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Fibonacci shift operation on Fibonacci numbers of order m^*

by

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Abstract: Fibonacci codes of different orders are of variable length. They are used for encoding of numbers and find applications in information and coding theory. The Fibonacci binary encoding and computation of its values are closely associated with Fibonacci shift operations. Walder et al. (2012) established a relation for kth Fibonacci left shift for Fibonacci numbers of order $m = 2, 3$. In this paper, we establish a relation for kth Fibonacci left shift for Fibonacci numbers of order $m = 4$ and we generalize the formula for kth Fibonacci left shift for the Fibonacci numbers of all orders.

Keywords: Fibonacci numbers, Fibonacci numbers of order m, golden mean, Fibonacci shift operation

1. Introduction

The Fibonacci numbers of order $m = 2$ are known as classical Fibonacci numbers (Stakhov, 1977). The classical Fibonacci numbers and golden mean, $\tau = \frac{1+\sqrt{5}}{2}$, find applications in coding and information theory particularly (see Esmaeeili, Gulliver and Kakhbod, 2009; Stakhov, 2007; or Basu and Prasad, 2009, 2010). The Fibonacci numbers of order m ($m \geq 2$) are defined by

$$F_i^{(m)} = F_{i-1}^{(m)} + F_{i-2}^{(m)} + F_{i-3}^{(m)} + \cdots + F_{i-m}^{(m)}, \quad \text{for} \quad i \geq 1$$

where

$$F_{-m+1}^{(m)} = F_{-m+2}^{(m)} = F_{-m+3}^{(m)} = \cdots = F_{-2}^{(m)} = 0,$$

and

$$F_{-1}^{(m)} = F_0^{(m)} = 1.$$

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Table 1. Fibonacci numbers of order $m = 2, 3, 4$

i	$F_i^{(2)}$	$F_i^{(3)}$	$F_i^{(4)}$
0	1	1	1
1	2	2	2
2	3	4	4
3	5	7	8
4	8	13	15
5	13	24	29
6	21	44	56
7	34	81	108
8	55	149	208
9	89	274	401

1.1. Fibonacci code of order $m = 2$

The encoding algorithm for a positive integer n by the Fibonacci code of order $m = 2$ is given by:

Step 1 Find the largest Fibonacci number of order $m = 2$ equal to or less than n, keeping track of the remainder.

Step 2 If the number we subtracted was the ith unique Fibonacci number of order $m = 2$, put a one in the ith digit of our output.

Step 3 Repeat the previous steps, substituting our remainder for n, until we reach a remainder of zero.

Step 4 Place a one after the last naturally occurring one in our output.

To decode a token in the code, remove the last 1, assign the remaining bits the values $1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \cdots$ (the Fibonacci number of order $m = 2$) and add.

Table 2. Fibonacci code of order $m = 2$ for some integers

n	$F^{(2)}(n)\prime$	$F^{(2)}(n)$
1	1	11
2	01	011
3	001	0011
4	101	1011
5	0001	00011
6	1001	10011
7	0101	01011
8	00001	000011
9	10001	100011
10	01001	010011

1.2. Fibonacci code of order $m \geq 3$

When generating the Fibonacci codes of order $m \geq 3$ it is not possible to append the sequence of $(m-1)$ 1-bits to the Fibonacci representation, $F^m(n)\prime$. For example, let us consider the Fibonacci representation, $F^m(n)\prime$ for integers 13 and 37 with respect to the Fibonacci numbers of order $m = 3$, which are $F^3(13)\prime = 00001$, $F^3(37)\prime = 000011$. We obtain Fibonacci codes 0000111 and 00001111 after appending the 11 sequence to the Fibonacci representations, respectively. The second code gives a sequence of four 1 bits at the end. These two codes do not yield clear results during the decoding by virtue of the fact that these two codes end with three 1-bits. We obtain the number 13 for both codes, but, the 1 bit at the end of the second code is taken as a part of the next number, which leads to an error. To solve this type of problem, we use the Fibonacci sum, $S_n^{(m)}$ (see Apostolico and Fraenkel, 1987). The Fibonacci sum, $S_n^{(m)}$, is defined by

$$S_n^{(m)} = \begin{cases} 0 & \text{for } n < -1 \\ \sum_{i=-1}^{n} F_i^{(m)} & \text{for } n \geq -1 \end{cases}.$$

Then, the Fibonacci code of order $m \geq 3$ for any positive integer n is given by:

Step 1 If $n = 1$ then $F^{(m)}(n) = 1_m$.
Step 2 If $n = 2$ then $F^{(m)}(n) = 01_m$.
Step 3 Find k such that $S_{k-2}^{(m)} < n \leq S_{k-1}^{(m)}$. Take $Q = n - S_{k-2}^{(m)} - 1$.
Step 4 Compute $F^{(m)}(Q)\prime$.
Step 5 Append 01_m as a suffix to $F^{(m)}(Q)\prime$. If necessary, append 0-bits between $F^{(m)}(Q)\prime$ and 01_m to make $F^{(m)}(n)$ of length $m + k$.

Table 3. Fibonacci code of order $m = 3,\ 4$ for some integers

n	$F^{(3)}(n)$	$F^{(4)}(n)$
1	111	1111
2	0111	01111
3	00111	001111
4	10111	101111
5	000111	0001111
6	100111	1001111
7	010111	0101111
8	110111	1101111
9	0000111	00001111
10	1000111	10001111

1.3. Fibonacci binary encoding and computation of its values

Let $F^{(m)}(n)\prime$ be the Fibonacci representation of the Fibonacci code, $F^{(m)}(n)$ and $F^{(m)}(n)\prime = a_0a_1a_2a_3\cdots a_k$ be the Fibonacci binary encoding denoted by

$V(F(n))$ and defined by

$$V(F(n)) = n = \sum_{i=0}^{k} a_i F_i^{(m)} (a_i \varepsilon \{0,1\}, \quad 0 \le i \le k).$$

Each bit represents a Fibonacci number, $F_i^{(m)}$, for Fibonacci binary encoding. The property of the numbers is that they do not contain any sequence of m consecutive 1-bits (Walder et al., 2012). The Fibonacci code, $F^{(m)}(n)$ maps the number, n onto a binary string in such a manner the string ends with a sequence of m consecutive 1-bits.

2. Fibonacci shift operation

Fibonacci shift operation is a type of operation, which is used for Fibonacci encoding. This operation may be left or right. The kth Fibonacci left shift, $F^{(m)}(n) \ll_F k$ is defined by $F^{(m)}(n) \ll_F k = \overbrace{00000\cdots0}^{k} a_1 a_2 \cdots a_p 1$ and the kth Fibonacci right shift, $F^{(m)}(n) \gg_F k$ is defined by $F^{(m)}(n) \gg_F k = a_k a_{k+1} a_{k+2} a_{k+3} \cdots a_p$, where k is an integer and $k \ge 0$. Walder et al. (2012) established a relation for kth Fibonacci left shift for Fibonacci numbers of order $m = 2, 3$.

In this paper, we establish the relation for kth Fibonacci left shift for Fibonacci numbers of order $m = 4$ and we generalize the formula for kth Fibonacci left shift for the Fibonacci numbers of all orders.

Theorem 1 *Let $F^{(4)}(n)$ be a Fibonacci binary encoding of order $m = 4$ for the n value. Then, $V(F^{(4)}(n) \ll_F k) = F_{k-1}^{(4)} V(F^{(4)}(n)) + (F_{k-2}^{(4)} + F_{k-3}^{(4)} + F_{k-4}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-2}^{(4)} + F_{k-3}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{k-2}^{(4)} V(F^{(4)}(n) \gg_F 3)$.*

Proof: We will prove this result by mathematical induction.
$V(F^{(4)}(n) \ll_F 0) = \sum_{i=0}^{p} a_i F_{i+0}^{(4)} = \sum_{i=0}^{p} a_i F_i^{(4)} = V(F^{(4)}(n)) =$
$F_{-1}^{(4)} V(F^{(4)}(n)) + (F_{-2}^{(4)} + F_{-3}^{(4)} + F_{-4}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{-2}^{(4)} +$
$F_{-3}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{-2}^{(4)} V(F^{(4)}(n) \gg_F 3)$.

$V(F^{(4)}(n) \ll_F 1) = \sum_{i=0}^{p} a_i F_{i+1}^{(4)} = \sum_{i=0}^{p} a_i (F_i^{(4)} + F_{i-1}^{(4)} + F_{i-2}^{(4)} + F_{i-3}^{(4)}) =$
$\sum_{i=0}^{p} a_i F_i^{(4)} + \sum_{i=0}^{p} a_i F_{i-1}^{(4)} + \sum_{i=0}^{p} a_i F_{i-2}^{(4)} + \sum_{i=0}^{p} a_i F_{i-3}^{(4)} =$
$V(F^{(4)}(n)) + V(F^{(4)}(n) \gg_F 1) + V(F^{(4)}(n) \gg_F 2) + V(F^{(4)}(n) \gg_F 3) =$
$F_0^{(4)} V(F^{(4)}(n)) + (F_{-1}^{(4)} + F_{-2}^{(4)} + F_{-3}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{-1}^{(4)} + F_{-2}^{(4)}) V(F^{(4)}(n) \gg_F$
$2) + F_{-1}^{(4)} V(F^{(4)}(n) \gg_F 3)$.

$V(F^{(4)}(n) \ll_F 2) = \sum_{i=0}^{p} a_i F_{i+2}^{(4)} = \sum_{i=0}^{p} a_i (F_{i+1}^{(4)} + F_i^{(4)} + F_{i-1}^{(4)} + F_{i-2}^{(4)}) =$

$\sum_{i=0}^{p} a_i F_{i+1}^{(4)} + \sum_{i=0}^{p} a_i F_{i}^{(4)} + \sum_{i=0}^{p} a_i F_{i-1}^{(4)} + \sum_{i=0}^{p} a_i F_{i-2}^{(4)} =$
$\sum_{i=0}^{p} a_i (F_{i}^{(4)} + F_{i-1}^{(4)} + F_{i-2}^{(4)} + F_{i-3}^{(4)}) + \sum_{i=0}^{p} a_i F_{i}^{(4)} + \sum_{i=0}^{p} a_i F_{i-1}^{(4)} + \sum_{i=0}^{p} a_i F_{i-2}^{(4)} =$
$2\sum_{i=0}^{p} a_i F_{i}^{(4)} + 2\sum_{i=0}^{p} a_i F_{i-1}^{(4)} + 2\sum_{i=0}^{p} a_i F_{i-2}^{(4)} + \sum_{i=0}^{p} a_i F_{i-3}^{(4)} =$
$2V(F^{(4)}(n)) + 2V(F^{(4)}(n) \gg_F 1) + 2V(F^{(4)}(n) \gg_F 2) + V(F^{(4)}(n) \gg_F 3) =$
$F_{1}^{(4)} V(F^{(4)}(n)) + (F_{0}^{(4)} + F_{-1}^{(4)} + F_{-2}^{(4)} + F_{-3}^{(4)}) V(F^{(4)}(n) \gg_F 1) +$
$(F_{0}^{(4)} + F_{-1}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{0}^{(4)} V(F^{(4)}(n) \gg_F 3).$

This means that the theorem holds for $k = 0, 1, 2$.
Let us suppose that this theorem holds for $0 \leq j < k$. We have to prove that
$V(F^{(4)}(n) \ll_F k) =$
$F_{k-1}^{(4)} V(F^{(4)}(n)) + (F_{k-2}^{(4)} + F_{k-3}^{(4)} + F_{k-4}^{(4)}) V(F^{(4)}(n) \gg_F 1) +$
$(F_{k-2}^{(4)} + F_{k-3}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{k-2}^{(4)} V(F^{(4)}(n) \gg_F 3).$

Now,
$V(F^{(4)}(n) \ll_F k) = \sum_{i=0}^{p} a_i F_{i+k}^{(4)}$
$= \sum_{i=0}^{p} a_i (F_{i+k-1}^{(4)} + F_{i+k-2}^{(4)} + F_{i+k-3}^{(4)} + F_{i+k-4}^{(4)})$
$= \sum_{i=0}^{p} a_i F_{i+k-1}^{(4)} + \sum_{i=0}^{p} a_i F_{i+k-2}^{(4)} + \sum_{i=0}^{p} a_i F_{i+k-3}^{(4)} + \sum_{i=0}^{p} a_i F_{i+k-4}^{(4)}$

$= V(F^{(4)}(n) \ll_F k-1) + V(F^{(4)}(n) \ll_F k-2) +$
$V(F^{(4)}(n) \ll_F k-3) + V(F^{(4)}(n) \ll_F k-4) = F_{k-2}^{(4)} V(F^{(4)}(n)) + (F_{k-3}^{(4)} + F_{k-4}^{(4)} + F_{k-5}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-3}^{(4)} + F_{k-4}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{k-3}^{(4)} V(F^{(4)}(n) \gg_F 3) + F_{k-3}^{(4)} V(F^{(4)}(n)) + (F_{k-4}^{(4)} + F_{k-5}^{(4)} + F_{k-6}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-4}^{(4)} + F_{k-5}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{k-4}^{(4)} V(F^{(4)}(n) \gg_F 3) + F_{k-4}^{(4)} V(F^{(4)}(n)) + (F_{k-5}^{(4)} + F_{k-6}^{(4)} + F_{k-7}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-5}^{(4)} + F_{k-6}^{(4)}) V(F^{(4)}(n) \gg_F 2) +$
$F_{k-5}^{(4)} V(F^{(4)}(n) \gg_F 3) + F_{k-5}^{(4)} V(F^{(4)}(n)) + (F_{k-6}^{(4)} + F_{k-7}^{(4)} + F_{k-8}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-6}^{(4)} + F_{k-7}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{k-6}^{(4)} V(F^{(4)}(n) \gg_F 3)$

$= (F_{k-2}^{(4)} + F_{k-3}^{(4)} + F_{k-4}^{(4)} + F_{k-5}^{(4)}) V(F^{(4)}(n)) + (F_{k-3}^{(4)} + F_{k-4}^{(4)} + F_{k-5}^{(4)} + F_{k-6}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-4}^{(4)} + F_{k-5}^{(4)} + F_{k-6}^{(4)} + F_{k-7}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-5}^{(4)} + F_{k-6}^{(4)} + F_{k-7}^{(4)} + F_{k-8}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-3}^{(4)} + F_{k-4}^{(4)} + F_{k-5}^{(4)} + F_{k-6}^{(4)}) V(F^{(4)}(n) \gg_F 2) + (F_{k-4}^{(4)} + F_{k-5}^{(4)} + F_{k-6}^{(4)} + F_{k-7}^{(4)}) V(F^{(4)}(n) \gg_F 2) + (F_{k-3}^{(4)} + F_{k-4}^{(4)} + F_{k-5}^{(4)} + F_{k-6}^{(4)}) V(F^{(4)}(n) \gg_F 3)$

$= F_{k-1}^{(4)} V(F^{(4)}(n)) + (F_{k-2}^{(4)} + F_{k-3}^{(4)} + F_{k-4}^{(4)}) V(F^{(4)}(n) \gg_F 1) + (F_{k-2}^{(4)} + F_{k-3}^{(4)}) V(F^{(4)}(n) \gg_F 2) + F_{k-2}^{(4)} V(F^{(4)}(n) \gg_F 3).$

Hence the theorem holds. $\square$

THEOREM 2 *Let $F^{(m)}(n)$ be a Fibonacci binary encoding of order m for the n*

value. Then
$V(F^{(m)}(n) \ll_F k) = F^{(m)}_{k-1} V(F^{(m)}(n)) + (F^{(m)}_{k-2} + F^{(m)}_{k-3} + F^{(m)}_{k-4} + \cdots + F^{(m)}_{k-m}) V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-2} + F^{(m)}_{k-3} + \cdots + F^{(m)}_{k-m+1}) V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{k-2} V(F^{(m)}(n) \gg_F (m-1))$.

Proof We will prove this result by mathematical induction.
$V(F^{(m)}(n) \ll_F 0) = V(F^{(m)}(n)) =$
$F^{(m)}_{-1} V(F^{(m)}(n)) + (F^{(m)}_{-2} + F^{(m)}_{-3} + F^{(m)}_{-4} + \cdots + F^{(m)}_{-m}) V(F^{(m)}(n) \gg_F 1) +$
$(F^{(m)}_{-2} + F^{(m)}_{-3} + \cdots + F^{(m)}_{-m+1}) V(F^{(m)}(n) \gg_F 2) + \cdots +$
$F^{(m)}_{-2} V(F^{(m)}(n) \gg_F (m-1))$.

$V(F^{(m)}(n) \ll_F 1) = \sum_{i=0}^{p} a_i F^{(m)}_{i+1} =$
$\sum_{i=0}^{p} a_i (F^{(m)}_{i} + F^{(m)}_{i-1} + F^{(m)}_{i-2} + \cdots + F^{(m)}_{i-(m-1)}) =$
$\sum_{i=0}^{p} a_i F^{(m)}_{i} + \sum_{i=0}^{p} a_i F^{(m)}_{i-1} + \sum_{i=0}^{p} a_i F^{(m)}_{i-2} + \cdots + \sum_{i=0}^{p} a_i F^{(m)}_{i-(m-1)} =$
$V(F^{(m)}(n)) + V(F^{(m)}(n) \gg_F 1) + V(F^{(m)}(n) \gg_F 2) + \cdots + V(F^{(m)}(n) \gg_F (m-1)) = F^{(m)}_{0} V(F^{(m)}(n)) + (F^{(m)}_{-1} + F^{(m)}_{-2} + F^{(m)}_{-3} + \cdots + F^{(m)}_{-m+1}) V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{-1} + F^{(m)}_{-2} + \cdots + F^{(m)}_{-m+2}) V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{-1} V(F^{(m)}(n) \gg_F (m-1))$.

$V(F^{(m)}(n) \ll_F 2) = \sum_{i=0}^{p} a_i F^{(m)}_{i+2} = \sum_{i=0}^{p} a_i (F^{(m)}_{i+1} + F^{(m)}_{i} + F^{(m)}_{i-1} + \cdots + F^{(m)}_{i-(m-2)}) = \sum_{i=0}^{p} a_i F^{(m)}_{i+1} + \sum_{i=0}^{p} a_i F^{(m)}_{i} + \sum_{i=0}^{p} a_i F^{(m)}_{i-1} + \cdots +$
$\sum_{i=0}^{p} a_i F^{(m)}_{i-(m-2)} = \sum_{i=0}^{p} a_i (F^{(m)}_{i} + F^{(m)}_{i-1} + \cdots + F^{(m)}_{i-(m-1)}) + \sum_{i=0}^{p} a_i F^{(m)}_{i} + \sum_{i=0}^{p} a_i F^{(m)}_{i-1} + \cdots + \sum_{i=0}^{p} a_i F^{(m)}_{i-(m-2)} = 2V(F^{(m)}(n)) + 2V(F^{(m)}(n) \gg_F 1) +$
$2V(F^{(m)}(n) \gg_F 2) + \cdots + 2V(F^{(m)}(n) \gg_F (m-2)) +$
$V(F^{(m)}(n) \gg_F (m-1)) = F^{(m)}_{1} V(F^{(m)}(n)) + (F^{(m)}_{0} + F^{(m)}_{-1} + F^{(m)}_{-2} + \cdots + F^{(m)}_{-m+2}) V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{0} + F^{(m)}_{-1} + \cdots + F^{(m)}_{-m+3}) V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{0} V(F^{(m)}(n) \gg_F (m-1))$.

This means that theorem holds for $k = 0, 1, 2$.
Let us suppose that this theorem holds for $0 \le j < k$. We have to prove that
$V(F^{(m)}(n) \ll_F k) =$
$F^{(m)}_{k-1} V(F^{(m)}(n)) + (F^{(m)}_{k-2} + F^{(m)}_{k-3} + F^{(m)}_{k-4} + \cdots + F^{(m)}_{k-m}) V(F^{(m)}(n) \gg_F 1)$
$+(F^{(m)}_{k-2} + F^{(m)}_{k-3} + \cdots + F^{(m)}_{k-m+1}) V(F^{(m)}(n) \gg_F 2) + \cdots$
$+F^{(m)}_{k-2} V(F^{(m)}(n) \gg_F (m-1))$.

Now,
$V(F^{(m)}(n) \ll_F k) = \sum_{i=0}^{p} a_i F^{(m)}_{i+k} = \sum_{i=0}^{p} a_i (F^{(m)}_{i+k-1} + F^{(m)}_{i+k-2} + \cdots + F^{(m)}_{i+k-m})$
$= \sum_{i=0}^{p} a_i F^{(m)}_{i+k-1} + \sum_{i=0}^{p} a_i F^{(m)}_{i+k-2} + \cdots + \sum_{i=0}^{p} a_i F^{(m)}_{i+k-m}$
$= V(F^{(m)}(n) \ll_F k-1) + V(F^{(m)}(n) \ll_F k-2) + V(F^{(m)}(n) \ll_F k-3) +$

$\cdots + V(F^{(m)}(n) \ll_F k-m) = F^{(m)}_{k-2}V(F^{(m)}(n)) + (F^{(m)}_{k-3} + F^{(m)}_{k-4} + \cdots +$
$F^{(m)}_{k-(m+1)})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-3} + F^{(m)}_{k-4} + \cdots +$
$F^{(m)}_{k-m})V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{k-3}V(F^{(m)}(n) \gg_F m-1) + F^{(m)}_{k-3}V(F^{(m)}(n)) +$
$(F^{(m)}_{k-4} + F^{(m)}_{k-5} + \cdots + F^{(m)}_{k-(m+2)})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-4} + F^{(m)}_{k-5} + \cdots +$
$F^{(m)}_{k-(m+1)})V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{k-4}V(F^{(m)}(n) \gg_F m-1) + F^{(m)}_{k-4}V(F^{(m)}(n)) +$
$(F^{(m)}_{k-5} + F^{(m)}_{k-6} + \cdots + F^{(m)}_{k-(m+3)})V(F^{(m)}(n) \gg_F 1) + (F^{(4)}_{k-5} + F^{(m)}_{k-6} + \cdots +$
$F^{(m)}_{k-(m+2)})V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{k-5}V(F^{(m)}(n) \gg_F m-1) +$
$F^{(m)}_{k-5}V(F^{(m)}(n)) + (F^{(m)}_{k-6} + F^{(m)}_{k-7} + \cdots +$
$F^{(m)}_{k-(m+4)})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-6} + F^{(m)}_{k-7} + \cdots +$
$F^{(m)}_{k-(m+3)})V(F^{(m)}(n) \gg_F 2) + \cdots +$
$F^{(m)}_{k-6}V(F^{(m)}(n) \gg_F m-1)$

$= (F^{(m)}_{k-2} + F^{(m)}_{k-3} + F^{(m)}_{k-4} + \cdots + F^{(m)}_{k-(m+1)})V(F^{(m)}(n)) + (F^{(m)}_{k-3} + F^{(m)}_{k-4} + F^{(m)}_{k-5} +$
$\cdots + F^{(m)}_{k-(m+2)})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-4} + F^{(m)}_{k-5} + F^{(m)}_{k-6} + \cdots +$
$F^{(m)}_{k-(m+3)})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-5} + F^{(m)}_{k-6} + F^{(m)}_{k-7} + \cdots +$
$F^{(m)}_{k-(m+4)})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-3} + F^{(m)}_{k-4} + F^{(m)}_{k-5} + \cdots +$
$F^{(m)}_{k-(m+2)})V(F^{(m)}(n) \gg_F 2) + (F^{(m)}_{k-4} + F^{(m)}_{k-5} + F^{(m)}_{k-6} + \cdots +$
$F^{(m)}_{k-(m+3)})V(F^{(m)}(n) \gg_F 2) + \cdots + (F^{(m)}_{k-3} + F^{(m)}_{k-4} + F^{(m)}_{k-5} + \cdots +$
$F^{(m)}_{k-(m+2)})V(F^{(m)}(n) \gg_F m-1)$

$= F^{(4)}_{k-1}V(F^{(m)}(n)) + (F^{(m)}_{k-2} + F^{(m)}_{k-3} + F^{(m)}_{k-4} + \cdots +$
$F^{(m)}_{k-m})V(F^{(m)}(n) \gg_F 1) + (F^{(m)}_{k-2} + F^{(m)}_{k-3} + \cdots +$
$F^{(m)}_{k-m+1})V(F^{(m)}(n) \gg_F 2) + \cdots + F^{(m)}_{k-2}V(F^{(m)}(n) \gg_F (m-1)).$

Hence the theorem holds. □

3. Conclusion

Fibonacci codes for different orders are of variable length and they are used for encoding of numbers and have applications in information theory, data compression, coding theory, number theory, cryptography etc. In this paper, we establish a relation for the kth Fibonacci left shift for Fibonacci numbers of order $m = 4$ and we generalize a formula for kth Fibonacci left shift for the Fibonacci numbers of all orders. In the future, we hope that these results can be used in cryptography, information and coding theory.

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