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AN INEQUALITY FOR IMAGINARY PARTS OF EIGENVALUES OF NON-COMPACT OPERATORS WITH HILBERT–SCHMIDT HERMITIAN COMPONENTS

Michael Gil'

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Abstract. Let A be a bounded linear operator in a complex separable Hilbert space, A^* be its adjoint one and $A_I := (A - A^*)/(2i)$. Assuming that A_I is a Hilbert–Schmidt operator, we investigate perturbations of the imaginary parts of the eigenvalues of A. Our results are formulated in terms of the "extended" eigenvalue sets in the sense introduced by T. Kato. Besides, we refine the classical Weyl inequality $\sum_{k=1}^{\infty}(\operatorname{Im}\lambda_k(A))^2 \leq N_2^2(A_I)$, where $\lambda_k(A)$ $(k = 1, 2, \ldots)$ are the eigenvalues of A and $N_2(\cdot)$ is the Hilbert–Schmidt norm. In addition, we discuss applications of our results to the Jacobi operators.

Keywords: Hilbert space, linear operators, eigenvalues.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(\cdot,\cdot)$ and the unit operator I, $\mathcal{B}(\mathcal{H})$ be the set of linear bounded operators in $\mathcal{H}$. For an $A \in \mathcal{B}(\mathcal{H})$, A^* is the adjoint one, $\|A\|$ is the operator norm, $\sigma(A)$ is the spectrum, $A_I = (A-A^*)/(2i)$ and $\lambda_k(A)$ $(k = 1, 2, \ldots)$ denote the nonreal eigenvalues of A taken with their multiplicities and enumerated in the order pointed below. By $\mathcal{S}_2$ we denote the Hilbert–Schmidt ideal of linear compact operators C in $\mathcal{H}$ with the finite norm $N_2(C) := [\operatorname{trace}\,(C^*C)]^{1/2}$.

If

$$A \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad A_I \in \mathcal{S}_2, \tag{1.1}$$

then due to the well known Lemma I.4.1 from [15] the non-real spectrum of A consists of the isolated eigenvalues, which satisfy the classical Weyl inequality

$$\sum_{j=1}^{\infty}(\operatorname{Im}\lambda_j(A))^2 \leq N_2^2(A_I), \tag{1.2}$$

cf. [15, Section II.6]. The literature devoted to estimates on imaginary or real parts of eigenvalues is rather rich, but mainly finite dimensional and compact operators were investigated, cf. [6–10, 13, 16, 18, 20, 23, 24]. Below we do not assume that the considered operators are compact and refine inequality (1.2).

Furthermore, let C be a compact operator, then following [19] (see also [2]), by *an enumeration of the eigenvalues of* C we shall mean a sequence $\lambda_1(C), \lambda_2(C), \ldots$ whose terms consist of all the eigenvalues of C each counted as often as its multiplicity. By an *extended enumeration of the eigenvalues of* C we shall mean a sequence $\lambda_1'(C), \lambda_2'(C), \ldots$ whose terms consist of all the nonzero eigenvalues of C each counted as often as its multiplicity and the term 0 repeated infinitely often.

Besides, if $C \in \mathcal{B}(\mathcal{H})$ has a compact hermitian component C_I by *an enumeration of the imaginary parts of the eigenvalues of* C we shall mean a sequence $\mathrm{Im}\ \lambda_1(C), \mathrm{Re}\ \mathrm{Im}\ \lambda_2(C), \ldots$ whose terms consist of all the imaginary parts of the eigenvalues of C each counted as often as its multiplicity. By an *extended enumeration of the imaginary parts of eigenvalues of* C we shall mean a sequence $\mathrm{Im}\ \lambda_1'(C), \mathrm{Im}\ \lambda_2'(C), \ldots$ whose terms consist of all imaginary parts of the eigenvalues of A each counted as often as its multiplicity and the term 0 repeated infinitely often.

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let condition* (1.1) *hold and* $S \in \mathcal{S}_2$ *be selfadjoint. Let* $\mathrm{Im}\ \lambda_1'(A)$*,* $\mathrm{Im}\ \lambda_2'(A)$*, ... be the extended enumeration of the imaginary parts of eigenvalues of* A *and* $\lambda_1'(S), \lambda_2'(S), \ldots$ *be the extended enumeration of the eigenvalues of* S*. Then there exists a permutation* π *of the natural numbers, such that*

$$\left[\sum_{k=1}^{\infty}(\mathrm{Im}\ \lambda_k'(A) - \lambda_{\pi(k)}'(S))^2\right]^{1/2} \leq N_2(A_I - S) + \left[N_2^2(A_I) - \sum_{k=1}^{\infty}(\mathrm{Im}\ \lambda_k(A))^2\right]^{1/2}. \tag{1.3}$$

The proof of this theorem is presented in the next section. Theorem 1.1 is sharp in the following sense: if $A = iA_I$, i.e. the real Hermitian component of A is equal to zero, then (1.3) coincides (in the case of selfadjoint operators) with the following well-known result, proved in [2, Theorem 2].

Theorem 1.2. *Let* Z *and* $\tilde{Z}$ *be normal Hilbert–Schmidt operators and let* $\{\lambda_1'(Z), \lambda_2'(Z), \ldots\}$ *and* $\{\lambda_1'(\tilde{Z}), \tilde{\lambda_2}'(\tilde{Z}), \ldots\}$ *be extended enumerations of their eigenvalues. Then there exists a permutation* π *of the natural numbers, such that*

$$\left[\sum_{k=1}^{\infty}(\lambda_k'(Z) - \lambda_{\pi(k)}'(\tilde{Z}))^2\right]^{1/2} \leq N_2(Z - \tilde{Z}).$$

This theorem is an infinite-dimensional analogues of the Hoffman–Wielandt Theorem on perturbations of finite normal matrices. So Theorem 1.1 extends Theorem 1.2 to the class of non-normal ones satisfying (1.1).

The perturbation theory of operators is very rich (see [1, 4, 12, 14, 17, 25] and the references therein), but to the best of our knowledge the perturbation bounds are derived mainly for the absolute values of eigenvalues. At the same time the variations of imaginary and real parts of eigenvalues are investigated considerably less than the absolute values.

Furthermore, take $S = A_I$. Then Theorem 1.1 gives us the following result.

Corollary 1.3. *Let condition* (1.1) *hold. Then there exists a permutation* π *of the natural numbers, such that*

$$\sum_{k=1}^{\infty}(\mathrm{Im}\ \lambda_k(A))^2 + \sum_{k=1}^{\infty}(\mathrm{Im}\ \lambda_k'(A) - \lambda_{\pi(k)}'(A_I))^2 \leq N_2^2(A_I). \tag{1.4}$$

Clearly, (1.4) refines (1.2). Corollary 1.3 is sharp: 1.4 is attained, when A is normal, since in this case $\mathrm{Im}\ \lambda_k(A) = \lambda_k(A_I)$ $(k = 1, 2, \ldots)$.

In the next section we also prove the following corollary of Theorem 1.1

Corollary 1.4. *Under the hypothesis of Theorem 1.1 one has*

$$\left[\sum_{k=1}^{\infty}(\mathrm{Im}\ \lambda_k'(A) - \lambda_{\pi(k)}'(S))^2\right]^{1/2}$$
$$\leq \frac{1}{3}\left[2N_2(S) + (6N_2^2(A_I) + 6N_2^2(A_I - S) - 2N^2(S))^{1/2}\right].$$

This corollary is less sharp but more convenient than Theorem 1.1.

Below we also discuss application of Theorem 1.1 to non-selfadjoint Jacobi operators. The spectral theory of Jacobi operators is a classical subject with many beautiful results, though the majority of results are related to selfadjoint and compact Jacobi operators, cf. [3, 5, 21, 22, 26]. Using Theorem 1.1, we are able to obtain new results on the eigenvalues of Jacobi operators.

2. PROOFS

Recall that a bounded linear operator V, satisfying the condition $r_s(V) = 0$ is called a quasi-nilpotent operator, cf. [15]. Here $r_s(\cdot)$ denotes the spectral radius. So any quasi-nilpotent operator V has the property $\sigma(V) = \{0\}$.

For two orthogonal projections P_1, P_2 in $\mathcal{H}$ we write $P_1 < P_2$ if $P_1\mathcal{H} \subset P_2\mathcal{H}$. A set $\mathcal{P}$ of orthogonal projections in $\mathcal{H}$ containing at least two orthogonal projections is called *a chain*, if from $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ it follows that either $P_1 < P_2$ or $P_1 > P_2$. For two chains $\mathcal{P}_1, \mathcal{P}_2$ we write $\mathcal{P}_1 < \mathcal{P}_2$ if from $P \in \mathcal{P}_1$ it follows that $P \in \mathcal{P}_2$. In this case we say that $\mathcal{P}_1$ precedes $\mathcal{P}_2$. The chain that precedes only itself is called *a maximal chain*. For more details about maximal chains see [11, Section 8.1] and the references given therein.

Let $A = D + V$, where $D \in \mathcal{B}(\mathcal{H})$ is a normal operator and V is a compact quasi-nilpotent operator in $\mathcal{H}$. Let V have a maximal invariant chain $\mathcal{P}$ and $PD = DP$ for all $P \in \mathcal{P}$. In addition, let the essential spectrum of A lie on an unclosed Jordan curve. Then following Definition 8.1 from [11] we will say that A is *a $\mathcal{P}$-triangular operator, equality $A = D + V$ is its triangular representation, D and V are the diagonal and nilpotent parts of A, respectively.*

Due to Corollary 8.2 from [11] for any $\mathcal{P}$-triangular operator, we have $\sigma(A) = \sigma(D)$, where D is the diagonal part of A. Due to [11, Theorem 8.8], under condition (1.1) A is a $\mathcal{P}$-triangular operator, and therefore

$$A = D + V \quad (\sigma(A) = \sigma(D)). \tag{2.1}$$

Since D is normal, $\text{Im } \lambda_k(A) = \text{Im } \lambda_k(D) = \lambda_k(D_I)$ $(D_I = (D - D^*))/(2i))$. So due to (1.1) and (1.2) $D_I \in \mathcal{S}_2$, and therefore $V_I = (V - V^*))/(2i) = A_I - D_I \in \mathcal{S}_2$. Moreover, Lemma 9.3 from [11, p. 149] yields

$$N_2(V_I) = g_I(A), \tag{2.2}$$

where

$$g_I(A) = \left[N_2^2(A_I) - \sum_{k=1}^{\infty}(\text{Im } \lambda_k(A))^2\right]^{1/2}.$$

Since D_I and S are selfadjoint Hilbert–Schmidt operators, due to Theorem 1.2, there exists a permutation π of natural numbers, such that

$$\sum_{k=1}^{\infty}(\lambda'_k(D_I) - \lambda'_{\pi(k)}(S))^2 \le N_2^2(D_I - S).$$

Furthermore, take into account that (2.1) implies

$$N_2(D_I - S) = N_2(A_I - V_I - S) \le N_2(A_I - S) + N_2(V_I).$$

We thus have proved the following lemma.

Lemma 2.1. *Let condition* (1.1) *hold and S be a selfadjoint Hilbert–Schmidt operator. Then there exists a permutation π of natural numbers, such that*

$$\left[\sum_{k=1}^{\infty}(\text{Im } \lambda'_k(A) - \lambda'_{\pi(k)}(S))^2\right]^{1/2} \le N_2(A_I - S) + N_2(V_I),$$

where V is the nilpotent part of A.

The assertion of Theorem 1.1 follows from Lemma 2.1 and (2.2).

Proof of Corollary 1.4. Since

$$(a + b)^2 \le 2a^2 + 2b^2 \quad (a, b > 0),$$

Theorem 1.1 yields

$$\sum_{k=1}^{\infty}(\text{Im } \lambda'_k(A) - \lambda'_{\pi(k)}(S))^2 + 2\sum_{k=1}^{\infty}(\text{Im } \lambda_k(A))^2 \le 2N_2^2(A_I - S) + 2N_2^2(A_I). \tag{2.3}$$

With $x_k = \text{Im } \lambda'_k(A) - \lambda'_{\pi(k)}(S)$, $c_k = \lambda'_{\pi(k)}(S)$, we have

$$\sum_{k=1}^{\infty}(\text{Im } \lambda_k(A))^2 = \sum_{k=1}^{\infty}(\text{Im } \lambda'_k(A))^2 = \sum_{k=1}^{\infty}(x_k + c_k)^2 = \sum_{k=1}^{\infty}(x_k^2 + 2x_k c_k + c_k^2).$$

Applying to the Schwarz inequality, we have

$$\sum_{k=1}^{\infty}|x_k c_k| \leq \hat{x}\hat{c},$$

where

$$\hat{x} = \left(\sum_{k=1}^{\infty} x_k^2\right)^{1/2} = \left(\sum_{k=1}^{\infty} \text{Im } \lambda'_k(A) - \lambda'_{\pi(k)}(S)\right)^{1/2}$$

and

$$\hat{c} = \left(\sum_{k=1}^{\infty} c_k^2\right)^{1/2} = \left(\sum_{k=1}^{\infty}\left(\lambda'_{\pi(k)}(S)^2\right)\right)^{1/2} = \left(\sum_{k=1}^{\infty}\left(\lambda_k(S)^2\right)\right)^{1/2} = N_2(S).$$

Now (2.3) implies

$$\hat{x}^2 - \frac{4}{3}N_2(S)\hat{x} \leq \gamma,$$

where

$$\gamma = \frac{2}{3}(N_2^2(A_I) + N_2^2(A_I - S) - N^2(S)).$$

Solving this inequality, we get

$$\begin{aligned}
\hat{x} &\leq \frac{2}{3}N_2(S) + \left[\frac{4}{9}N_2^2(S) + \gamma\right]^{1/2} \\
&= \frac{2}{3}N_2(S) + \left[\frac{4}{9}N_2^2(S) + \frac{2}{3}N_2^2(A_I) + \frac{2}{3}N_2^2(A_I - S) - \frac{2}{3}N^2(S))^{1/2}\right] \\
&= \frac{1}{3}\left[2N_2(S) + (6N_2^2(A_I) + 6N_2^2(A_I - S) - 2N_2(S))^{1/2}\right],
\end{aligned}$$

as claimed. □

3. JACOBI OPERATORS

Let l^2 be the Hilbert space of complex number sequences $x = \{x_k\}_{k=1}^{\infty}, y = \{y_k\}_{k=1}^{\infty}$ with the scalar product

$$(x, y) = \sum_{k=1}^{\infty} x_k \overline{y}_k.$$

Given bounded sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ $(k = 1, 2, \dots)$, we define the associated Jacobi operator $J : l^2 \to l^2$ as follows:

$$(Jx)_1 = a_1x_1 + c_1x_2, \quad (Jx)_k = b_{k-1}x_{k-1} + a_kx_k + c_kx_{k+1}$$

$(k = 2, 3, \dots; x = (x_j) \in l^2)$. In the matrix form we have

$$J = \begin{pmatrix} a_1 & c_1 & 0 & 0 & 0 & \dots \\ b_1 & a_2 & c_2 & 0 & 0 & \dots \\ 0 & b_2 & a_3 & c_3 & 0 & \dots \\ 0 & 0 & b_3 & a_4 & c_4 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix}.$$

Put $w_k = (c_k - \overline{b}_k)/(2i)$. Then

$$J_I := \frac{1}{2i}(J - J^*) = \begin{pmatrix} \text{Im } a_1 & w_1 & 0 & 0 & 0 & \dots \\ \overline{w}_1 & \text{Im } a_2 & w_2 & 0 & 0 & \dots \\ 0 & \overline{w}_2 & \text{Im } a_3 & w_3 & 0 & \dots \\ 0 & 0 & \overline{w}_3 & \text{Im } a_4 & w_4 & \dots \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \end{pmatrix}.$$

It is assumed that

$$N_2^2(J_I) := \sum_{k=1}^{\infty} 2|w_k|^2 + (\text{Im } a_k)^2 < \infty. \tag{3.1}$$

Taking $S = \text{diag } (\text{Im } a_k)_{k=1}^{\infty}$, we have $\lambda_k(S) = \text{Im } a_k$, and due to Corollary 1.4 we get the following result

Corollary 3.1. *Let condition* (3.1) *hold. Then there exists a permutation π of natural numbers, such that*

$$\left(\sum_{k=1}^{\infty} \text{Im } \lambda_k'(J) - a_{\pi(k)}' \right)^{1/2}$$
$$\leq \frac{1}{3}[2N_2(S) + (6N_2^2(J_I) + 12 \sum_{k=1}^{\infty} |w_k|^2 - 2N^2(S))^{1/2}],$$

where

$$N_2^2(S) = \sum_{k=1}^{\infty} |\text{Im } a_k|^2, \quad N_2^2(J_I - S) = 2 \sum_{k=1}^{\infty} |w_k|^2,$$

and $\{\text{Im } a_{\pi(k)}'\}$ *and* $\{\text{Im } \lambda_k'(J)\}$ *are the extended enumerations of the sets* $\{\lambda_k(S) = \text{Im } a_k\}$ *and* $\{\text{Im } \lambda_k(J)\}$*, respectively.*

REFERENCES

[1] P. Aiena, S. Triolo, *Weyl-type theorems on Banach spaces under compact perturbations*, Mediterr. J. Math. **15** (2018), no. 3, Article no. 126.

[2] R. Bhatia, L. Elsner, *The Hoffman–Wielandt inequality in infinite dimensions*, Proc. Indian Acad. Sci. (Math. Sci.) **104** (1994), no. 3, 483–494.

[3] V.S. Budyka, M.M. Malamud, *Deficiency indices and discreteness property of block Jacobi matrices and Dirac operators with point interactions*, J. Math. Anal. Appl. **506** (2022), no. 1, Article no. 125582.

[4] W. Chaker, A. Jeribi, B. Krichen, *Demicompact linear operators, essential spectrum and some perturbation results*, Math. Nachr. **288** (2015), no. 13, 1476–1486.

[5] W. Geng, K. Tao, *Large deviation theorems for Dirichlet determinants of analytic quasi-periodic Jacobi operators with Brjuno-Rüssmann frequency*, Commun. Pure Appl. Anal. **19** (2020), no. 12, 5305–5335.

[6] M.I. Gil', *Lower bounds for eigenvalues of Schatten-von Neumann operators*, J. Inequal. Pure Appl. Math. **8** (2007), no. 3, Article no. 66.

[7] M.I. Gil', *Sums of real parts of eigenvalues of perturbed matrices*, J. Math. Inequal. **4** (2010), no. 4, 517–522.

[8] M.I. Gil', *Bounds for eigenvalues of Schatten-von Neumann operators via self-commutators*, J. Funct. Anal. **267** (2014), no. 9, 3500–3506.

[9] M.I. Gil', *A bound for imaginary parts of eigenvalues of Hilbert–Schmidt operators*, Funct. Anal. Approx. Comput. **7** (2015), no. 1, 35–38.

[10] M.I. Gil', *Inequalities for eigenvalues of compact operators in a Hilbert space*, Commun. Contemp. Math. **18** (2016), no. 1, Article no. 1550022.

[11] M.I. Gil', *Operator Functions and Operator Equations*, World Scientific, New Jersey, 2018.

[12] M.I. Gil', *Norm estimates for resolvents of linear operators in a Banach space and spectral variations*, Adv. Oper. Theory **4** (2019), no. 1, 113–139.

[13] M.I. Gil', *Bounds for absolute values and imaginary parts of matrix eigenvalues via traces*, Proyecciones **41** (2022), no. 5, 1229–1237.

[14] M.I. Gil', *On matching distance between eigenvalues of unbounded operators*, Constr. Math. Anal. **5** (2022), no. 1, 46–53.

[15] I.C. Gohberg, M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Mathem. Monographs, vol. 18, Amer. Math. Soc., Providence, R. I., 1969.

[16] I.H. Gumus, O. Hirzallah, F. Kittaneh, *Eigenvalue localization for complex matrices*, Electron. J. Linear Algebra **27** (2014), 892–906.

[17] A. Jeribi, *Perturbation Theory for Linear Operators. Denseness and Bases with Applications*, Springer-Verlag Singapore, 2021.

[18] W. Kahan, *Spectra of nearly Hermitian matrices*, Proc. Amer. Math. Soc. **48** (1975), 11–17.

[19] T. Kato, *Variation of discrete spectra*, Commun. Math. Phys. **111** (1987), 501–504.

[20] M. Kian, M. Bakherad, *A new estimation for eigenvalues of matrix power functions*, Anal. Math. **45** (2019), no. 3, 527–534.

[21] M. Malejki, *Approximation of eigenvalues of some unbounded self-adjoint discrete Jacobi matrices by eigenvalues of finite submatrices*, Opuscula Math. **27** (2007), no. 1, 37–49.

[22] M. Malejki, *Approximation and asymptotics of eigenvalues of unbounded self-adjoint Jacobi matrices acting in* l^2 *by the use of finite submatrices*, Cent. Eur. J. Math. **8** (2010), 114–128.

[23] O. Rojo, *Inequalities involving the mean and the standard deviation of nonnegative real numbers*, J. Inequal. Appl. (2006), Article no. 43465, 1–15.

[24] O. Rojo, R.L. Soto, H. Rojo, *New eigenvalue estimates for complex matrices*, Comput. Math. Appl. **25** (1993), no. 3, 91–97.

[25] M.L. Sahari, A.K. Taha, L. Randriamihamison, *A note on the spectrum of diagonal perturbation of weighted shift operator*, Matematiche (Catania) **74** (2019), no. 1, 35–47.

[26] M. Webb, S. Olver, *Spectra of Jacobi operators via connection coefficient matrices*, Comm. Math. Phys. **382** (2021), no. 2, 657–707.

Michael Gil'
gilmi@bezeqint.net

Department of Mathematics
Ben Gurion University of the Negev
P.O. Box 653, Beer-Sheva 84105, Israel

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