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CONTROL SYSTEM DEFINED BY SOME INTEGRAL OPERATOR

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Abstract. In the paper we consider a nonlinear control system governed by the Volterra integral operator. Using a version of the global implicit function theorem we prove that the control system under consideration is well-posed and robust, i.e. for any admissible control u there exists a uniquely defined trajectory x_u which continuously depends on control u and the operator $u \mapsto x_u$ is continuously differentiable. The novelty of this paper is, among others, the application of the Bielecki norm in the space of solutions which allows us to weaken standard assumptions.

Keywords: Volterra equation, implicit function theorem, sensitivity.

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1. INTRODUCTION

In the paper we investigate the following system described by a nonlinear integral equation of the Volterra type with a functional parameter u

$$x(t) + \int_0^t V\left(t, \tau, x(\tau), u(\tau)\right) d\tau = 0, \quad t \in [0, 1]. \tag{1.1}$$

The parameter u can be referred to as a control.

We shall consider system (1.1) in the space of absolutely continuous trajectories $\bar{H}^1$ and in the set $\mathcal{U}$ of controls. Since the trajectories of control system (1.1) are absolutely continuous on interval $[0, 1]$, we see that it reduces to the classical control system described by ordinary differential equations, provided the function V does not depend on t (see Example 4.2)

Under some appropriate assumption, to be specified in the next section we prove that for any admissible control u there exists exactly one trajectory x_u to system (1.1) and the operator $u \mapsto x_u$ is differentiable in the Fréchet sense. It implies that control system (1.1) is stable and robust. Here, the stability means that solution x_u depends continuously on parameter u. In engineering robustness is usually understood as *the ability of a system to resist change without adapting its initial stable configuration* [5]. If the operator $u \mapsto x_u$ is differentiable then the change of the solution caused by the change of the parameter is not rapid. The speed of this change can be expressed by the derivative of the operator $u \mapsto x_u$, which measures the resistance of the system to the change of u.

Integral operators and integral equations have essential applications in technical sciences, physics, biology etc. In particular, Volterra operators in the form of convolution

$$x(t) + \int_0^t w(t-\tau) \cdot z(x(\tau), u(\tau))\, d\tau = 0$$

with nonlinear function z (especially with respect to x) has many applications in general control theory with feedback loops, see [8, 13, 15] and references therein, in the mathematical model of nuclear reactors, see [11, 12, 17], in the study of viscoelastic materials with memory, see [18] and in epidemiology see for example [4] and the book [8] by Gripenberg *et al.*

The paper is organised as follows. In section 2 we introduce the problem, assumptions and discuss the space of solutions and parameters. Section 3 includes the main result of the paper, that is, Theorem 3.4. Finally, in section 4 we calculate a simple Example 4.1 and discuss the application of the result to the classical Cauchy problem (Example 4.2).

The problem of the smooth dependence of solutions on the parameter of integral or integro-differential equations was discussed in papers [2, 6, 7, 9, 10].

The novelty of this paper, in contrast to papers [9, 10] is, among others, the application of the Bielecki norm in the space of solutions which allows us to weaken one of the assumptions (see Remark 2.3 for details).

2. BASIC NOTIONS AND ASSUMPTIONS

Let us consider the integro-differential system (1.1), where $V : P_\triangle \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, I is the unit interval in $\mathbb{R}$, i.e. $I := [0,1]$ and $P_\triangle := \left\{(t,\tau) \in \mathbb{R}^2 : 0 \le \tau \le t \le 1\right\}$. We consider system (1.1) in the space $\bar{H}^1 = \bar{H}^1(I, \mathbb{R}^n)$ of absolutely continuous functions $x : I \to \mathbb{R}^n$ such that $x(0) = 0$ and the derivative $\dot{x}$ is a square integrable function, i.e. $\dot{x} \in L^2(I, \mathbb{R}^n)$. We assume that the parameter (control) u belongs to the set $\mathcal{U} := L^\infty\left(I, \mathbb{R}^k\right)$.

For $m \ge 0$, let

$$\|x\|_m := \left(\int_0^1 e^{-mt} \left|\dot{x}(t)\right|^2 dt \right)^{\frac{1}{2}}. \tag{2.1}$$

It is easy to see that for any $m \geq 0$ the function $\|\cdot\|_m$ defines a norm in $\bar{H}^1$ (the so-called Bielecki norm, see [1]) and

$$e^{-\frac{m}{2}} \|x\|_0 \leq \|x\|_m \leq \|x\|_0 . \tag{2.2}$$

Thus, for any $m \geq 0$ the norms $\|\cdot\|_0$ and $\|\cdot\|_m$are equivalent and therefore any two norms $\|\cdot\|_{m_1}$ and $\|\cdot\|_{m_1}$ are equivalent.

It is worth pointing out the in $\bar{H}^1$ we can define the inner product by the formula

$$\langle x, y\rangle_m := \int_0^1 e^{-mt} \langle \dot{x}(t), \dot{y}(t)\rangle_{\mathbb{R}^n} \, dt, \tag{2.3}$$

where $m \geq 0$ is a given number. The above inner product defines the norm $\|\cdot\|_m$ and the space $\bar{H}^1$ is a Hilbert space.

Remark 2.1. It can be proved (see [14]) that the weak convergence $x_s \rightharpoonup x_0$ in $\bar{H}^1$ implies the uniform convergence $x_s \rightrightarrows x_0$ and the weak convergence $\dot{x}_s \rightharpoonup \dot{x}_0$ of derivatives in L^2. Also, it is worth pointing out that in [14] the proof of the uniform convergence (cf. [14, Lemma 1.2]) is based on the Arzeli theorem which gives the convergence only for a subsequence. However, the uniform convergence is true for the whole sequence. This can be justified by a standard argument. Suppose that $x_s \rightharpoonup x_0$ in $\bar{H}^1$ and x_s does not tend to x_0 uniformly (but the convergence is true for a subsequence). Since x_s does not tend to x_0 uniformly we can choose a subsequence whose terms are "far from x_0", but this subsequence also tends weakly to x_0 and we can choose a subsequence which tends uniformly to x_0. So, we get a contradiction.

For $m \geq 0$, let

$$\|x\|_{L^2,m} := \left(\int_0^1 e^{-mt} |x(t)|^2 \, dt \right)^{\frac{1}{2}}$$

for $x \in L^2(I, \mathbb{R}^n)$. It is obvious, that the above formula defines the family of equivalent norms in $L^2(I, \mathbb{R}^n)$. Let us now state and prove a technical lemma which we will simplify our further considerations.

Lemma 2.2. *For any $x \in \bar{H}^1$ and $m > 0$ one has*

$$\|x\|_{L^2,m} \leq \frac{\|x\|_m}{\sqrt{m}}$$

and

$$\left\| \int_0^{\cdot} |x(\tau)| \, d\tau \right\|_{L^2,m} = \left(\int_0^1 e^{-mt} \left(\int_0^t |x(\tau)|^2 \, d\tau \right)^2 dt \right)^{\frac{1}{2}} \leq \frac{\|x\|_m}{m},$$

where for a given integrable function z the symbol $\int_0^{\cdot} z(\tau)d\tau$ denotes the function $[0,1] \ni t \mapsto \int_0^t z(\tau)d\tau$.

Proof. Fix $m > 0$. Of course, if $x \in \bar{H}^1$ then $x \in L^2(I, \mathbb{R}^n)$. Using the Schwartz inequality and integrating by parts we get

$$\|x\|^2_{L^2,m} = \int_0^1 e^{-mt} |x(t)|^2 dt = \int_0^1 e^{-mt} \left| \int_0^t \dot{x}(\tau) d\tau \right|^2 dt \leq \int_0^1 e^{-mt} \int_0^t |\dot{x}(\tau)|^2 d\tau dt$$
$$= \left[-\frac{1}{m} e^{-mt} \int_0^t |\dot{x}(\tau)|^2 d\tau \right]_{t=0}^{t=1} + \frac{1}{m} \int_0^1 e^{-mt} |\dot{x}(t)|^2 dt \leq \frac{\|x\|^2_m}{m}$$

and from the above inequality

$$\left\| \int_0^{\cdot} |x(\tau)| d\tau \right\|^2_{L^2,m} = \int_0^1 e^{-mt} \left(\int_0^t |x(\tau)| d\tau \right)^2 dt \leq \int_0^1 e^{-mt} \left(\int_0^t |x(\tau)|^2 d\tau \right) dt$$
$$= \left[-\frac{1}{m} e^{-mt} \int_0^t |x(\tau)|^2 d\tau \right]_{t=0}^{t=1} + \frac{1}{m} \int_0^1 e^{-mt} |x(t)|^2 dt$$
$$\leq \frac{1}{m} \|x\|^2_{L^2,m} \leq \frac{\|x\|^2_m}{m^2}.$$

□

We impose the following assumptions on the function V defining system (1.1):

(A1) the function $V(\cdot, \tau, \cdot, \cdot)$ is continuous on the set $G := I \times \mathbb{R}^n \times \mathbb{R}^k$ for a.e. $\tau \in I$; there exist derivatives V_t, V_x, V_{xt}, V_u, V_{ut} and the functions $V_t(\cdot, \tau, \cdot, \cdot)$, $V_x(\cdot, \tau, \cdot, \cdot)$, $V_{xt}(\cdot, \tau, \cdot, \cdot)$, $V_u(\cdot, \tau, \cdot, \cdot)$, $V_{ut}(\cdot, \tau, \cdot, \cdot)$ are continuous on G for a.e. $\tau \in I$,

(A2) the functions $V, V_t, V_x, V_{xt}, V_u, V_{ut}$ are measurable with respect to τ for $(t, x, u) \in I \times \mathbb{R}^n \times \mathbb{R}^k$ and locally bounded with respect to x and u, i.e. for any $\rho_1, \rho_2 > 0$ there exists a function $M \in L^2(I, \mathbb{R})$ such that

$$|V(t,\tau,x,u)|, |V_t(t,\tau,x,u)|, |V_x(t,\tau,x,u)|, |V_{xt}(t,\tau,x,u)|,$$
$$|V_u(t,\tau,x,u)|, |V_{ut}(t,\tau,x,u)| \leq M(\tau)$$

for $t \in I$, $|x| \leq \rho_1$, $|u| \leq \rho_2$ and a.e. $\tau \in I$.

(A3) for any $u \in L^\infty(I, \mathbb{R}^k)$ there exists a constant $\bar{A} > 0$ and functions $B_1, C \in L^2(I, \mathbb{R}^+)$ and $A, B_2 \in L^2(P_\triangle, \mathbb{R}^+)$ such that

$$\int_0^t A^2(t,\tau) d\tau \leq \bar{A}^2$$

for a.e. $t \in [0,1]$ and

$$|V(t,t,x,u(t))| \leq C(t)|x| + B_1(t)$$

for $x \in \mathbb{R}^n$ and a.e. $t \in I$ and

$$|V_t(t, \tau, x, u(\tau))| \leq A(t,\tau)|x| + B_2(t,\tau)$$

for $x \in \mathbb{R}^n$ and a.e. $(t,\tau) \in P_\triangle$.

Remark 2.3. It should be emphasized that we do not make any assumption on the constant $\bar{A}$ in (A3). If we compare this with the corresponding assumption in papers [6,9] or [7] we see that it is usually assumed that the coefficient in the linearity should satisfy a specific additional condition connected with the diagonal of the square $[0,1]^2$. This condition is removed in our paper thanks to the application of the Bielecki norm.

3. MAIN RESULT

In this section we will prove the main result of this paper, i.e. that for any $u \in L^\infty(I, \mathbb{R}^k)$ there exists exactly one solution $x_u \in \bar{H}^1$ to (1.1) and the solution depends in a differentiable way on the parameter u.

Define the operator $F : \bar{H}^1 \times L^\infty(I, \mathbb{R}^k) \to \bar{H}^1$ by

$$F(x,u)(t) := x(t) + \int_0^t V(t, \tau, x(\tau), u(\tau))\, d\tau, \tag{3.1}$$

where $x \in \bar{H}^1, u \in L^\infty(I, \mathbb{R}^k)$. Based on Assumption (A2), it can easily be shown (in a similar manner as in [3]) that for any $x \in \bar{H}^1$ and $u \in L^\infty(I, \mathbb{R}^k)$, the function $y : I \to \mathbb{R}^n$ given by

$$y(t) := x(t) + \int_0^t V(t, \tau, x(\tau), u(\tau))\, d\tau$$

is absolutely continuous, $y(0) = 0$ and $\dot{y} \in L^2(I, \mathbb{R}^n)$. Thus the operator F is well-posed. Moreover, we have the following result.

Proposition 3.1. *Assume* (A1)–(A2)*, then the operator* F *defined by* (3.1) *is continuously Fréchet differentiable at every point* $(x,u) \in \bar{H}^1 \times L^\infty(I, \mathbb{R}^k)$ *and for any* $(h,v) \in \bar{H}^1 \times L^\infty(I, \mathbb{R}^k)$ *we have*

$$\begin{aligned}&(F'(x,u)(h,v))(t)\\ &\quad = h(t) + \int_0^t V_x(t, \tau, x(\tau), u(\tau))\, h(\tau) d\tau + \int_0^t V_u(t, \tau, x(\tau), u(\tau))\, v(\tau) d\tau\end{aligned}$$

for $t \in [0,1]$.

Proof. Based on assumptions (A1)–(A2) it can be easily shown that

$$\frac{d}{d\theta}F\left(x+\theta h,u+\theta v\right)\Big|_{\theta=0} = h(t)+\int_0^t V_x\left(t,\tau,x(\tau),u(\tau)\right)h(\tau)d\tau+\int_0^t V_u\left(t,\tau,x(\tau),u(\tau)\right)v(\tau)d\tau \quad (3.2)$$

and the operator

$$\bar{H}^1\times L^\infty\left(I,\mathbb{R}^k\right)\ni(h,v)\mapsto\frac{d}{d\theta}F\left(x+\theta h,u+\theta v\right)\Big|_{\theta=0}$$

is linear and bounded with values in $\bar{H}^1$. Therefore F is Gâteaux differentiable with the Gâteaux differential F_G defined by (3.2). To prove that F is continuously Fréchet differentiable it is enough to show that F_G is continuous. Let $(x_0,u_0)\in\bar{H}^1\times L^\infty\left(I,\mathbb{R}^k\right)$ and let $(x_n,u_n)\to(x_0,u_0)$ in $\bar{H}^1\times L^\infty\left(I,\mathbb{R}^k\right)$ as $n\to\infty$. Note that from the Schwartz inequality we get immediately that for $x\in\bar{H}^1$

$$|x(t)|\leq\int_0^t|x\left(s\right)|\,ds\leq\|x\|^2,\quad t\in[0,1]\,. \quad (3.3)$$

Taking into account the above and again applying the Schwarz inequality we get

$$\begin{aligned}&\|F_G\left(x_n,u_n\right)(h,v)-F_G\left(x_0,u_0\right)(h,v)\|\\&\leq\|h\|\left(\int_0^1\left|V_x\left(t,t,x_n(t),u_n(t)\right)-V_x\left(t,t,x_0(t),u_0(t)\right)\right|^2dt\right)^{\frac{1}{2}}\\&\quad+\|h\|\left(\int_0^1\int_0^t\left|V_{xt}\left(t,\tau,x_n(\tau),u_n(\tau)\right)-V_{xt}\left(t,\tau,x_0(\tau),u_0\left(\tau\right)\right)\right|^2d\tau dt\right)^{\frac{1}{2}}\\&\quad+\|v\|_\infty\left(\int_0^1\int_0^t\left|V_{ut}\left(t,\tau,x_n(\tau),u_n(\tau)\right)-V_{ut}\left(t,\tau,x_0(\tau),u_0\left(\tau\right)\right)\right|^2d\tau dt\right)^{\frac{1}{2}}\\&\quad+\|v\|_\infty\left(\int_0^1\left|V_u\left(t,t,x_n(t),u_n(t)\right)-V_u\left(t,t,x_0(t),u_0(t)\right)\right|^2dt\right)^{\frac{1}{2}}.\end{aligned}$$

By (3.3), the sequence (x_n,u_n) converges pointwise to (x_0,u_0) (almost everywhere on $[0,1]$) and, in view of (A1)–(A2), we can apply Lebesgue's dominated convergence theorem (the iterated integral can be considered as the double integral on the set $P_\triangle$). Consequently, we get that $\|F_G\left(x_n,u_n\right)(h,v)-F_G\left(x_0,u_0\right)(h,v)\|\to0$ as $n\to\infty$ and F is continuously Fréchet differentiable. □

Remark 3.2. From Proposition 3.1 it follows that for any $h \in \bar{H}^1$ we have

$$\left(F'_x(x,u)\,h\right)(t) = h(t) + \int_0^t V_x\left(t,\tau,x(\tau),u(\tau)\right)h(\tau)d\tau. \tag{3.4}$$

The proof of the main result is based on the following global implicit function theorem [9].

Theorem 3.3. *Let X, Y be real Banach spaces and H be a real Hilbert space. If $F : X \times Y \to H$ is of class C^1 and*

(a) *the differential $F_x(x,y) : X \to H$ is bijective for any $(x,y) \in X \times Y$,*
(b) *the functional*

$$f : X \ni x \mapsto \frac{1}{2}\left\|F(x,y)\right\|^2 \in \mathbb{R}$$

satisfies the Palais–Smale condition for any $y \in Y$,

then there exists a unique function $\lambda : Y \to X$ such that the equation $F(x,y) = 0$ and $\lambda(y) = x$ are equivalent in the set $X \times Y$. The above function λ is of class C^1 with the derivative given by formula

$$\lambda'(y) = -\left[F_x(\lambda(y), y)\right]^{-1} \circ F_y(\lambda(y), y) \tag{3.5}$$

for $y \in Y$.

Let us recall that a sequence $(x_s) \subset X$ is said to be a Palais–Smale sequence (PS) for functional $f : X \to \mathbb{R}$ if there is a constans $R > 0$ such that $\|f(x_s)\| \le R$ and $f'(x_s) \to 0$ as $s \to \infty$. We say that f satisfies Palais–Smale (PS) condition if any (PS) sequence admits a convergent subsequence.

Now we are ready to formulate the main result.

Theorem 3.4. *If the function V satisfies assumptions* (A1)–(A3) *then for any admissible parameter* (*control*) $u \in L^\infty\left(I, \mathbb{R}^k\right)$ *there exists a uniquely defined solution* (*trajectory*) x_u *to* (1.1). *Moreover, the operator*

$$\mathcal{U} \ni u \mapsto x_u \in \bar{H}^1$$

is of C^1 class (*it is continuously differentiable in the Fréchet sense*).

We have divided the proof into a sequence of lemmas. To begin with let us prove the following

Lemma 3.5. *If the function V satisfies assumptions* (A1)–(A2) *then for any given $x_0 \in \bar{H}^1$ and $u_0 \in L^\infty\left(I, \mathbb{R}^k\right)$ the linear equation*

$$h(t) + \int_0^t V_x\left(t,\tau,x_0(\tau),u_0(\tau)\right)h(\tau)d\tau = g(t) \tag{3.6}$$

possesses a unique solution $h \in \bar{H}^1$, where $g \in \bar{H}^1$ is an arbitrarily fixed function.

Proof. Fix $x_0 \in \bar{H}^1$ and $u_0 \in L^\infty\left(I, \mathbb{R}^k\right)$. Let us denote by K the kernel of integral equation (3.6), i.e.

$$K(t,\tau) := V_x\left(t,\tau,x_0(\tau),u_0(\tau)\right). \tag{3.7}$$

By (A2), the kernel $K \in L^2\left(P_\triangle, \mathbb{R}\right)$ therefore by the well-known sufficient condition for the existence of a solution to a linear integral equation with square integrable kernel (see [16, 19]), there exists a unique solution $h \in L^2(I, \mathbb{R}^n)$ to (3.6). Moreover, directly from (3.6) and (A2) it follows that h is absolutely continuous, $h(0) = 0$ and $\dot{h} \in L^2\left(I, \mathbb{R}^n\right)$. Thus equation (3.6) possesses a unique solution in the space $\bar{H}^1$ for any given $g \in \bar{H}^1$. □

Let

$$\varphi_u(x) := \frac{1}{2}\left\|F(x,u)\right\|_0^2, \quad x \in \bar{H}^1,$$

where $u \in L^\infty\left(I, \mathbb{R}^k\right)$ is fixed and F is defined by (3.1).

Lemma 3.6. *If V satisfies* (A1)–(A3) *then for any fixed $u \in L^\infty\left(I, \mathbb{R}^k\right)$ the functional φ_u is coercive, i.e. $\varphi_u(x_s) \to \infty$ if $\|x_s\|_0 \to \infty$.*

Proof. Fix $u \in L^\infty\left(I, \mathbb{R}^k\right)$. It is sufficient to show that there is a constant $m \geq 0$ such that

$$\left\|F(x_s,u)\right\|_m \to \infty \text{ if } \left\|x_s\right\|_m \to \infty.$$

By (A3) and thanks to Lemma 2.2, we have

$$\begin{aligned}
&\left\|F(x,u)\right\|_m \\
&= \left(\int_0^1 e^{-mt}\left|\dot{x}(t) + V\left(t,t,x(t),u(t)\right) + \int_0^t V_t\left(t,\tau,x(\tau),u(\tau)\right)d\tau\right|^2 dt\right)^{\frac{1}{2}} \\
&= \left\|\dot{x} + V\left(\cdot,\cdot,x(\cdot),u(\cdot)\right) + \int_0^{\cdot} V_t\left(\cdot,\tau,x(\tau),u(\tau)\right)d\tau\right\|_{L^2,m} \\
&\geq \left\|\dot{x}\right\|_{L^2,m} - \left\|V\left(\cdot,\cdot,x(\cdot),u(\cdot)\right)\right\|_{L^2,m} - \left\|\int_0^{\cdot} V_t\left(\cdot,\tau,x(\tau),u(\tau)\right)d\tau\right\|_{L^2,m} \\
&\geq \left\|\dot{x}\right\|_{L^2,m} - \left\|C\right\|_{L^2,m} \cdot \left\|x\right\|_{L^2,m} - \left\|B_1\right\|_{L^2,m} \\
&\quad - \bar{A}\left\|\int_0^{\cdot} \left|x(\tau)\right| d\tau\right\|_{L^2,m} - \left\|\int_0^{\cdot} B_2\left(\cdot,\tau\right)d\tau\right\|_{L^2,m} \\
&\geq \left\|x\right\|_m - \left\|C\right\|_{L^2,m}\frac{\left\|x\right\|_m}{\sqrt{m}} - \bar{A}\frac{\left\|x\right\|_m}{m} + D,
\end{aligned}$$

where $D := -\left\|B_1\right\|_{L^2,m} - \left\|\int_0^{\cdot} B_2\left(\cdot,\tau\right)d\tau\right\|_{L^2,m}$ and does not depend on (x,u). Consequently, for a sufficiently large $m > 0$, that is $m \geq \max\left\{1, \left\|C\right\|_{L^2,0} + \bar{A}\right\}$, we have that $\left\|F(x_s,u)\right\|_m \to \infty$ if $\left\|x_s\right\|_m \to \infty$ and φ_u is coercive. □

Lemma 3.7. *If V satisfies* (A1)–(A3) *then the functional φ_u satisfies the* (PS) *condition for any $u \in L^\infty\left(I, \mathbb{R}^k\right)$.*

Proof. Fix $u \in \mathcal{U}$. Let $(x_s) \subset \bar{H}^1$ be a (PS) sequence for φ_u. We have that (x_s) is bounded. Suppose it were false. Passing to a subsequence if necessary, we may assume that $\|x_s\| \to \infty$ as $s \to \infty$. Consequently, by Lemma 3.6, $\varphi_u(x_s) \to \infty$, but this contradicts the fact that (x_s) is a (PS) sequence for φ_u. Since $\bar{H}^1$ is a Hilbert space and (x_s) is bounded, therefore it admits a subsequence (still denoted by (x_s)) such that $x_s \rightharpoonup x_0$ weakly in $\bar{H}^1$. By (2.3) and (3.4), we obtain that

$$\begin{aligned}\varphi_u'(x_s)h &= \langle F(x_s,u), F_x'(x_s,u)h\rangle_0 \\ &= \int_0^1 \Big\langle \dot{x}_s(t) + V(t,t,x_s(t),u(t)) + \int_0^t V_t(t,\tau,x_s(\tau),u(\tau))\,d\tau, \\ &\dot{h}(t) + V_x(t,t,x_s(t),u(t))\,h(t) + \int_0^t V_{xt}(t,\tau,x_s(\tau),u(\tau))\,h(t)\,d\tau \Big\rangle_{\mathbb{R}^n} dt \end{aligned} \tag{3.8}$$

for $s = 0,1,2,\ldots$, where $h \in \bar{H}^1$. Let us put $h_s = x_s - x_0$, $s = 1,2,\ldots$. In view of (3.8) direct calculations lead to the equality

$$\begin{aligned}(\varphi_u'(x_s) - \varphi_u'(x_0))\,h_s = \|x_s - x_0\|_0^2 + \phi^1(x_s,h_s) - \phi^1(x_0,h_s) + \phi^2(x_s,\dot{h}_s) \\ - \phi^2(x_0,\dot{h}_s) + \phi^3(x_s,h_s) - \phi^3(x_0,h_s)\end{aligned} \tag{3.9}$$

for $s = 1,2,\ldots$, where

$$\begin{aligned}\phi^1(x,h) &= \int_0^1 \Big\langle \dot{x}(t), V_x(t,t,x(t),u(t))\,h(t) + \int_0^t V_{xt}(t,\tau,x(\tau),u(\tau))\,h(\tau)d\tau \Big\rangle_{\mathbb{R}^n} dt, \\ \phi^2(x,\dot{h}) &= \int_0^1 \Big\langle V(t,t,x_s(t),u(t)) + \int_0^t V_t(t,\tau,x(\tau),u(\tau))\,d\tau, \dot{h}(t) \Big\rangle_{\mathbb{R}^n} dt, \\ \phi^3(x,h) &= \int_0^1 \Big\langle V(t,t,x_s(t),u(t)) + \int_0^t V_t(t,\tau,x_s(\tau),u(\tau))\,d\tau, \\ &V_x(t,t,x_s(t),u(t))\,h_s(t) + \int_0^t V_{xt}(t,\tau,x_s(\tau),u(\tau))\,h_s(\tau)d\tau \Big\rangle_{\mathbb{R}^n} dt.\end{aligned} \tag{3.10}$$

By (A2) and thanks to the Schwarz inequality, we have

$$\begin{aligned}\left|\phi^1(x,h)\right| &\le 2\max_{t\in I}|h(t)| \cdot \|x\|_0 \cdot \|M\|_{L^2}, \\ \left|\phi^3(x,h)\right| &\le 4\max_{t\in I}|h(t)|\,\|M\|_{L^2}^2.\end{aligned}$$

Since the weak convergence (x_s) implies the uniform convergence (see Remark 2.1), we have that $h_s = x_s - x_0 \rightrightarrows 0$ uniformly on I as $s \to \infty$ and (x_s) is bounded, therefore $\phi^1(x_s, h_s), \phi^1(x_0, h_s), \phi^3(x_s, h_s)$ and $\phi^3(x_0, h_s)$ tend to zero as $s \to \infty$. To prove that $(\phi^2(x_s, \dot{h}_s) - \phi^2(x_0, \dot{h}_s)) \to 0$ as $s \to 0$ we proceed in a slightly different way. Let

$$g_s(t) := V(t, t, x_s(t), u(t)) + \int_0^t V_t(t, \tau, x(\tau), u(\tau))\, d\tau \quad \text{for } t \in I \text{ and } s = 0, 1, \ldots.$$

Thanks to (A1)–(A2), the fact that $x_s \rightrightarrows x_0$ uniformly on I as $s \to \infty$ and applying the Lebesgue dominated convergence theorem we get that g_s tends to g_0. By (3.10), we have

$$\left|\phi^2(x_s, \dot{h}_s) - \phi^2(x_0, \dot{h}_s)\right| = \left|\int_0^1 \left\langle g_s(t) - g_0(t), \dot{h}_s(t)\right\rangle dt\right| \le \left\|\dot{h}_s\right\|_{L^2} \|g_s - g_0\|_{L^2,0}.$$

Applying the Lebesgue dominated convergence theorem (thanks to (A2)) once again and making use of the fact that $(\dot{h}_s)$ is bounded in L^2 (conf. the definition of the norm in H^1), we infer that $(\phi^2(x_s, \dot{h}_s) - \phi^2(x_0, \dot{h}_s)) \to 0$ as $s \to 0$. Finally, thanks to the fact that (x_s) is the (PS) sequence and $(\dot{h}_s)$ is bounded we see that $(\varphi_u'(x_s) - \varphi_u'(x_0))\, h_s \to 0$ as $s \to 0$. Consequently, in view of (3.9), we have that $\|x_s - x_0\|_0^2 \to 0$ as $s \to 0$ and therefore φ_u satisfies the (PS) condition. □

Now we can prove Theorem 3.4.

Proof of Theorem 3.4. Let $X := \bar{H}^1$, $Y := L^\infty\left(I, \mathbb{R}^k\right)$ and $H := \bar{H}^1$. In a standard way we check that the operator F defined by (3.1) is of C^1 class. Thanks to Lemmas 3.5 and 3.7, we know that all the assumptions of Theorem 3.3 are satisfied, therefore the assertion of Theorem 3.4 is satisfied. □

4. EXAMPLES

In this section we present an illustrative example and discuss the application of the main result to the problem of existence and smooth dependence on the parameter of a classical Cauchy problem.

Example 4.1. Let $V : P_\triangle \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be defined by the formula

$$V(t, \tau, x, u) := \frac{x^T Q_1(t, \tau) x}{1 + x^T Q_2 x} x + g(\tau, u) \cos |x|^2,$$

where the matrix-valued function $Q_1 : P_\triangle \to \mathbb{R}^{n \times n}$ is continuous together with its derivative $(Q_1)_t$, the (constant) matrix $Q_2 \in \mathbb{R}^{n \times n}$ is positively defined, $g : I \times \mathbb{R}^k \to \mathbb{R}^n$ is continuous together with its derivative g_u. Consider a control system of the form (1.1) with defined above function V, where $x \in \bar{H}^1(I, \mathbb{R}^n)$, $u \in L^\infty\left(I, \mathbb{R}^k\right)$. It is easy to notice that under assumptions imposed on Q_1, Q_2 and g there exists a function $M : I \to \mathbb{R}$ such that all assumptions (A1)–(A2) are satisfied. Next, (A3) is satisfied if we put

$$A := \max\left\{|Q_1(t,\tau)|, |(Q_1)_t(t,\tau)| : (t,\tau) \in P_\triangle\right\}$$

and $C := A/\alpha$, where $\alpha > 0$ is a constant satisfying the inequality $x^T Q_2 x \geq \alpha|x|$, $x \in \mathbb{R}^n$. As a result, we get, applying Theorem 3.4, that system (1.1) has the following properties:

1. for any $u \in L^\infty(I, \mathbb{R}^k)$ there exists a unique solution $x_u \in \bar{H}^1(I, \mathbb{R}^n)$ to (1.1), in particular x is absolutely continuous, $x(0) = 0$ and $\dot{x} \in L^2(I, \mathbb{R}^n)$,
2. the solution x_u depends continuously on the parameter $u \in \mathcal{U}$,
3. the operator
$$\mathcal{U} \ni u \mapsto x_u \in \bar{H}^1(I, \mathbb{R}^n)$$
is well possed, and continuously (in the Fréchet sense) differentiable.

Example 4.2. Consider the classical Cauchy problem with a parameter u of the form

$$\dot{x} = f(t, x, u) \tag{4.1}$$

$$x(0) = 0. \tag{4.2}$$

We say that for a given parameter u a function $x : I \to \mathbb{R}^n$, where $n \geq 1$ and $I := [0,1]$ solves problem (4.1)–(4.2) if it satisfies equation (4.1) for almost every $t \in I$ and $x(0) = 0$. Therefore, we look for solutions in the space of absolutely continuous functions $x : I \to \mathbb{R}^n$ such that $x(0) = 0$ and the derivative $\dot{x}$ is a square integrable function, i.e. $\dot{x} \in L^2(I, \mathbb{R}^n)$. We assume that the parameter $u \in L^\infty(I, \mathbb{R}^k)$ with $k \geq 1$.

In a standard manner we transform problem (4.1)–(4.2) to the following integral problem

$$x(t) = \int_0^t f(s, x(s), u(s))\, ds. \tag{4.3}$$

Our goal is to apply Theorem 3.4 to the above problem. We assume that

(1) the function $f(\tau, \cdot, \cdot)$ is continuous on the set $G := \mathbb{R}^n \times \mathbb{R}^k$ for a.e. $\tau \in I$; there are continuous derivatives $f_x(\tau, \cdot, \cdot), f_u(\tau, \cdot, \cdot)$ are continuous on G for a.e. $\tau \in I$,

(2) the functions f, f_x, f_u are measurable with respect to τ for $(x, u) \in \mathbb{R}^n \times \mathbb{R}^k$ and locally bounded with respect to x, i.e. for any $\rho > 0$ and $u \in \mathcal{U}$ there exists a function $M \in L^2(I, \mathbb{R})$ such that

$$|f(\tau, x, u(\tau))|, |f_x(\tau, x, u(\tau))|, |f_u(\tau, x, u(\tau))| \leq M(\tau)$$

and $|x| \leq \rho$ and a.e. $\tau \in I$.

(3) for any $u \in L^\infty(I, \mathbb{R}^k)$ there exist a constant $C > 0$ and a function $B_1 \in L^2(I, \mathbb{R}^+)$ such that

$$|f(t, x, u(t))| \leq C|x| + B_1(t).$$

Applying Theorem 3.4, we get that for any $u \in L^\infty\left(I, \mathbb{R}^k\right)$ there exists a unique solution x_u to (4.3), and consequently to (4.1)–(4.2), which depends continuously on u. It should be noticed that the solution x_u is a global solution, that it is defined on the whole interval I.

The classical result for the existence of solution to (4.1)–(4.2) requires assumptions of Lipschitz continuity of f with respect to x (for the existence and uniqueness of a local solution) and sublinearity of f for the global solution. For the continuous dependence it is usually assumed that f is Lipschitz continuous with respect to (x, u) and f is smooth with respect to (x, u) for the smooth dependence. To sum up, assumptions (1)–(3) for f are comparable with classical ones.

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