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THIRD ORDER SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATION OF REACTION DIFFUSION TYPE WITH INTEGRAL BOUNDARY CONDITION

Elango Sekar, Ayyadurai Tamilselvan

Bharathidasan University
Tiruchirappalli, India
sekarmaths036@gmail.com, mathats@bdu.ac.in

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Abstract. A class of third order singularly perturbed delay differential equations of reaction diffusion type with an integral boundary condition is considered. A numerical method based on a finite difference scheme on a Shishkin mesh is presented. The method suggested is of almost first order convergent. An error estimate is derived in the discrete norm. Numerical examples are presented, which validate the theoretical estimates.

MSC 2010: *65L11, 74S20, 65N12*
Keywords: *singular perturbation problem, finite difference scheme, Shishkin mesh, delay, integral boundary condition, error estimate*

1. Introduction

We consider the following class of third order singularly perturbed delay differential equations of reaction diffusion type with integral boundary condition:

$$\begin{cases} -\varepsilon u'''(x)+b(x)u'(x)+c(x)u(x)+d(x)u'(x-1)=f(x),\ x\in(0,2), \\ u(x)=\phi(x),\ x\in[-1,0],\ \phi\in C^1[-1,0]\ \ u'(2)=\varepsilon\int\limits_0^2 g(x)u'(x)dx+l, \end{cases} \tag{1}$$

where $0<\varepsilon<<1$, $b(x)\geq\alpha\geq 0$, $\theta\leq c(x)\leq\theta_0\leq 0$, $\gamma\leq d(x)\leq\gamma_0\leq 0$, $\alpha+\theta+\gamma>0$, $g(x)$ is nonnegative and monotone with $\int\limits_0^2 g(x)dx<1$ and $b(x),c(x),d(x),f(x),g(x)$ are sufficiently smooth on $\bar{\Omega}=[0,2]$ and l be a real number. Define $\Omega_1=(0,1)$, $\Omega_2=(1,2)$, $\Omega^*=\Omega_1\cup\Omega_2$, $\bar{\Omega}^{2N}=\{0,1,2,\ldots,2N\}$, $\Omega_1^{2N}=\{1,2,\ldots,N-1\}$, $\Omega_2^{2N}=$ $=\{N+1,N+2,\ldots,2N-1\}$ and $\Omega^{*2N}=\Omega_1^{2N}\cup\Omega_2^{2N}$.

A differential equation is said to be singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions with various different arguments and also the highest derivative term is multiplied by a small parameter ε. It is well known that the standard numerical methods used for solving singularly perturbed differential equations are not well posed and fail to give an analytical solution when the perturbation parameter ε is small. Therefore, it is necessary to improve suitable numerical methods which are uniformly convergent to solve the problem. Some authors have worked on singularly perturbed differential equations with delay using uniformly convergent numerical methods [1–6].

Differential equations with integral boundary conditions is an important class of problems arising in the fields of electro chemistry [7], thermo elasticity [8], heat conduction [9] etc.

In the present paper, motivated by the works of [6, 10–12], we analyze a fitted finite difference scheme on a piecewise uniform mesh for the numerical solution of third order singularly perturbed delay differential equation of reaction diffusion type with an integral boundary condition.

This paper is arranged in the following manner. In Section 2, the maximum principle, stability result and derivative estimate are derived for the continuous problem. The discretized problem is discussed in Section 3. An error estimate for the numerical method is established in Section 4. We carried out numerical experiments in Section 5. The paper concludes with a discussion given in Section 6.

2. Statement of the problem

The boundary value problem (1) can be transformed into the following equivalent problem:

Find $\bar{u} = (u_1, u_2)$, $u_1 \in X_1 = C^0(\bar{\Omega}) \cap C^1(\Omega \cup \{2\})$ and $u_2 \in X_2 = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$:

$$L_1\bar{u}(x) = u_1'(x) - u_2(x) = 0, \; x \in \Omega \cup \{2\} \tag{2}$$

$$L_2\bar{u}(x) = \begin{cases} -\varepsilon u_2''(x) + b(x)u_2(x) + c(x)u_1(x) = f(x) - d(x)\phi'(x-1), & x \in \Omega_1 \\ -\varepsilon u_2''(x) + b(x)u_2(x) + c(x)u_1(x) + d(x)u_2(x-1) = f(x), & x \in \Omega_2 \end{cases} \tag{3}$$

where $\bar{u}(x) = (u_1(x), u_2(x))$ with the boundary conditions

$$u_1(0) = \phi(0), \quad u_2(0) = \phi'(0), \; u_2(1^-) = u_2(1^+),$$
$$u_2'(1^-) = u_2'(1^+), \; Ku_2(2) = u_2(2) - \varepsilon \int_0^2 g(x)u_2(x)dx = l. \tag{4}$$

The norm used for studying the convergence of the numerical solution is supremum norm defined by $\|u\|_{\Omega^*} := \sup_{x\in\Omega^*} |u(x)|$.

Theorem 1 ***(Maximum Principle)*** *Let $\bar{u}(x) = (u_1(x), u_2(x))$ be any function satisfying $u_1(0) \geq 0, u_2(0) \geq 0$, $Ku_2(2) \geq 0$, $L_1\bar{u}(x) \geq 0$, $x \in \Omega \cup \{2\}$, $L_2\bar{u}(x) \geq 0, \forall\, x \in \Omega^*$ and $[u_2'](1) \leq 0$. Then $\bar{u}(x) \geq 0$, $\forall\, x \in \bar{\Omega}$.* □

PROOF Define $\bar{s}(x) = (s_1(x), s_2(x))$ as $s_1(x) = 1 + x,\ \ x \in \bar{\Omega}$ and

$$s_2(x) = \begin{cases} \dfrac{1}{8} + \dfrac{x}{2}, & x \in [0,1] \\ \dfrac{3}{8} + \dfrac{x}{4}, & x \in [1,2]. \end{cases}$$

Note that $\bar{s}(x) > 0,\ x \in \bar{\Omega},\ L_1\bar{s}(x) > 0,\ L_2\bar{s}(x) > 0,\ s_1(0) > 0,\ s_2(0) > 0$ and $Ks_2(1) > 0$. Further we define

$$\mu = \max\left\{ \max_{x\in\bar{\Omega}} \left(\frac{-u_1(x)}{s_1(x)} \right), \max_{x\in\bar{\Omega}} \left(\frac{-u_2(x)}{s_2(x)} \right) \right\}.$$

Then there exists at least one $x_0 \in \Omega$, such that $\left(\dfrac{-u_1(x_0)}{s_1(x_0)} \right) = \mu$ or $\left(\dfrac{-u_2(x_0)}{s_2(x_0)} \right) = \mu$ or both. Also $(\bar{u} + \mu\bar{s})(x) \geq \bar{0},\ x \in \bar{\Omega}$. Therefore either $(u_1 + \mu s_1)$ or $(u_2 + \mu s_2)$ attains minimum at $x = x_0$. Suppose the theorem does not hold true, then $\mu > 0$.
Case (i): Assume that $(u_1 + \mu s_1)(x_0) = 0$, for $x_0 = 0$. Therefore $(u_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$0 = (u_1 + \mu s_1)(0) = u_1(0) + \mu s_1(0) > 0.$$

Case (ii): Assume that $(u_1 + \mu s_1)(x_0) = 0$, for $x_0 \in \Omega \cup \{2\}$. Therefore $(u_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$0 < L_1(\bar{u} + \mu\bar{s})(x_0) \quad = \quad (u_1 + \mu s_1)'(x_0) - (u_2 + \mu s_2)(x_0) \leq 0.$$

The proof for the operators $L_2(\bar{u} + \mu\bar{s})(x_0)$ and $K(\bar{u} + \mu\bar{s})(x_0)$ are similar cases refer [6].

Observe that in all the cases we arrived at a contradiction. Therefore $\mu > 0$ is not possible. Hence $\bar{u}(x) \geq 0, \forall\, x \in \bar{\Omega}$. ■

Corollary 1 *(Stability Result) The solution $\bar{u}(x)$ of problem $(2)-(4)$ satisfies the bound*

$$|u_i(x)| \leq C \max\{|u_1(0)|, |u_2(0)|, |Ku_2(1)|, ||L_1\bar{u}||_{\bar{\Omega}}, ||L_2\bar{u}||_{\Omega^*}\},\ x \in \bar{\Omega},\ i = 1,2.$$

PROOF Refer [6]. ■

Bounds for the derivatives of the solution $\bar{u}(x)$ are given in the following lemma.

Lemma 1 *Let $\bar{u}(x)$ be the solution of $(2)-(4)$. Then we have the following bounds:*

$$||u_1^{(k)}|| \leq C\varepsilon^{(1-k)/2},\ k = 1,2,3,$$
$$||u_2^{(k)}|| \leq C\varepsilon^{-k/2},\ k = 1,2,3.$$ □

PROOF Refer [6]. ∎

The Shishkin decomposition of the solution $\bar{u}(x)$ of $(2)-(4)$ is $\bar{u}(x)=\bar{v}(x)+\bar{w}(x)$, where $\bar{v}(x)=(v_1(x),v_2(x))$ and $\bar{w}(x)=(w_1(x),w_2(x))$ are regular and singular components respectively. The regular component $\bar{v}(x)$ can be written as $\bar{v}(x)=\bar{v}_0(x)+$ $+\,\varepsilon\bar{v}_1(x)$, where $\bar{v}_0=(v_{01},v_{02})$ and $\bar{v}_1=(v_{11},v_{12})$ satisfy the following equations:

$$\begin{cases} L_1 v(x) = v'_{11}(x) - v_{12}(x) = 0, & x\in\bar{\Omega} \\ v_{11}(0)=\phi(0), \end{cases} \tag{5}$$

$$\begin{cases} L_2\bar{v}_1(x) = -\varepsilon v''_{12}(x)+b(x)v_{12}(x)+c(x)v''_{11}(x) = v''_{02}(x), & x\in\Omega_1 \\ L_2\bar{v}_1(x) = -\varepsilon v''_{12}(x)+b(x)v_{12}(x)+c(x)v''_{11}(x)+d(x)v_{12}(x-1) = v''_{02}(x), & x\in\Omega_2 \\ \bar{v}_1(0)=0,\ \bar{v}_1(1)=0,\ K\bar{v}_1(2)=0. \end{cases} \tag{6}$$

Layer component $\bar{w}(x)=(w_1(x),w_2(x))$ is the solution of

$$\begin{cases} L_1\bar{w}(x) = w'_1(x)-w_2(x)=0, \\ w_1(0)=0. \end{cases} \tag{7}$$

$$\begin{cases} L_2\bar{w}(x) = -\varepsilon w''_2(x)+b(x)w_2(x)+c(x)w_1(x)=0, & x\in\Omega_1 \\ L_2\bar{w}(x) = -\varepsilon w''_2(x)+b(x)w_2(x)+c(x)w_1(x)+d(x)w_2(x-1)=0, & x\in\Omega_2 \\ \bar{w}(0)=\bar{u}(0)-\bar{v}(0),\quad [\bar{w}](1)=-[\bar{v}](1),\ K\bar{w}_2(2)=K\bar{u}_2(2)-K\bar{v}_2(2). \end{cases} \tag{8}$$

We further decompose $\bar{w}(x)$ as $\bar{w}(x)=\bar{w}_L(x)+\bar{w}_R(x)$.

The left layer components $\bar{w}_L(x)$ are the solutions of the following problems:
Find $\bar{w}_L(x)\in Y$ such that

$$\begin{cases} L_1\bar{w}_L(x)=0, \quad x\in\bar{\Omega}, \quad \bar{w}_L(0)=0 \\ L_2\bar{w}_{L1}(x)=0, \quad x\in\Omega_1, \quad \bar{w}_{L1}(0)=w(0),\ \bar{w}_{L1}(1)=0 \\ L_2\bar{w}_{L2}(x)=0, \quad x\in\Omega_2, \quad \bar{w}_{L2}(1)=\bar{A},\ \bar{w}_L(2)=0. \end{cases} \tag{9}$$

Further the right layer components $\bar{w}_R(x)$ are the solutions of the following problems:
Find $\bar{w}_R(x)\in Y$ such that

$$\begin{cases} L_1\bar{w}_R(x)=0, \quad x\in\bar{\Omega}, \quad \bar{w}_R(0)=0 \\ L_2\bar{w}_{R1}(x)=0, \quad x\in\Omega_1, \quad \bar{w}_{R1}(0)=0,\ \bar{w}_{R1}(1)=\bar{A}_1. \\ L_2\bar{w}_{R2}(x)=0, \quad x\in\Omega_2, \quad \bar{w}_{R2}(1)=0,\ K\bar{w}_{R2}(2)=K\bar{w}(2). \end{cases} \tag{10}$$

where $\bar{A}$ and $\bar{A}_1$ are constants to be chosen in order to satisfy the jump conditions at the point $x=1$.

Lemma 2 *The regular component $\bar{v}(x)$ satisfies the following bounds.*

$$\|v_1^k\|_{\Omega^*} \leq C(1+\varepsilon^{-(k-3)/2}), \quad \text{for k} = 0,1,2,3 \tag{11}$$

$$\|v_2^k\|_{\Omega^*} \leq C(1+\varepsilon^{-(k-2)/2}), \quad \text{for k} = 0,1,2,3 \tag{12}$$

□

PROOF Integrating the reduced problem of $(2)-(4)$ and $(5)-(6)$ and using the Corollary 1, the inequality $(11)-(12)$ can be proved easily. ■

Lemma 3 *The singular component $\bar{w}(x)$ satisfies the following bounds.*

$$|w_{R1}^k(x)| \leq C\varepsilon^{-(k-1)/2}exp(\frac{-(2-x)\sqrt{\alpha}}{\sqrt{\varepsilon}}), \quad x \in \bar{\Omega}, \ k = 0,1,2,3 \tag{13}$$

$$|w_{R2}^k(x)| \leq C\varepsilon^{-k/2}\begin{cases} exp(\frac{-(1-x)\sqrt{\alpha}}{\sqrt{\varepsilon}}), & x \in \Omega_1, \ k = 0,1,2,3 \\ exp(\frac{-(2-x)\sqrt{\alpha}}{\sqrt{\varepsilon}}), & x \in \Omega_2, \ k = 0,1,2,3 \end{cases} \tag{14}$$

$$|w_{L1}^k(x)| \leq C\varepsilon^{-(k-1)/2}exp(\frac{-x\sqrt{\alpha}}{\sqrt{\varepsilon}}), \quad x \in \bar{\Omega}, \ k = 0,1,2,3 \tag{15}$$

$$|w_{L2}^k(x)| \leq C\varepsilon^{-k/2}\begin{cases} exp(\frac{-x\sqrt{\alpha}}{\sqrt{\varepsilon}}), & x \in \Omega_1, \ k = 0,1,2,3 \\ exp(\frac{-(x-1)\sqrt{\alpha}}{\sqrt{\varepsilon}}), & x \in \Omega_2, \ k = 0,1,2,3 \end{cases} \tag{16}$$

□

PROOF To prove the inequalities $(13)-(14)$, consider the barrier functions $\bar{\Phi}^{\pm}(x) = = (\Phi_1^{\pm}(x), \Phi_2^{\pm}(x))$, where

$$\Phi_1^{\pm}(x) = C\sqrt{\varepsilon}exp(\frac{-(2-x)\sqrt{\alpha}}{\sqrt{\varepsilon}}) \pm w_{R1}(x), \quad x \in \bar{\Omega}$$

$$\Phi_2^{\pm}(x) = C\begin{cases} exp(\frac{-(1-x)\sqrt{\alpha}}{\sqrt{\varepsilon}}), & x \in \Omega_1 \\ exp(\frac{-(2-x)\sqrt{\alpha}}{\sqrt{\varepsilon}}), & x \in \Omega_2 \end{cases} \pm w_{R2}(x)$$

It is easy to see that $\Phi_1^{\pm}(0) \geq 0$ and $\Phi_2^{\pm}(0) \geq 0$, for a suitable choice of $C > 0$. Further,

$$K\Phi_2^{\pm}(2) = \Phi_2^{\pm}(2) - \varepsilon\int_0^2 g(x)\Phi_2^{\pm}(x)dx \geq C[1-\varepsilon\int_0^2 g(x)dx] \pm Kw_{R2}(2) \geq 0.$$

Also $L_1\Phi_1(x) \geq 0$ and $L_2\Phi_2(x) \geq 0$. By Theorem 1, we have a right layer bound. Integration of (10) yields the estimates of $|w'_R(x)|$. From the differential equations (10), one can derive the rest of the derivative estimates (13)-(14).

Similarly, the bounds for left layer components $(15)-(16)$ can be derived. ∎

3. The discrete problem

3.1. Mesh selection procedure

The boundary value problem $(2)-(4)$ exhibits strong boundary layers at $x = 0$, $x = 2$ and interior layers(left and right) at $x = 1$.

The interval $[0,1]$ is partitioned into three piecewise uniform Shishkin meshes as: $[0,1] = [0,\sigma] \cup [\sigma, 1-\sigma] \cup [1-\sigma, 1]$. Similarly, $[1,2]$ is partitioned into three piecewise uniform Shishkin meshes as: $[1,2] = [1,1+\sigma] \cup [1+\sigma, 2-\sigma] \cup [2-\sigma, 2]$, where σ is the transition parameter defined by $\sigma = \min\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N\}$. The discrete problem corresponding to $(2)-(4)$ is:

Find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))$ such that

$$\begin{cases} L_1^N\bar{U}(x_i) = D^-U_1(x_i) - U_2(x_i) = 0, \\ L_2^N\bar{U}(x_i) = -\varepsilon\delta^2U_2(x_i) + b(x_i)U_2(x_i) + c(x_i)U_1(x_i) + d(x_i)U_2^*(x_i) = f^*(x_i), \end{cases} \tag{17}$$

$$\begin{cases} U_1(x_0) = \phi(0),\ U_2(x_0) = \phi(0),\ D^-U_2(x_N) = D^+U_2(x_N), \\ K^NU_2(x_{2N}) = U_2(x_{2N}) - \varepsilon\displaystyle\sum_{i=1}^{2N}\frac{g(x_{i-1})U_2(x_{i-1}) + g(x_i)U_2(x_i)}{2}h_i = l, \forall x_i \in \bar{\Omega}^{2N}. \end{cases} \tag{18}$$

where:

$$\begin{aligned} \delta^2U_2(x_i) &= \frac{1}{\bar{h}_i}\left(\frac{U_2(x_{i+1}) - U_2(x_i)}{h_{i+1}} - \frac{U_2(x_i) - U_2(x_{i-1})}{h_i}\right), \\ D^-U_2(x_i) &= \frac{U_2(x_i) - U_2(x_{i-1})}{h_i}. \\ f^*(x_i) &= \begin{cases} f(x_i) - d(x_i)\phi'(x_i - 1), & x_i \in {\Omega_1}^{2N} \cap \bar{\Omega}^{2N} \\ f(x_i), & x_i \in {\Omega_2}^{2N} \cap \bar{\Omega}^{2N}. \end{cases} \\ U_2^*(x_i) &= \begin{cases} 0, & x_i \in {\Omega_1}^{2N} \cap \bar{\Omega}^{2N} \\ U_2(x_{i-N}) & x_i \in {\Omega_2}^{2N} \cap \bar{\Omega}^{2N}. \end{cases} \end{aligned}$$

4. Analysis of the method

Theorem 2 ***(Discrete Maximum Principle)*** *Let $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))$ be the mesh function satisfying $\Psi_1(x_0) \geq 0$, $\Psi_2(x_0) \geq 0$, $K^N\Psi_2(x_{2N}) \geq 0$, $L_1^N\bar{\Psi}(x_i) \geq 0$, $L_2^N\bar{\Psi}(x_i) \geq 0$, and $[D]\Psi_2(x_N) \leq 0$. Then $\bar{\Psi}(x_i) \geq 0$, $x_i \in \bar{\Omega}^{2N}$.* □

PROOF Define $\bar{S}(x_i) = (S_1(x_i), S_2(x_i))$, where $S_1(x_i) = 1 + x_i$, $x_i \in \bar{\Omega}^{2N}$ and

$$S_2(x_i) = \begin{cases} \dfrac{1}{8} + \dfrac{x_i}{2}, & x_i \in \Omega_1^{2N} \cap \bar{\Omega}^{2N} \\ \dfrac{3}{8} + \dfrac{x_i}{4}, & x_i \in \Omega_2^{2N} \cap \bar{\Omega}^{2N}. \end{cases}$$

Note that $S_k(x_i) > 0$, $x_i \in \bar{\Omega}^{2N}$, $k = 1,2$, $L_1^N\bar{S}(x_i) > 0, \forall x_i \in \bar{\Omega}^{2N} \cap \Omega \cup \{x_{2N}\}$, $L_2^N\bar{S}(x_i) > 0, \forall x_i \in \bar{\Omega}^{2N} \cap \Omega^*$. Let

$$\gamma = \max\left\{ \max_{x_i \in \bar{\Omega}^{2N}} \left(\frac{-\Psi_1(x_i)}{S_1(x_i)} \right), \max_{x_i \in \bar{\Omega}^{2N}} \left(\frac{-\Psi_2(x_i)}{S_2(x_i)} \right) \right\}.$$

Then there exists one $x_k \in \bar{\Omega}^{2N}$ such that $(\Psi_1 + \gamma S_1)(x_k) = 0$ or $(\Psi_2 + \gamma S_2)(x_k) = 0$ or both. We have $(\Psi_j + \gamma S_j)(x_i) \geq 0, x_i \in \bar{\Omega}^{2N}, j = 1,2$. Therefore either $(\Psi_1 + \gamma S_1)$ or $(\Psi_1 + \gamma S_1)$ attains minimum at $x_i = x_k$. Suppose the theorem does not hold true, then $\gamma > 0$.
Case (i): Assume that $(\Psi_1 + \gamma S_1)(x_k) = 0$, for $x_k = 0$. Therefore $(\Psi_1 + \gamma S_1)$ attains its minimum at $x_i = x_k$. Then,

$$0 = (\Psi_1 + \gamma S_1)(x_0) = \Psi_1(x_0) + \gamma S_1(x_0) > 0.$$

Case (ii): Assume that $(\Psi_1 + \gamma S_1)(x_k) = 0$, for $x_k \in \Omega^{2N} \cup \{x_{2N}\}$. Therefore $(\Psi_1 + + \gamma S_1)$ attains its minimum at $x_i = x_k$. Then,

$$0 < L_1^N(\bar{\Psi} + \gamma\bar{S})(x_i) = D^-(\Psi_1 + \gamma S_1)(x_i) - (\Psi_2 + \gamma S_2)(x_i) \leq 0.$$

Refer [6] for the remaining part of the proof for the operator $L_2^N(\bar{\Psi} + \gamma\bar{S})(x_i)$ and $K^N(\bar{\Psi} + \gamma\bar{S})(x_i)$. ■

Lemma 4 ***(Discrete Stability Result)*** *Let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))$ be any mesh function. Then*

$$|U_k(x_i)| \leq C \max\Big\{ |U_1(x_0)|, |U_2(x_0)|, |KU_2(x_{2N})|, \max_{x_j \in \bar{\Omega}^{2N}} |L_1^N\bar{U}(x_j)| \max_{x_j \in \Omega_1^{2N} \cup \Omega_2^{2N}} |L_2^N\bar{U}(x_j)| \Big\}, \quad x_i \in \bar{\Omega}^{2N}, \quad k = 1,2.$$ □

PROOF By choosing suitable barrier functions and using Theorem 2, one can establish the above inequality. ■

Analogous to the continuous case, the discrete solution $\bar{U}(x_i)$ can be decomposed as

$$\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i),$$

where $V(x_i)$ and $W(x_i)$ are respectively the solutions of the problems:

$$\begin{cases} L_1^N \bar{V}(x_i) = D^- V_1(x_i) - V_2(x_i) = 0, \ \ x_i \in \Omega^{2N} \setminus \{0\}, \ \ V_1(x_0) = v_1(0), \\ L_2^N \bar{V}(x_i) = -\varepsilon\delta^2 V_2(x_i) + b(x_i)V_2(x_i) + c(x_i)V_1(x_i) + d(x_i)V_2^*(x_i) = f^*(x_i), \\ x_i \in \Omega^{2N} \setminus \{0, N, 2N\}, \\ V_2(x_0) = v_2(0), V_2(x_{N-1}) = (v_2)(1^-), V_2(x_{N+1}) = (v_2)(1^+), \ K^N V_2(x_{2N}) = Kv_2(2) \end{cases} \tag{19}$$

and

$$\begin{cases} L_1^N \bar{W}(x_i) = D^- W_1(x_i) - W_2(x_i) = 0, \ \ x_i \in \Omega^{2N} \setminus \{0\}, \ \ W_1(x_0) = w_1(0), \\ L_2^N \bar{W}(x_i) = -\varepsilon\delta^2 W_2(x_i) + b(x_i)W_2(x_i) + c(x_i)W_1(x_i) + d(x_i)W_2^*(x_i) = 0, \\ x_i \in \Omega^{2N} \setminus \{0, N, 2N\} \\ W_2(x_0) = w_2(0), \ \ V_2(x_{N+1}) + W_2(x_{N+1}) = V_2(x_{N-1}) + W_2(x_{N-1}), \\ D^- W_2(x_N) + D^- V_2(x_N) = D^+ W_2(x_N) + D^+ V_2(x_N, \ K^N W_2(x_{2N}) = Kw_2(2). \end{cases} \tag{20}$$

We obtain error estimates separately for each component of the numerical solution.

Lemma 5 *Let $\bar{V}(x_i)$ be a numerical solution of $(5)-(6)$ defined by (19). Then*

$$|(v_j(x_i) - V_j(x_i))| \leq CN^{-1}, \ x_i \in \bar{\Omega}^{2N}, \ j = 1, 2.$$

PROOF Now

$$L_1^N(\bar{v}(x_i) - \bar{V}(x_i)) \ = \ L_1^N \bar{v}(x_i) - L_1^N \bar{V}(x_i) = \left(D^- - \frac{d}{dx}\right) v_1(x_i),$$

$$L_2^N(\bar{v}(x_i) - \bar{V}(x_i)) = -\varepsilon\left(\delta^2 - \frac{d^2}{dx^2}\right) v_2(x_i) + d(x_i) \begin{cases} 0, \ i = 1, 2, ..., N-1 \\ v_2^* - v_2(x_{i-N}), \ i = N+1, N+2, \\ ..., 2N-1 \end{cases}.$$

Therefore

$$L_j^N(\bar{v}(x_i) - \bar{V}(x_i)) \leq CN^{-1}, \ x_i \in \Omega^{2N}, \ j = 1, 2.$$

Further

$$\begin{aligned} K^N(v_2 - V_2)(x_{2N}) \ &= \ K^N v_2(x_{2N}) - K^N V_2(x_{2N}) = K^N v_2(x_{2N}) - Kv_2(2) \\ |K^N(v_2 - V_2)(x_{2N})| \ &\leq \ C\varepsilon(h_1^3 v''(\chi_1) + \cdots + h_N^3 v''(\chi_N)) \leq CN^{-2} \ \leq CN^{-1}. \end{aligned}$$

where $x_{i-1} \leq \chi_i \leq x_i,\ 1 \leq i \leq 2N$. Then by the discrete stability result, we have $|(v_j(x_i) - V_j(x_i))| \leq CN^{-1},\ x_i \in \bar{\Omega}^{2N},\ j = 1,2.$ ■

Lemma 6 *Let $\bar{W}(x_i)$ be a numerical solution of* $(7) - (8)$ *defined in* (20)*. Then*

$$|(w_j - W_j)(x_i)| \leq CN^{-1}(\ln N)^2,\ x_i \in \bar{\Omega}^{2N},\ j = 1,2.$$

PROOF Following the technique as in [13], we have $|L_1^N(w_j(x_i) - W_j(x_i))| \leq CN^{-1}\ln N$, $x_i \in \bar{\Omega}^{2N}$ and $|L_2^N(w_j(x_i) - W_j(x_i))| \leq CN^{-1}\ln N,\ x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$.

By the Lemma 4, we have

$$|w_j(x_i) - W_j(x_i)| \leq CN^{-1},\ x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}.$$

At the point $x_i = x_{2N}$, firstly the estimate for $\bar{W}_L - \bar{w}_L$ is given. The argument depends on whether $\sigma = \dfrac{1}{4}$ or $\sigma = 2\sqrt{\dfrac{\varepsilon}{\alpha}}\ln N < \dfrac{1}{4}$

Case (i): $\sigma = \dfrac{1}{4}$

In this case the mesh is uniform and $2\sqrt{\dfrac{\varepsilon}{\alpha}}\ln N \geq \dfrac{1}{4}$, it is clear that $x_i - x_{i-1} = N^{-1}$ and $\varepsilon^{-\frac{1}{2}} \leq C\ln N$. From [13] it follows that

$$\begin{aligned}
K_j^N(\bar{W}_L - \bar{w}_L)(x_{2N}) &= K_j^N\bar{W}_L(x_{2N}) - K_j^N\bar{w}_L(x_{2N}) \\
&= l_j - K_j^N\bar{w}_L(x_{2N}) \\
&= K_j\bar{w}_L(x_{2N}) - K_j^N\bar{w}_L(x_{2N}) \\
|K_j^N(\bar{W}_L - \bar{w}_L)(x_{2N})| &\leq C\varepsilon((h_1^3\bar{w}_L''(\chi_1) + \cdots + h_{2N}^3\bar{w}_L''(\chi_{2N})) \\
&\leq C\varepsilon^{-1}(h_1^3 + \cdots + h_{2N}^3) \\
&\leq CN^{-1},
\end{aligned}$$

where $x_{i-1} \leq \chi_i \leq x_i$. Applying Lemma 4 to the function $(\bar{W}_L - \bar{w}_L)(x_i)$ gives

$$|(\bar{W}_L - \bar{w}_L)(x_i)| \leq C(N^{-1}\ln N).$$

Case (ii): $\sigma < \dfrac{1}{4}$

The mesh is piecewise uniform, with the mesh spacing $2(1-2\sigma)/N$ in the subinterval $[\sigma, 1-\sigma]$ and $[1+\sigma, 2-\sigma]$ and $4\sigma/N$ in each of the subintervals $[0,\sigma]$, $[1-\sigma, 1]$, $[1, 1+\sigma]$ and $[2-\sigma, 2]$. From [13] it follows that

$$|K_j^N(\bar{W}_L - \bar{w}_L)(x_i)| \leq C(N^{-1}\ln N)$$

and

$$\begin{aligned}
|K_j^N(\bar{W}_L - \bar{w}_L)(x_N)| &\leq \varepsilon|C(h_1^3 w''(\chi_1) + \cdots + h_N^3 w''(\chi_{2N}))| \\
&\leq C(h_1^3 + \cdots + h_{2N}^3) \leq CN^{-1},
\end{aligned}$$

where $x_{i-1} \leq \chi_i \leq x_i$. Applying Lemma 4 to the function $(\bar{W}_L - \bar{w}_L)(x_i)$ gives

$$|(\bar{W}_L - \bar{w}_L)(x_i)| \leq C(N^{-1}\ln N).$$

Analogous arguments are used to establish the error estimate for $\bar{W}_R$. This completes the proof. ∎

Theorem 3 *Let $\bar{U}(x_i)$ be the solution of $(2)-(4)$ defined in $(17)-(18)$. Then*

$$|u_j(x_i) - U_j(x_i)|_{\bar{\Omega}^{2N}} \leq CN^{-1}(\ln N), \text{ where } j = 1,2.$$

PROOF Combining Lemma 5 and Lemma 6, the proof gets completed. ∎

5. Numerical result

The ε-uniform convergence of the numerical method proposed in this paper is illustrated through one example presented in this section.

Example 1

$$\begin{cases} -\varepsilon u'''(x) + 5u'(x) - 2u(x) - u'(x-1) = 1, \;\; x \in \Omega^* \\ u(x) = 1, \; x \in [-1,0], \quad u'(2) = \varepsilon \int_0^2 \frac{x}{3} u(x)dx + 2, \end{cases}$$

Example 2

$$\begin{cases} -\varepsilon u'''(x) + (x^2+1)u'(x) - xu(x) - u'(x-1) = e^x, \;\; x \in \Omega^* \\ u(x) = 1, \; x \in [-1,0], \quad u'(2) = \varepsilon \int_0^2 \frac{x}{3} u(x)dx + 5, \end{cases}$$

Table 1. Maximum pointwise errors and order of convergence for Example 1

	Number of mesh points 2N					
	32	64	128	256	512	1024
D_1^N	5.5241e-02	2.5460e-02	1.2190e-02	6.1527e-03	3.1160e-03	1.5641e-03
P_1^N	1.1175e+00	1.0626e+00	9.8636e-01	9.8150e-01	9.9438e-01	-
D_2^N	2.6336e-02	1.2465e-02	5.4961e-03	2.5944e-03	1.3281e-03	6.7400e-04
P_2^N	1.0792e+00	1.1814e+00	1.0830e+00	9.6608e-01	9.7853e-01	-

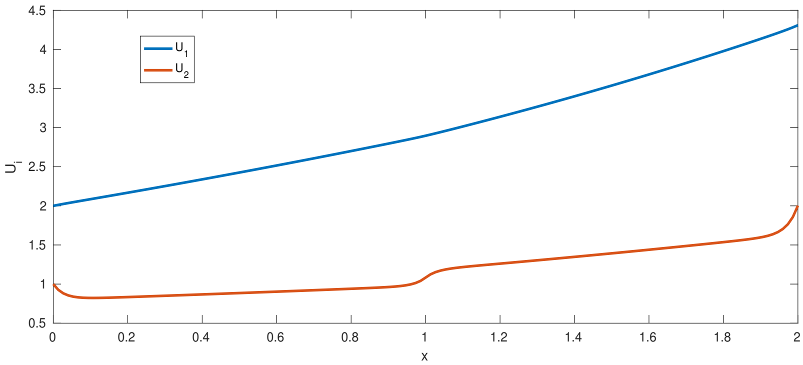

Fig. 1. Numerical solution graph of Example 1

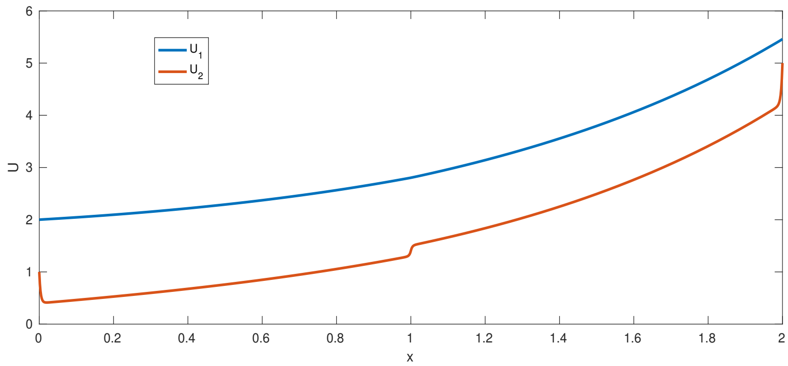

Fig. 2. Numerical solution graph of Example 2

Table 2. Maximum pointwise errors and order of convergence for Example 2

	Number of mesh points 2N					
	32	64	128	256	512	1024
D_1^N	9.5017e-02	4.2218e-02	2.0233e-02	1.0037e-02	4.9987e-03	2.4944e-03
P_1^N	1.1703e+00	1.0612e+00	1.0114e+00	1.0057e+00	1.0028e+00	-
D_2^N	3.4660e-02	1.5527e-02	7.0288e-03	2.7895e-03	1.3470e-03	6.3924e-04
P_2^N	1.1585e+00	1.1435e+00	1.3332e+00	1.0503e+00	1.0753e+00	-

6. Conclusions

We have solved a class of third order singularly perturbed delay differential equations with an integral boundary condition using the finite difference method on a piecewise uniform mesh. One example is presented which authenticates our proposed numerical method. We have proved that the order of our numerical method is $O(N^{-1}\ln N)$ (see Tables 1 and 2). Graph of numerical solution of Examples 1 and 2 is given in Figures 1 and 2.

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