frac{(\gamma\mu)^{1/2}}{\sqrt{2}}P_r^{1/4}|\nabla\theta|_{L^2}\Big((\lambda_1^{-3}+1)\sum_{i=1}^{m}\|\phi_i\|^2+\frac{\gamma}{P_r}\sum_{i=1}^{m}\|\psi_i\|^2\Big)^{1/2}&\\
\le \frac{1}{2}\mu\gamma P_r^{1/2}\delta^{-1}(\lambda_1^{-3}+2)\|\theta\|^2+\frac{\delta}{4}\sum_{i=1}^{m}\Big(\|\phi_i\|^2+\frac{\gamma}{P_r}\|\psi_i\|^2\Big).&
\end{aligned}
$$

It follows from (5.14) and (5.15) that

$$
\begin{aligned}
(5.16)\qquad \sum_{i=1}^{m}|b(\Psi^{\varphi_i},z,\varphi_i)| &\le \frac{1}{2}\mu\delta^{-1}(\lambda_1^{-3}+2)(\|\omega\|^2+\gamma P_r^{1/2}\|\theta\|^2)\\
&\quad+\frac{\delta}{2}\sum_{i=1}^{m}\Big(\|\phi_i\|^2+\frac{\gamma}{P_r}\|\psi_i\|^2\Big).
\end{aligned}
$$

We recall that the dependence on s has been omitted and in fact $z = z(s,x), \rho = \rho(s,x)$, etc. Using this inequality in (5.10) we obtain

$$\mathrm{Tr}_m\big(F'(Z(s,\tau)z_0,s)\big) \le \frac{1}{2}\mu\delta^{-1}(\lambda_1^{-3}+2)(\|\omega\|^2+\gamma P_r^{1/2}\|\theta\|^2) - \frac{\delta}{2}\sum_{i=1}^m\bigg(\|\phi_i\|^2+\frac{\gamma}{P_r}\|\psi_i\|^2\bigg). \tag{5.17}$$

Since $\{\varphi_i\}_{i=1,\dots,m}$ is orthonormal in H, $|\phi_i|^2=\gamma|\psi_i|^2=1/2$. Using this and the Poincaré inequality (1.5) in (5.17) we have

$$\mathrm{Tr}_m\big(F'(Z(s,\tau)z_0,s)\big) \le \frac{1}{2}\mu\delta^{-1}(\lambda_1^{-3}+2)(\|\omega\|^2+\gamma P_r^{1/2}\|\theta\|^2) - m\frac{\delta\lambda_1}{4}\bigg(1+\frac{1}{P_r}\bigg).$$

Hence

$$\begin{aligned}\tilde q_m &= \limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\mathrm{Tr}_m\big(F'(Z(s,\tau-T)z_0,s)\big)\,ds\\ &\le \frac{\mu\delta^{-1}}{2}(\lambda_1^{-3}+2)\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}(\|\omega\|^2+\sqrt{P_r}\,\gamma\|\theta\|^2)\,ds\\ &\quad - m\frac{\delta\lambda_1}{4P_r}(1+P_r),\end{aligned}$$

for all $m\in\mathbb{N}$.

Let us now estimate the last term of the inequality above. From (1.2) and using (2.5), we obtain the energy estimates

$$\frac{d}{dt}|\omega|^2+\frac{1}{2}\|\omega\|^2 \le \bigg(\frac{R_a}{P_r}\bigg)^2\frac{1}{\lambda_1}\|\theta\|^2+2\|f\|_{H^{-1}}^2, \tag{5.18}$$

$$\frac{d}{dt}|\theta|^2+\frac{3}{2P_r}\|\theta\|^2 \le 2P_r\|g\|_{H^{-1}}^2. \tag{5.19}$$

Multiplying (5.19) by γ, adding to (5.18) and using (2.1), we obtain

$$\frac{d}{dt}|z|^2+\frac{1}{2}\|\omega\|^2+\frac{\gamma}{2P_r}\|\theta\|^2 \le 2(\|f\|_{H^{-1}}^2+P_r\gamma\|g\|_{H^{-1}}^2).$$

Then

$$\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\|\omega\|^2\,ds \le \delta_1,$$

$$\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{\gamma}{T}\int_{\tau-T}^{\tau}\|\theta\|^2\,ds \le P_r\delta_1,$$

where

$$\delta_1 := 4(\|f\|_{L^\infty(-\infty,T^*;H^{-1})}^2+P_r\gamma\|g\|_{L^\infty(-\infty,T^*;H^{-1})}^2).$$

Therefore,

$$\limsup_{T\to\infty} \sup_{z_0\in A(\tau-T)} \frac{1}{T}\int_{\tau-T}^{\tau} (\|\omega\|^2 + \gamma\sqrt{P_r}\,\|\theta\|^2)\,ds \le (1+P_r^{3/2})\delta_1.$$

Since γ satisfies (2.1) and we want to minimize the last term, we set henceforth $\gamma = R_a^2/(P_r\lambda_1)$. Then

$$\limsup_{T\to\infty} \sup_{z_0\in A(\tau-T)} \frac{1}{T}\int_{\tau-T}^{\tau} (\|\omega\|^2 + \gamma\sqrt{P_r}\,\|\theta\|^2)\,ds \le 4(1+P_r^{3/2})\Big(\|f\|^2_{L^\infty(-\infty,T^*;H^{-1})} + \frac{R_a^2}{\lambda_1}\|g\|^2_{L^\infty(-\infty,T^*;H^{-1})}\Big).$$

Hence

$$\tilde{q}_m \le -m\frac{\delta\lambda_1}{4}\Big(1+\frac{1}{P_r}\Big) + 2\Lambda,$$

where

$$\Lambda := \mu\delta^{-1}(\lambda_1^{-3}+2)(1+P_r^{3/2})\Big(\|f\|^2_{L^\infty(-\infty,T^*;H^{-1})} + \frac{R_a^2}{\lambda_1}\|g\|^2_{L^\infty(-\infty,T^*;H^{-1})}\Big). \tag{5.20}$$

We now consider two cases:

Case 1: If $\Lambda < (\delta\lambda_1/8)(1+1/P_r)$, then taking

$$q_m = -\frac{\delta\lambda_1}{4}\Big(1+\frac{1}{P_r}\Big)(m-1), \qquad m=1,2,\ldots,$$

and $n_0 = 1$, we can apply Theorem 2.2 to obtain

$$d_F(A(\tau)) \le 1 \quad \text{for all } \tau \le T^*.$$

Case 2: If $\Lambda \ge (\delta\lambda_1/8)(1+1/P_r)$, then taking

$$q_m = -m\frac{\delta\lambda_1}{4}\Big(1+\frac{1}{P_r}\Big) + 2\Lambda, \qquad m=1,2,\ldots,$$

and

$$n_0 = 1 + \Big[\frac{8\Lambda P_r}{\delta\lambda_1(1+P_r)} - 1\Big],$$

where $[r]$ denotes the integer part of the real number r, we obtain

$$d_F(A(\tau)) \le \frac{8\Lambda P_r}{\delta\lambda_1(1+P_r)} \quad \text{for all } \tau \le T^*.$$

Finally, since $Z(t,\tau)$ is Lipschitz in $A(\tau)$, it follows from [14, Proposition 13.9] that $d_F(A(t))$ is bounded for every $t \ge \tau$ with the same bound. ■

Acknowledgements. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2012.04.

References

[1] J. M. Ball, *Global attractor for damped semilinear wave equations*, Discrete Contin. Dynam. Systems 10 (2004), 31–52.
[2] T. Caraballo, G. Łukaszewicz and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Anal. 64 (2006), 484–498.
[3] A. Carvalho, J. A. Langa and J. C. Robinson, *Attractors for Infinite-Dimensional Non-autonomous Dynamical Systems*, Springer, New York, 2013.
[4] S. Chen, *Symmetry analysis of convection on patterns*, Comm. Theor. Phys. 1 (1982), 413–426.
[5] S. Fang, L. Jin and B. Guo, *Global existence of solutions to the periodic initial value problems for two-dimensional Newton–Boussinesq equations*, Appl. Math. Mech. English Ed. 31 (2010), 405–414.
[6] M. J. Feigenbaum, *The onset spectrum of turbulence*, Phys. Lett. A 74 (1979), 375–378.
[7] G. Fucci, B. Wang and P. Singh, *Asymptotic behavior of the Newton–Boussinesq equation in a two-dimensional channel*, Nonlinear Anal. 70 (2009), 2000–2013.
[8] B. Guo, *Spectral method for solving the two dimensional Newton–Boussinesq equations*, Acta Math. Appl. 5 (1989), 208–218.
[9] B. Guo, *Nonlinear Galerkin methods for solving the two-dimensional Newton–Boussinesq equations*, Chin. Ann. of Math. 16 (1995), 379–390.
[10] B. Guo and B. Wang, *Approximate inertial manifolds for the two-dimensional Newton–Boussinesq equations*, J. Partial Differential Equations 9 (1996), 237–250.
[11] B. Guo and B. Wang, *Gevrey class regularity and approximate inertial manifolds for the Newton–Boussinesq equations*, Chin. Ann. of Math. 19 (1998), 179–188.
[12] A. A. Ilyin, *Lieb–Thirring integral inequalities and their applications to the attractors of the Navier–Stokes equations*, Sb. Math. 196 (2005), 29–61.
[13] J. A. Langa, G. Łukaszewicz and J. Real, *Finite fractal dimension of pullback attractors for non-autonomous 2D Navier–Stokes equations in some unbounded domains*, Nonlinear Anal. 66 (2007), 735–749.
[14] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge Univ. Press, Cambridge, 2001.
[15] R. Rosa, *The global attractor for the 2D Navier–Stokes flow on some unbounded domains*, Nonlinear Anal. 32 (1998), 71–85.
[16] X. L. Song and Y. R. Hou, *Pullback D-attractors for the non-autonomous Newton–Boussinesq equation in two-dimensional bounded domains*, Discrete Contin. Dynam. Systems 32 (2012), 991–1009.
[17] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, 2nd ed., North-Holland, Amsterdam, 1979.
[18] B. Wang, *Attractors for reaction-diffusion equations in unbounded domains*, Phys. D 128 (1999), 41–52.

Cung The Anh (corresponding author)
Department of Mathematics
Hanoi University of Education
136 Xuan Thuy, Cau Giay
Hanoi, Vietnam
E-mail: anhctmath@hnue.edu.vn

Dang Thanh Son
Foundation Sciences Faculty
Telecommunication University
101 Mai Xuan Thuong, Nha Trang
Khanh Hoa, Vietnam
E-mail: dangthanhson1810@gmail.com

Received October 30, 2014 (7996)