Zeszyty Naukowe Wydziału Elektrotechniki i Automatyki Politechniki Gdańskiej Nr 47

Politechnika Gdańska

Wydział Elektrotechniki i Automatyki

Gdańsk 2015

SOME APPLICATIONS OF FRACTIONAL CALCULUS IN MODELLING OF ACCELEROMETER AND PRESSURE TRANSDUCER

Daniel PIETRUSZCZAK[1], **Mirosław LUFT**[2], **Piotr LESIAK**[3]

1. Faculty of Transport and Electrical Engineering, Kazimierz Pułaski University of Technology and Humanities in Radom phone: 48 361-77-13, e-mail: d.pietruszczak@uthrad.pl
2. Faculty of Transport and Electrical Engineering, Kazimierz Pułaski University of Technology and Humanities in Radom phone: 48 361-77-10, e-mail: m.luft@uthrad.pl
3. Faculty of Transport and Electrical Engineering, Kazimierz Pułaski University of Technology and Humanities in Radom phone: 48 361-77-09, e-mail: p.lesiak@uthrad.pl

Abstract: The paper outlines applications of fractional calculus for dynamic measurements while developing a method for description of transducer dynamic properties, which the authors consider to be the original and unique achievement of their work. The main objective of the paper is to show the implementation of a fractional calculus-based method that allows for description of dynamic properties of transducers of arbitrary orders (not only integer). The paper presents possibilities of using fractional calculus in dynamic measurements for modelling an accelerometer and a pressure transducer. Tests are executed in the MATLAB&Simulink environment. New methods of modelling transducers are important particularly in the case of the development of new technologies and materials the performance of which is beyond the scope of dynamic behaviour modelled by means of differential equations of integer orders. An example here can be a pair: a capacitor (integer order model) and supercapacitor (fractional order model).

Key words: accelerometer, fractional calculus, dynamic measurements, MATLAB&Simulink, pressure transducer.

1. INTRODUCTION

The dynamic development of recent research into the use of fractional calculus for the analysis of dynamic systems [1] and [2] encouraged the authors of this paper to attempt the make use of it for the analysis and modelling of an accelerometer [3] and a pressure transducer [4].

The main objective of this work is the implementation of a fractional calculus-based method [5], and [6] that allows for the description of dynamic properties of signal processing measuring transducers with integer-order and fractional-order. Fractional calculus is a generalisation of integral-order differential calculus – this is confirmed by laboratory testing of dynamic systems [3], [6] and [7].

Modelling measurement transducers by derivative of arbitrary orders opens up a number of possibilities in the field of the dynamic system identification and the development of new, earlier unattainable control algorithms for intelligent measurement systems [8].

2. SELECTED ISSUES OF THE FRACTIONAL CALCULUS

In the fractional calculus a derivative of arbitrary order is treated as an interpolation of a sequence of operators of discrete orders with operators of continuous orders. A notation introduced by H.D. Davis [2] is used here in which a fractional order derivative of $f(t)$ function is represented as:

$$ {}_{t_0}D_t^{\nu} f(t) \tag{1}$$

where t_0 and t define the integration or differentiation interval, ν is the order of the derivative.

As the problem has been continually developed, there are many definitions of fractional derivative [1] and [2]. Describing dynamic properties of the measuring transducers using fractional arithmetic, we can use one of three definitions: Grünwald-Letnikov, Riemann-Liouville and Caputo.

The function of a real variable $f(t)$ defined in the $[t_0, t]$ interval is given. Assuming that the function increment $h > 0$ is such that: $h = \frac{t - t_0}{k}$ provided that $h \to 0$ causes that $k \to +\infty$ for the established $t - t_0$, then Grünwald-Letnikov fractional derivative of discrete function for $f(hi), i = 0,1,2,...$ is defined as:

$$ {}_{t_0}D_t^{\nu} f(t) = \lim_{\substack{h \to 0 \\ t - t_0 = kh}} \left[\frac{1}{h^{\nu}} \sum_{i=1}^{k} a_i^{(\nu)} f(t - hi) \right]. \tag{2}$$

The Riemann-Liouville's fractional derivative (3) is the function described by the formula:

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt} \int_a^t (t-\tau)^{k-\alpha-1} f(\tau) d\tau \tag{3}$$

where ν is the order of integration within the $[t_0, t]$ interval of $f(t)$ function, $k - 1 \le \alpha \le k$, $\alpha \in R^+$, $\Gamma(x)$ is defined as:

$$ \Gamma(x) = \lim_{n \to \infty} \frac{n! n^x}{x(x+1)...(x+n)} \tag{4}$$

for $x \in C$.

The Caputo's definition of fractional derivative is described as:

$$ {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(k-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \qquad (5) $$

where: $n-1 \le \alpha \le n$.

3. ACCELEROMETER. CONCEPT, RESEARCH METHODOLOGY AND RESULTS

Transducers measuring accelerations (accelerometers) are tested [3], [5], [6] and [10], treated as a representative group of measuring transducers. In the classic notation, accelerometers are described with second-order differential equations (6), like many other groups of measuring transducers. Simulation and laboratory testing of a second-order measuring transducer (Fig.1.) - accelerometer has been tested.

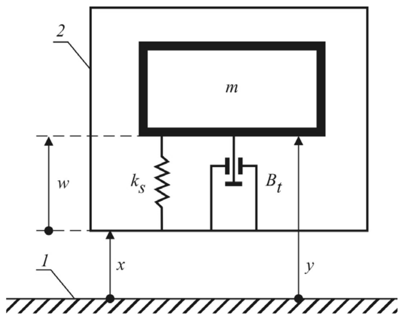


Fig. 1. Kinetic diagram of an accelerometer: m – seismic mass, k_s – spring constant, B_t – damping coefficient, x – object motion relative to a fixed system of coordinates, – motion of a vibrating mass relative to a fixed system of coordinates, w – motion of a vibrating mass relative to a vibrating object, 1- base, 2- transducer housing [3] and [7]

Dynamic behaviour of the accelerometer is written down in a form of a differential equation of the second order [3]:

$$ \frac{d^2}{dt^2} w(t) + 2\zeta\omega_0 \frac{d}{dt} w(t) + \omega_0^2 w(t) = -\frac{d^2}{dt^2} x(t) \qquad (6) $$

where: w - motion of a vibrating mass relative to a vibrating object, x - object motion relative to a fixed system of coordinates, parameters characteristic of accelerometers k - amplification coefficient: ω_0 - natural pulsation and ζ - damping degree. Introducing a non-integral order to the measuring transducer's equation (6) converts it into:

$$ \frac{d^2}{dt^2} w(t) + 2\zeta\omega_0 \frac{d^{(\nu)}}{dt^{(\nu)}} w(t) + \omega_0^2 w(t) = -\frac{d^2}{dt^2} x(t) \qquad (7) $$

where ν – fractional order derivatives.

The concept of authors' work [3], [5], [6] and [7] is based on a comparison of different models of an accelerometer's dynamic behaviour (based on differential equations of integer and fractional orders) with the processing characteristics of a real accelerometer so as to obtain an unambiguous answer to the question about which method of modelling is more accurate and whether there are any criteria for which a certain model is better at reproducing the dynamic behaviour of the real accelerometer.

The research plan has included the following algorithm of proceedings:

1. Investigating processing characteristics of real accelerometers over the entire range of the measuring signal processing with the highest possible measurement accuracy.
2. Developing models describing dynamic behaviour of real accelerometers by means of differential equations of integer order on the basis of characteristics of the measuring signal processing.
3. Developing models describing dynamic behaviour of real accelerometers by means of fractional calculus on the basis of characteristics of the measuring signal processing.
4. Comparing processing characteristics of the accelerometer models from points 2 and 3 with their real counterparts and comparing processing characteristics of different models with each other.

It presents research results of acceleration measurements in the measurement system shown in Fig.2. Table 1 includes some research results.

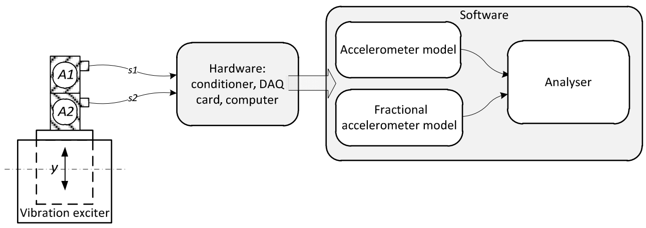


Fig. 2. Measurement system: *A1*, *A2* - model and tested measuring transducers [3]

Signals received from accelerometers of different sensitivities were compared in the system. Sensitivity of accelerometer *A1* which was adopted as a model was ca. 30 times higher than that of the investigated accelerometer *A2*. Equations of integer and fractional orders describing dynamic behaviour of the investigated accelerometer were determined by means of the ARX method (AutoRegressive with eXogenous input identification method) [3], [4], [6] and [10] on the basis of the data from accelerometers.

The signals from determined models were compared to the signal from the model accelerator. The relative errors of measurements were determined adopting the signal from the model accelerator as a reference value. The median of the series of 500 successive measurement samples was adopted as the error measure. Measurements were taken separately for the following frequencies of the vibration exciter: 100 Hz, 200 Hz, 300 Hz, 400 Hz and 500 Hz. On the basis of preliminary investigations it was found out that in the examined cases the models described by means of fractional order equations convey the accelerometer processing characteristics more accurately than integer order equations. Depending on examined frequencies the accuracy of reproduced dynamic behaviour of an accelerometer by a model is between ca. 5% to ca. 10%. These values can be increased if we developed a more accurate model of fractional orders. Table 1 presents results of laboratory tests. Theoretical and simulation tests are included in works [3], [5], [6] and [7].

Table 1. Values of median relative error for the transducer's model of integer and fractional order. [5]

Frequency [Hz]	Median relative error for the integer order model [%]	Median relative error for the fractional order model [%]	Difference [%]
100	30.8089	20.8040	10.0049
200	30.2997	20.8041	9.4956
300	29.5564	20.8042	8.7522
400	28.3097	20.8039	7.5058
500	26.0184	20.8040	5.2144

Conclusions from this research are compatible with conclusions from the laboratory tests [5]. Results of the research suggest that the in the future for new models of accelerometers fractional calculus model of measuring transducer will more accurate at reflecting the dynamics of the input signal processing than the model described by the classical differential equations.

4. MEMBRANE PRESSURE TRANSDUCER

This chapter presents attempt at a mathematical description and frequency analysis of a transmitter of continuous quantities, like for example pressure, with the use of the fractional order differential equations. To examine dynamic properties of the pressure transducer, a model of a pressure chamber with an inlet pipe was made (Fig. 3).

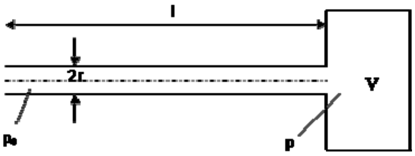


Fig. 3. Pressure chamber with an inlet pipe: *r*, *l* – pipe dimensions, p_0 – inlet pressure, *p* – pressure in the transmitter's chamber [4]

The differential equation constituting the mathematical model of the analysed pneumatic system, in which the fractional order differential equation is applied, looks as follows:

$$\frac{d^{(v_2)}}{dt^{(v_2)}} p(t) + 2\xi\omega_0 \frac{d^{(v_1)}}{dt^{(v_1)}} p(t) + {\omega_0}^2 \frac{d^{(v_0)}}{dt^{(v_0)}} p(t) = = {\omega_0}^2 p_0(t) \quad (8)$$

where: $p(t)$ – pressure in the transmitter's chamber, $p_0(t)$ – inlet pressure, $v > 0$ – order of derivative.

To determine the derivative of a continuous function, i.e. pressure in the transmitter's chamber, we used the Riemann-Liouville definition of fractional derivative (3). The Laplace transform for the Riemann-Liouville fractional derivative is [2]:

$$L\left[{}^{R-L}_{0}D^{\alpha}_{t} f(t)\right] = s^{\alpha} F(s) - \sum_{k=0}^{j-1} s^{k}\, {}^{R-L}_{0}D^{\alpha-k-1}_{t} f(0) \quad (9)$$

where: $j - 1 \le \alpha \le j \in N$

Applying the Laplace transform to equation (8), for zero initial conditions, we obtain:

$$s^{2v} p(s) + 2\xi\omega_0 s^{v} p(s) + {\omega_0}^2 p(s) = {\omega_0}^2 p_0(s) \quad (10)$$

Hence:

$$p(s) = \frac{{\omega_0}^2}{\left(s^{2v} + 2\xi\omega_0 s^{v} + {\omega_0}^2\right)} p_0(s) \quad (11)$$

From equation (11) we obtain the transfer function of the analysed pressure transmitter:

$$G^{(v)}(s) = \frac{p(s)}{p_0(s)} = \frac{{\omega_0}^2}{s^{2v} + 2\xi\omega_0 s^{v} + {\omega_0}^2} \quad (12)$$

Substituting:

$$s = j\omega = \omega e^{j\frac{\pi}{2}} = \omega\left[\cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)\right] \quad (13)$$

in the formula (12), we obtain the spectral transfer of the transmitter:

$$G^{(v)}(j\omega) = \frac{{\omega_0}^2}{(j\omega)^{2v} + 2\xi\omega_0 (j\omega)^{v} + {\omega_0}^2} \quad (14)$$

Owing to elementary transformations we can calculate the real and imaginary parts of the spectral transform function:

$$G^{(v)}(j\omega) = P^{(v)}(\omega) + jQ^{(v)}(\omega) \quad (15)$$

Knowing the real and imaginary part of the spectral transform of the transmitter, we can determine the equation describing the logarithmic amplitude function:

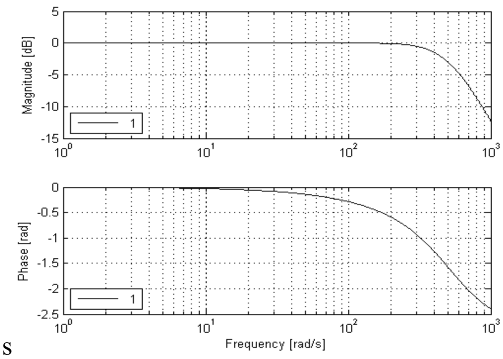
$$M^{(v)}(\omega) = 20\log\sqrt{\left[P^{(v)}(\omega)\right]^2 + \left[Q^{(v)}(\omega)\right]^2} \quad (16)$$

as well as the equation describing the logarithmic phase characteristic:

$$\varphi^{(v)}(\omega) = arctg\left[\frac{Q^{(v)}(\omega)}{P^{(v)}(\omega)}\right] \quad (17)$$

5. NUMERICAL TESTS AND SIMULATIONS

In order to verify the dependencies describing logarithmic functions of amplitude (16) and phase (17) of the tested trasducers, a pneumatic pressure transducer was modelled in the MATLAB&Simulink described by means of an ordinary differential equation and a fractional order differential equation.



s

Fig. 4. Logarithmic frequency responses of the pressure transducer described by the ordinary and fractional order equation [4]

While describing the transmitter with the use of the fractional order differential equation was adopted parameter v=1 and compared the obtained logarithmic functions of the amplitude and phase with the logarithmic functions of amplitude and phase obtained from a description of the pressure transmitter made with an ordinary differential equation. In simulations was adopted: pulsation $\omega = 500$ rad/s and damping coefficient $\xi = 0.7$.

The transfer function of the pneumatic pressure transmitter described with the use of the ordinary differential equation looks as follows:

$$G(s) = \frac{p(s)}{p_0(s)} = \frac{\omega_0^{\ 2}}{s^2 + 2\xi\omega_0 s + \omega_0^{\ 2}} \qquad (18)$$

While conducting simulation of equation (18), which represents dynamics of phenomena occurring in the analysed pneumatic system, in MATLAB environment, we obtained the frequency responses outlined in Fig.4:

When simulating in MATLAB&Simulink environment equations (16) and (17), describing the pneumatic pressure transducer with the use of the fractional order differential equation and adopting the parameter $v = 0$, we obtained the the same functions outlined in Fig. 4. The comparison of classic and fractional models' described by frequency diagrams have the same course in the tested frequency ranges as the classic models. This means that fractional calculus is a generalisation of integral-order differential calculus [3].

6. CONCLUSIONS

In this paper the authors presented (proven by them) thesis: *method of description of the dynamic properties of accelerometer and pressure transducer in terms of signal processing, based on fractional calculus, allows for a description of dynamic properties of broader class of measuring transducers, i.e. integer-order and fractional-order*.

The paper presented possibilities of using fractional calculus in modelling of accelerometer. It describes a laboratory measurement system for investigating dynamic properties of it. This paper presented also attempt at a mathematical description a measuring transducer of continuous quantities, like for example pressure, with the use of the fractional order differential equations.

The authors wants to continue their work on the use of fractional calculus in dynamic measurements for measuring transducers different from those that are introduced in this paper, especially those requiring fast and accurate measurements. Further research will be conducted to verify whether: *the model of dynamic properties of real accelerometer and pressure transducer determined by means of fractional calculus conveys the dynamic performance of the real accelerometer over the entire processing range more accurately than modelling of the same transducer by means of integer order differential equations*.

7. REFERENCES

1. Kaczorek T.: Selected Problems of Fractional Systems Theory, Springer-Verlag GmbH, ISBN 978-3-642-20501-9, Berlin, Germany 2011.
2. Ostalczyk P.: Epitome of the fractional calculus. Theory and its applications in automatics, Wydawnictwo Politechniki Łódzkiej, p. 430, ISBN 978-83-7283-245-0, Lodz, 2008 (Published in Polish).
3. Luft M., Szychta E., Cioć R., Pietruszczak D.: Effect of fractional orders in differential equation describing damping in the measuring transducer, Communications in Computer and Information Science, ISSN 1865-0929, e-ISSN 1865-0937, Springer Heidelberg Dordrecht London New York, 2011.
4. Luft M., Szychta L., Nowocień A., Cioć R., Pietruszczak D.: Charakterystyki częstotliwościowe modelu przetwornika ciśnienia opisanego równaniem różniczkowym niecałkowitego rzędu, Logistyka 4(2015), ISSN 1231-5478, Poznań, 2015 (Published in Polish).
5. Luft M., Pietruszczak D., Szychta E.: Analysis of dynamic properties of accelerometer using fractional derivatives, COMMUNICATIONS 4(2015) - Scientific Letters of the University of Zilina, ISSN 1335-4205, pp. 40-44, EDIS – Publishing Institution of the University of Zilina, Zilina, Slovak Republic, 2015.
6. Pietruszczak D., Szychta E.: Analysis of selected dynamic properties of fractional order accelerometers for application in telematics equipment, Communications in Computer and Information Science, Springer-Verlag Berlin Heidelberg, ISSN 1865-0929, e-ISSN 1865-0937, 2013.
7. Luft M., Szychta E., Cioć R., Pietruszczak D.: Application of fractional calculus in identification of the measuring system, Transport Systems and Processes, CRC Press Balkema, Taylor & Francis Group, ISBN 978-0-415-69120-8, London, UK, 2011.
8. Lesiak P.: Inteligentna technika pomiarowa, ISBN 83-88001-68-X, Wydawnictwo Politechniki Radomskiej, Radom, 2001 (Published in Polish).

O PEWNYCH ZASTOSOWANIACH RACHUNKU RÓŻNICZKOWEGO NIECAŁKOWITYCH RZĘDÓW W MODELOWANIU AKCELEROMETRU I PRZETWORNIKA CIŚNIENIA

W artykule przedstawiono możliwości zastosowania rachunku różniczkowego niecałkowitych rzędów (ang. fractional calculus) do opisu właściwości dynamicznych akcelerometru i przetwornika ciśnienia dowolnych rzędów, co autorzy uważają za swoje oryginalne osiągnięcie w pracy naukowej. Badania symulacyjne wykonano w środowisku MATLAB&Simulink. Autorzy zakładają, że przyszłości właściwości dynamiczne modeli przetworników o nowych rozwiązaniach konstrukcyjnych i technologicznych będą wymagały opisu za pomocą rachunku różniczkowego niecałkowitych rzędów. Takie założenie, że „teoria" powinna wyprzedzać „praktykę" wydaje się być słusznym gdyż wielokrotnie sprawdzała się w przeszłości. Przykładem może być tutaj klasyczny kondensator i jego „ułamkowy" odpowiednik: superkondensator.

Słowa kluczowe: akcelerometr, miernictwo dynamiczne, MATLAB&Simulink, przetwornik ciśnienia, rachunek różniczkowo-całkowy niecałkowitych rzędów.