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De la Vallée Poussin Summability, the Combinatorial Sum $\sum_{k=n}^{2n-1} \binom{2k}{k}$ and the de la Vallée Poussin Means Expansion

Ziad S. Ali

Abstract: In this paper we apply the de la Vallée Poussin sum to a combinatorial Chebyshev sum by Ziad S. Ali in [1]. One outcome of this consideration is the main lemma proving the following combinatorial identity: with $Re(z)$ standing for the real part of z we have

$$\sum_{k=n}^{2n-1} \binom{2k}{k} = Re\left(\binom{2n}{n} {}_2F_1(1, 1/2 + n; 1 + n; 4) - \binom{4n}{2n} {}_2F_1(1, 1/2 + 2n; 1 + 2n; 4) \right).$$

Our main lemma will indicate in its proof that the hypergeometric factors

$${}_2F_1(1, 1/2 + n; 1 + n; 4), \quad \text{and} \quad {}_2F_1(1, 1/2 + 2n; 1 + 2n; 4)$$

are complex, each having a real and imaginary part.

As we apply the de la Vallée Poussin sum to the combinatorial Chebyshev sum generated in the Key lemma by Ziad S. Ali in [1], we see in the proof of the main lemma the extreme importance of the use of the main properties of the gamma function. This represents a second important consideration.

A third new outcome are two interesting identities of the hypergeometric type with their new Meijer G function analogues. A fourth outcome is that by the use of the Cauchy integral formula for the derivatives we are able to give a different meaning to the sum:

$$\sum_{k=n}^{2n-1} \binom{2k}{k}.$$

A fifth outcome is that by the use of the Gauss-Kummer formula we are able to make better sense of the expressions

$$\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4), \quad \text{and} \quad \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)$$

by making use of the series definition of the hypergeometric function. As we continue we notice a new close relation of the Key lemma, and the de la Vallée Poussin means. With this close relation we were able to talk about the de la Vallée Poussin summability of the two infinite series $\sum_{n=0}^{\infty} \cos n\theta$, and $\sum_{n=0}^{\infty}(-1)^n \cos n\theta$.

Furthermore the application of the de la Vallée Poussin sum to the Key lemma has created two new expansions representing the following functions:

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)}, \quad \text{where} \quad x = \cos\theta,$$

and

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)}, \quad \text{where} \quad x = \cos\theta$$

in terms of the de la Vallée Poussin means of the two infinite series

$$\sum_{n=0}^{\infty} \cos n\theta \ ,$$

and

$$\sum_{n=0}^{\infty}(-1)^n \cos n\theta \ .$$

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1. Introduction

The Gauss' Hypergeometric function is given by:

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{1\cdot c}z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2 + \ldots = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n\,!},$$

where the above series converges for $|z| < 1$ and $(a)_n$ is is the Pochhammer symbol defined by:

$$(a)_0 = 0, \quad \text{and} \quad (a)_n = a(a+1)(a+2)\ldots(a+n-1) \ .$$

We further note that:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

where the symbol Γ refers to the gamma function.

Now we like to bring up two definitions related to de la Vallée Poussin; one is related to the de la Vallée Poussin means, and the other is related to the de la Vallée Poussin sum. We have: the de la Vallée Poussin means of the infinite series

$$\sum_{n=0}^{\infty} a_n$$

are defined by (see [3]):

$$V(n, a_n) = \sum_{j=0}^{n} \frac{(n!)^2}{(n-j)!(n+j)!} a_j \ .$$

Let T_i be the Chebyshev polynomials. Then Charles Jean de la Vallée-Poussin defines (see [2]) the de la Vallée Poussin sum SV_n as follows:

$$SV_n = \frac{S_n + S_{n+1} + \ldots + S_{2n-1}}{n},$$

where

$$S_n = \frac{1}{2}c_0(f) + \sum_{j=1}^{n} c_j(f)T_j \ .$$

We shall refer to the S_n as the Chebyshev sum or the Chebyshev expansion of f. The very important properties of the gamma functions that were used in this work, and helped immensely in the proof of the main lemma are:

$$\Gamma(z+1) = z\Gamma(z), \quad \text{and} \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \ .$$

Let f by holomorphic in an open subset U of the complex plane $\mathbb{C}$, and let $|z - z_0| \leq r$, be contained completely in U. Now let a be any point interior to $|z - z_0| \leq r$. Then the Cauchy integral formula for the derivative says that the n-th derivative of f at a is given by:

$$f^n(a) = \frac{n!}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-a)^{n+1}} dz \ .$$

The Meijer G function is defined by

$$G_{p,q}^{m,n}\begin{pmatrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{pmatrix}; z\Bigg) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds \ .$$

The L in the integral represents the path of integration. We may choose L to run from $-i\infty$ to $+i\infty$ such that all poles of $\Gamma(b_j - s), j = 1, 2, \ldots, m$, are on the right of the path, while all poles of $\Gamma(1 - a_k + s), k = 1, 2, \ldots, n$, are on the left.

The Meijer G function, and the Hypergeometric function ${}_pF_q$ are related by:

$${}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{p} \Gamma(a_k)} \, G_{p,q-1}^{1,p} \left(\begin{matrix} 1 - a_1, \ldots, 1 - a_p \\ 0, 1 - b_1, \ldots 1 - b_q \end{matrix} ; -z \right) .$$

2. The close relation between the Key lemma and the de la Vallée Poussin means

In [1] we have the following Key lemma:

Lemma 2.1. *For $1 \leq r \leq n$, and θ real we have:*

(i) $$\sum_{r=1}^{n} \binom{2n}{n-r} \left(\cos r\theta + (-1)^{r+1} \right) = 2^{n-1} (1 + \cos\theta)^n.$$

(ii) $$\sum_{r=1}^{n} (-1)^r \binom{2n}{n-r} \cos r\theta + \frac{1}{2} \binom{2n}{n} = 2^{n-1} (1 - \cos\theta)^n .$$

Before we move to the main lemma and its proof we state two theorems which are direct consequences of the Key lemma. We like to indicate before we state them that they are related to the de la Vallée Poussin means of the the following two infinite series:

$$\sum_{n=0}^{\infty} \cos n\theta, \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \cos n\theta .$$

Clearly from the Key lemma we have :

$$\sum_{r=0}^{n} \binom{2n}{n-r} \cos r\theta - \frac{1}{2} \binom{2n}{n} = 2^{n-1} (1 + \cos\theta)^n ,$$

$$\sum_{r=0}^{n} (-1)^r \binom{2n}{n-r} \cos r\theta - \frac{1}{2} \binom{2n}{n} = 2^{n-1} (1 - \cos\theta)^n .$$

Accordingly with

$$V(n, \cos n\theta)$$

being the de la Vallée Poussin means of

$$\sum_{n=0}^{\infty} \cos n\theta,$$

and

$$V(n,(-1)^n \cos n\theta)$$

being the de la Vallée Poussin means of

$$\sum_{n=0}^{\infty}(-1)^n \cos n\theta \ .$$

We have:

$$V(n,\cos n\theta) = \frac{2^{n-1}(1+\cos\theta)^n}{\binom{2n}{n}} + \frac{1}{2} \ ,$$

and

$$V(n,(-1)^n \cos n\theta) = \frac{2^{n-1}(1-\cos\theta)^n}{\binom{2n}{n}} + \frac{1}{2} \ .$$

Now it can be shown that for each real $\theta \neq 2k\pi$, where k is an integer the following limit:

$$\lim_{n\to\infty} \frac{2^{n-1}(1+\cos\theta)^n}{\binom{2n}{n}} = 0 \ , \quad \text{and}$$

for each real $\theta \neq k\pi$,where k is an odd integer the following limit

$$\lim_{n\to\infty} \frac{2^{n-1}(1-\cos\theta)^n}{\binom{2n}{n}} = 0 \ .$$

One way of showing the above limits is the use of the Stirling's formula for the gamma function which says:

$$\Gamma(x) \sim e^{-x}\sqrt(2\pi)x^{-1/2+x} \quad \text{as} \quad x \to \ \infty \ .$$

Accordingly we have the following two theorems:

Theorem 2.2. *For each real $\theta \neq 2k\pi$, where k is an integer the infinite series*

$$\sum_{n=0}^{\infty} \cos n\theta$$

is summable in the sense of de la Vallée Poussin to $\frac{1}{2}$.

Theorem 2.3. *For each real $\theta \neq k\pi$, where k is an odd integer the infinite series*

$$\sum_{n=0}^{\infty}(-1)^n \cos n\theta$$

is summable in the sense of de la Vallée Poussin to $\frac{1}{2}$.

3. The main lemma

There are two versions of the main lemma. We give now one version, and we will provide the other one at the end of the current section.

Lemma 3.1. *Let ${}_2F_1(a,b;c;z)$ be the Gauss' Hypergeometric function, then we have:*

$$\sum_{k=n}^{k=2n-1} \binom{2k}{k} = \binom{2n}{n} {}_2F_1(1,1/2+n;1+n;4) - \binom{4n}{2n} {}_2F_1(1,1/2+2n;1+2n;4) \ .$$

Before we prove the above lemma we like to indicate that by the Gauss-Kummer formula it is easily seen that the hypergeometric functions ${}_2F_1(1,1/2+n;1+n;4)$, and ${}_2F_1(1,1/2+2n;1+2n;4)$ are both discrete complex valued functions, and they are interesting as for exmple the left hand side of the identity given above is a natural number, which leads us to say that the imaginary part of the right hand side of the identity above is actually zero. Another interesting view is that the second term in the above lemma is obtained by replacing n in the first term by $2n$. The further interesting ideas involved are coming as we continue to prove the above lemma, and continue further.

Proof of the case $n = 1$ of Lemma 3.1. By induction on n. We note first that for $n = 1$ we have:

$$2 = 2\ {}_2F_1(1,3/2;2;4) - 6\ {}_2F_1(1,5/2;3;4) \ .$$

Now one way to evaluate the above expression is by using an already known technique, where the numbers 1, 3/2, 2, 4 inside the hypergeometric function ${}_2F_1$ for example in this case are fed into their proper slot of a computer program or a plug in algorithm which evaluates the hypergeometric functions of the type given above to get the complex number representing ${}_2F_1(1,3/2;2;4)$. Similarly the numbers 1, 5/2, 3, 4 are fed into each slot in the computer program to get the complex number representing ${}_2F_1(1,5/2;3;4)$. For example we see by using this method we have:

$$2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{6}\right) - 6\left(-\frac{1}{2} - i\frac{\sqrt{3}}{18}\right) ,$$

and the identity given in the lemma is true for $n = 1$.

One way to prove the case for $n = 1$ without the use of a computer is by the use of the **Gauss-Kummer formula**, which states: for $(a-b)$ not an integer, and z not in the unit interval $(0,1)$ we have

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2F_1(a,a-c+1;a-b+1;1/z) \\ &+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2F_1(b,b-c+1;-a+b+1;1/z) \ . \end{aligned}$$

We will now do basic calculations for the proof of Corollary 3.2, and Corollary 3.3 coming up. Accordingly we have:

$$
{}_2F_1(1,3/2;2;4) = \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(3/2)\Gamma(1)}(-4)^{-1}\ {}_2F_1(1,0;1/2;1/4)
$$

$$
+\ \frac{\Gamma(-1/2)\Gamma(2)}{\Gamma(1)\Gamma(1/2)}(-4)^{-3/2}\ {}_2F_1(3/2,1/2;3/2;1/4)\ .
$$

Now since

$$
\Gamma(1/2) = \sqrt{\pi}, \Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \Gamma(-1/2) = -2\sqrt{\pi}
$$

we have:

$$
{}_2F_1(1,3/2;2;4) = \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(3/2)\Gamma(1)}(-4)^{-1}\ {}_2F_1(1,0;1/2;1/4) = \frac{-1}{2}\ .
$$

Note that

$$
\sum_{n=0}^{\infty} \frac{(0)_n(1)_n}{(1/2)_n 4^n n!} = 1\ .
$$

Now since

$$
{}_2F_1(3/2,1/2;3/2;1/4) = \frac{1}{(1-(1/4))^{1/2}} = \frac{2\sqrt{3}}{3},
$$

and $(-4)^{-3/2} = \frac{i}{8}$, we have:

$$
\frac{\Gamma(-1/2)\Gamma(2)}{\Gamma(1)\Gamma(1/2)}(-4)^{-3/2}\ {}_2F_1(3/2,1/2;3/2;1/4) = -i\frac{\sqrt{3}}{6}\ .
$$

We like to note now that by using the binomial expansion formula

$$
(x+a)^{\gamma} = \sum_{k=0}^{\infty} \binom{\gamma}{k} x^k a^{\gamma-k},
$$

we have for $|z| < 1$:

$$
(-z+1)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k z^k \binom{\frac{-1}{2}}{k} = {}_2F_1(3/2,1/2;3/2;1/4)\ .
$$

Similarly we have:

$$
{}_2F_1(1,5/2;3;4) = \frac{\Gamma(3/2)\Gamma(3)}{\Gamma(5/2)\Gamma(2)}(-4)^{-1}\ {}_2F_1(1,-1;-1/2;1/4)
$$

$$
+\ \frac{\Gamma(-3/2)\Gamma(3)}{\Gamma(1)\Gamma(1/2)}(-4)^{-5/2}\ {}_2F_1(5/2,1/2;5/2;1/4)\ ,
$$

which simplifies to:

$$-\frac{4}{3}\frac{1}{4}\frac{3}{2}-\frac{i}{32}\frac{8}{3}\frac{2\sqrt{3}}{3}=\frac{-1}{2}-\frac{i\sqrt{3}}{18}\ .$$

Note that

$$_2F_1(1,-1;-1/2;1/4)=\sum_{k=0}^{\infty}\frac{(1)_k(-1)_k}{(-1/2)_k4^kk!}=\frac{3}{2}\ .$$

For $k \geq 2$ the term $(-1+1)$ is present in each product of $(-1)_k$: accordingly we are adding the first two terms only; furthermore

$$_2F_1(5/2,1/2;5/2;1/4)=\frac{2\sqrt{3}}{3}\ .$$

This is the case where a = c = 5/2, and in this case the sum is

$$\frac{1}{\sqrt{1-1/4}}=\frac{2\sqrt{3}}{2}\ .$$

This completes the induction proof for the case $n = 1$ without the use of a computer. □

We can clearly see from above that $_2F_1(1,3/2;2;4)$, and $_2F_1(1,5/2;3;4)$ are complex numbers, while $_2F_1(1,0;1/2;1/4)$, $_2F_1(3/2,1/2;3/2;1/4)$, $_2F_1(1,-1;-1/2;1/4)$, and $_2F_1(5/2,1/2;5/2;1/4)$ are real numbers.

From above we have the following Corollaries resulting from the induction proof when n = 1, and in relation to the real part, and the imaginary part of $_2F_1(1,3/2;2;4)$, and $_2F_1(1,5/2;3;4)$.

Corollary 3.2. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,3/2;2;4)=\frac{\Gamma(1/2)\Gamma(2)}{\Gamma(3/2)\Gamma(1)}(-4)^{-1}\ {}_2F_1(1,0;1/2;1/4)=-\frac{1}{2}\ ,$$

and

$$i\ Im\ {}_2F_1(1,3/2;2;4)=\frac{\Gamma(-1/2)\Gamma(2)}{\Gamma(1)\Gamma(1/2)}(-4)^{-3/2}\ {}_2F_1(3/2,1/2;3/2;1/4)=-i\frac{\sqrt{3}}{6}\ .$$

Corollary 3.3. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,5/2;3;4) = \frac{\Gamma(3/2)\Gamma(3)}{\Gamma(5/2)\Gamma(2)}(-4)^{-1}\ {}_2F_1(1,-1;-1/2;1/4) = -\frac{1}{2},$$

and

$$i\ Im\ {}_2F_1(1,5/2;3;4) = \frac{\Gamma(-3/2)\Gamma(3)}{\Gamma(1)\Gamma(1/2)}(-4)^{-5/2}\ {}_2F_1(5/2,1/2;5/2;1/4) = -i\frac{2\sqrt{3}}{18}\ .$$

Accordingly we have the following general lemmas:

Lemma 3.4. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,1/2+n;1+n;4) = \frac{\Gamma(n-1/2)\Gamma(n+1)}{\Gamma(n+1/2)\Gamma(n)}(-4)^{-1}{}_2F_1(1,1-n;3/2-n;1/4)$$

$$i\ Im\ {}_2F_1(1,1/2+n;1+n;4) = \frac{\Gamma(1/2-n)\Gamma(n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-n} \times\ {}_2F_1(1/2+n,1/2;1/2+n;1/4)\ .$$

The proof of the above lemma follows by using the Gauss-Kummer formula, and by the presence of $(-4)^{-1/2-n}$ in the second equation.

Lemma 3.5. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,1/2+2n;1+2n;4) = \frac{\Gamma(2n-1/2)\Gamma(2n+1)}{\Gamma(2n+1/2)\Gamma(2n)}(-4)^{-1}{}_2F_1(1,1-2n;3/2-2n;1/4)$$

$$i\ Im\ {}_2F_1(1,1/2+2n;1+2n;4) = \frac{\Gamma(1/2-2n)\Gamma(2n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-2n} \times\ {}_2F_1(1/2+2n,1/2;1/2+2n;1/4)\ .$$

The proof of the above lemma follows by using the Gauss-Kummer formula, and by the presence of $(-4)^{-1/2-2n}$ in the second equation.

Now we have the following lemma showing that the imaginary part of the right hand side of the main lemma above equals to zero, i.e.:

Lemma 3.6. *Let i be the imaginary unit, and let $Im(z)$ denote the imaginary part of z. We have:*

$$Im\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right)=0\ .$$

Proof of Lemma 3.6.

$$\binom{2n}{n}\left(\frac{\Gamma(1/2-n)\Gamma(n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-n}{}_2F_1(1/2+n,1/2;1/2+n;1/4)\right)$$

$$-\binom{4n}{2n}\left(\frac{\Gamma(1/2-2n)\Gamma(2n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-2n}{}_2F_1(1/2+2n,1/2;1/2+2n;1/4)\right)=0\ ,$$

clearly

$${}_2F_1(1/2+n,1/2;1/2+n;1/4))={}_2F_1(1/2+2n,1/2;1/2+2n;1/4))=\frac{1}{\sqrt{1-1/4}}=\frac{2\sqrt{3}}{3}\ .$$

Now by noting that

$$\binom{2n}{n}=\frac{2^{2n}\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\quad\text{and}\quad\binom{4n}{2n}=\frac{2^{4n}\Gamma(2n+1/2)}{\sqrt{\pi}\Gamma(2n+1)},$$

what we do now is to substitute the just above identities for $\binom{2n}{n}$, and $\binom{4n}{2n}$, in the above equation whose right hand side is zero; then we use the complement formula of the gamma function which is:

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

to handle both of the following products: $\Gamma(1/2+n)\Gamma(1/2-n)$ and $\Gamma(1/2+2n)\Gamma(1/2-2n)$.

Finally by writing

$$(-4)^{-1/2-jn}=(2^2e^{i\pi})^{-1/2-jn},\quad\text{with}\quad j=1,2\ .$$

We can easily show that the stated lemma above is correct, and that the imaginary part of the right hand side of the main lemma is identically equal to zero. This completes the proof of the above lemma. □

Proof of the induction step. Since the imaginary part of the right hand side of the main lemma given above equals zero. We have:

$$\begin{aligned}
&\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\\
&=\binom{2n}{n}\frac{\Gamma(n-1/2)\Gamma(n+1)}{\Gamma(n+1/2)\Gamma(n)}(-4)^{-1}{}_2F_1(1,1-n;3/2-n;1/4)\\
&-\binom{4n}{2n}\frac{\Gamma(2n-1/2)\Gamma(2n+1)}{\Gamma(2n+1/2)\Gamma(2n)}(-4)^{-1}{}_2F_1(1,1-2n;3/2-2n;1/4)\ .
\end{aligned}$$

Now assume that the identity we are to prove is true for n, we need to show that it is true for $n+1$; i.e. we need to show that:

$$\sum_{k=n+1}^{k=2n+1}\binom{2k}{k}=\binom{2n+2}{n+1}{}_2F_1(1,3/2+n;2+n;4)-\binom{4n+4}{2n+2}{}_2F_1(1,5/2+2n;3+2n;4)$$

equivalently we need to show that:

$$\begin{aligned}
\sum_{k=n+1}^{k=2n+1}\binom{2k}{k}&=\binom{2n+2}{n+1}\frac{\Gamma(n+1/2)\Gamma(n+2)}{\Gamma(n+3/2)\Gamma(n+1)}(-4)^{-1}{}_2F_1(1,-n;1/2-n;1/4)\\
&-\binom{4n+4}{2n+2}\frac{\Gamma(2n+3/2)\Gamma(2n+3)}{\Gamma(2n+5/2)\Gamma(2n+2)}(-4)^{-1}{}_2F_1(1,-1-2n;-1/2-2n;1/4)
\end{aligned}$$

equivalently again we need to show that:

$$\begin{aligned}
&\binom{2n+2}{n+1}\frac{\Gamma(n+1/2)\Gamma(n+2)}{\Gamma(n+3/2)\Gamma(n+1)}(-4)^{-1}{}_2F_1(1,-n;1/2-n;1/4)\\
&-\binom{4n+4}{2n+2}\frac{\Gamma(2n+3/2)\Gamma(2n+3)}{\Gamma(2n+5/2)\Gamma(2n+2)}(-4)^{-1}{}_2F_1(1,-1-2n;-1/2-2n;1/4)\\
&=\binom{2n}{n}\frac{\Gamma(n-1/2)\Gamma(n+1)}{\Gamma(n+1/2)\Gamma(n)}(-4)^{-1}{}_2F_1(1,1-n;3/2-n;1/4)\\
&-\binom{4n}{2n}\frac{\Gamma(2n-1/2)\Gamma(2n+1)}{\Gamma(2n+1/2)\Gamma(2n)}(-4)^{-1}{}_2F_1(1,1-2n;3/2-2n;1/4)\\
&-\binom{2n}{n}+\binom{4n}{2n}+\binom{4n+2}{2n+1}\ .
\end{aligned}$$

The above induction step can be shown by noting the following two new hypergeometric identities:

$${}_2F_1(1,-n;1/2-n;1/4)=\frac{-n}{4(\frac{1}{2}-n)}{}_2F_1(1,1-n;3/2-n;1/4)+1\ ,$$

$$ {}_2F_1(1,1-2n;3/2-2n;1/4) = \binom{8n+2}{2n+1}\left(\frac{4n-1}{n}\right) $$

$$ \times\ {}_2F_1(1,-1-2n;-(1/2)-2n;1/4) - \binom{10n+3}{2n+1}\left(\frac{4n-1}{n}\right)\ . $$

The proof of the first hypergeometric identity above is simple, and the proof of the second hypergeometric identity above is one where one really appreciates the simple law of cancellation. □

Based from above, we provide the second version of the main lemma:

Lemma 3.7. *Let Re(z) be the real part of z, then we have:*

$$ \sum_{k=n}^{2n-1}\binom{2k}{k} = Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4) - \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right)\ . $$

An interesting corollary is the following:

Corollary 3.8. *Let*

$$ f(n) = \frac{2^{2n}\Gamma(n+1/2)\Gamma(1/2-n)(-4)^{-1/2-n}}{\sqrt{\pi}}, \quad \textit{then} \quad f(n) = f(2n)\ . $$

4. Presentation of the two new hypergeometric identities above in terms of the Meijer G function

Since the following identities relating the hypergeometric function ${}_2F_1$, and the Meijer $G_{2,2}^{1,2}$ function, hold:

$$ {}_2F_1(1,-n;1/2-n;1/4) = \frac{\Gamma(1/2-n)}{\Gamma(-n)}\ G_{2,2}^{1,2}\left(\begin{matrix}0 & (1+n)\\ 0 & (1/2+n)\end{matrix}; -1/4\right)\ , $$

$$ {}_2F_1(1,1-n;3/2-n;1/4) = \frac{\Gamma(3/2-n)}{\Gamma(1-n)}\ G_{2,2}^{1,2}\left(\begin{matrix}0 & n\\ 0 & (-1/2+n)\end{matrix}; -1/4\right)\ , $$

$$ {}_2F_1(1,1-2n;3/2-2n;1/4) = \frac{\Gamma(3/2-n)}{\Gamma(1-2n)}\ G_{2,2}^{1,2}\left(\begin{matrix}0 & 2n\\ 0 & (-1/2+2n)\end{matrix}; -1/4\right)\ , $$

$$ {}_2F_1(1,-1-2n;-1/2-2n;1/4) = \frac{\Gamma(-1/2-2n)}{\Gamma(-1-2n)}\ G^{1,2}_{2,2}\begin{pmatrix} 0 & (2n+2) \\ 0 & (3/2+2n) \end{pmatrix}; -1/4 \Bigg) \ , $$

then we have the following two new identities relating the Meijer function $G^{1,2}_{2,2}$:

$$ \frac{\Gamma(1/2-n)}{\Gamma(-n)} G^{1,2}_{2,2}\left(\begin{matrix} 0 & (1+n) \\ 0 & (1/2+n) \end{matrix}; -1/4\right) = \frac{-n}{4(1/2-n)}\frac{\Gamma(3/2-n)}{\Gamma(1-n)} $$

$$ \times\ G^{1,2}_{2,2}\left(\begin{matrix} 0 & n \\ 0 & (-1/2+n) \end{matrix}; -1/4\right) + 1 \ , $$

$$ \frac{\Gamma(3/2-n)}{\Gamma(1-2n)} G^{1,2}_{2,2}\left(\begin{matrix} 0 & 2n \\ 0 & (-1/2+2n) \end{matrix}; -1/4\right) = \left(\frac{8n+2}{2n+1}\right)\left(\frac{4n-1}{n}\right)\frac{\Gamma(-1/2-2n)}{\Gamma(-1-2n)} $$

$$ \times\ G^{1,2}_{2,2}\left(0 \quad (2n+2)0 \quad (3/2+2n); -1/4\right) - \left(\frac{10n+3}{2n+1}\right)\left(\frac{4n-1}{n}\right) \ . $$

5. A different meaning of $\sum_{k=n}^{2n-1}\binom{2k}{k}$

Let $r > 0$ be given:

$$ \binom{m}{r} = \frac{1}{2\pi i}\int_{|z|=r}\frac{(1+z)^m}{z^{r+1}}dz \ . $$

Accordingly:

$$ \binom{2k}{k} = \frac{1}{2\pi i}\int_{|z|=r}\frac{(1+z)^{2k}}{z^{k+1}}dz \ . $$

Hence

$$ \sum_{k=n}^{2n-1}\frac{1}{2\pi i}\int_{|z|=r}\frac{(1+z)^{2k}}{(z-0)^{k+1}}dz = \binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4) $$

$$ -\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4) \ . $$

Accordingly by the Cauchy integral formula for derivatives we have the following new, and different meaning of $\sum_{k=n}^{2n-1}\binom{2k}{k}$:

$$ \sum_{k=n}^{2n-1}\binom{2k}{k} = \sum_{k=n}^{2n-1}\frac{1}{k!}\frac{\mathrm{d}^k\left((1+z)^{2k}\right)}{\mathrm{d}z^k}\Bigg|_{z=0} \ . $$

The above formula clearly states the different meaning of the sum $\sum_{k=n}^{2n-1}\binom{2k}{k}$. We remark that

$$ \frac{\mathrm{d}^k\left((1+z)^{2k}\right)}{\mathrm{d}z^k}\Bigg|_{z=0} $$

stands for the k^{th} derivative of of $(1+z)^{2k}$ evaluated at $z=0$.

6. The de la Vallée Poussin sum, and the de la Vallée Poussin means expansion

In this section, we apply the de la Vallée Poussin sum SV_n to the Key lemma parts (i), and (ii) to obtain an implicit, and an explicit de la Vallée Poussin means expansions DVP_i, and DVP_{ii} of two particular functions, which will be defined in the statements of the theorems. We remark that from the Key lemma we can easily see:

$$S_k = \frac{\binom{2k}{k}}{2} + \sum_{j=1}^{k} \binom{2k}{k-j} \cos j\theta \ .$$

Accordingly, we have the following two theorems:

Theorem 6.1. *The function*

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)}, \quad \textit{where} \quad x = \cos\theta$$

has an implicit de la Vallée Poussin means expansion DVP_i of the form

$$= \frac{1}{2n} \sum_{k=n}^{2n-1} \binom{2k}{k} + \frac{1}{n} \sum_{k=n}^{2n-1} \left(\sum_{j=1}^{k} \binom{2k}{k-j} T_j(x) \right)$$

and an explicit de la Vallée Poussin expansion DVP_i of the form

$$= \frac{1}{2n} \sum_{k=n}^{2n-1} \binom{2k}{k} + \frac{1}{n} \sum_{k=n}^{2n-1} \binom{2k}{k} V_{k1} \ ,$$

where V_{k1} are the de la Vallée Poussin means of

$$\sum_{k=1}^{\infty} \cos k\theta \ .$$

Theorem 6.2. *The function*

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)}, \quad \textit{where} \quad x = \cos\theta$$

has an implicit de la Vallée Poussin means expansion DVP_{ii} of the form

$$= \frac{1}{2n} \sum_{k=n}^{2n-1} \binom{2k}{k} + \frac{1}{n} \sum_{k=n}^{2n-1} \left(\sum_{j=1}^{k} (-1)^j \binom{2k}{k-j} T_j(x) \right)$$

and an explicit de la Vallée Poussin expansion DVP_{ii} of the form

$$= \quad \frac{1}{2n}\sum_{k=n}^{2n-1}\binom{2k}{k}+\frac{1}{n}\sum_{k=n}^{2n-1}\binom{2k}{k}V_{k2}\ ,$$

where V_{k2} are the de la Vallée Poussin means of

$$\sum_{k=1}^{\infty}(-1)^k\cos k\theta\ .$$

7. Concluding remark

In this paper, we were able to show that

$$\sum_{k=n}^{2n-1}\binom{2k}{k}=Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right).$$

Moreover, we were able to create two new hypergeometric identities to prove the induction step. After, it was interesting to find their Meijer G function analogue. Further, by the use of the Key lemma and the definition of the de la Vallée Poussin means, we were able to find two new expansions representing the following functions:

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)},\quad \text{where}\quad x=\cos\theta$$

and

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)},\quad \text{where}\quad x=\cos\theta.$$

The general form of the expansions can be put into a more familiar form as:

$$\frac{A_0}{2}+\sum_{K=1}^{n}A_{(K,n)}V_{(K,n)},\quad \text{where}$$

$$A_0 = \quad \frac{1}{n}\sum_{K=1}^{n}\binom{2K+2n-2}{K+n-1},\quad \text{and}$$

$$A_{(K,n)}=\frac{1}{n}\binom{2K+2n-2}{K+n-1}\ .$$

We like to add that the strong connection of the Key lemma, and the de la Vallée Poussin means has given us two theorems about the de la Vallée Poussin summability of the two infinite series

$$\sum_{n=0}^{\infty}\cos n\theta \qquad \theta\neq 2k\pi \quad k \quad \text{integer},\quad \text{and}$$

$$\sum_{n=0}^{\infty}(-1)^n \cos n\theta \qquad \theta \neq k\pi \quad k \quad \text{odd} \quad \text{integer.}$$

Moreover we note for example that

$$\sum_{n=0}^{\infty}(-1)^n$$

is also de la Vallée Poussin summable to $\frac{1}{2}$ just like it's Cesàro sum, and it's Abel sum.
We remark finaly that:

$$\sum_{K=1}^{n}\binom{2K+2n-2}{K+n-1} = Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4) - \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right).$$

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Ziad S. Ali
email: alioppp@yahoo.com
American College of Switzerland
Leysin 1854
SWIZERLAND

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