# **Stabilization of positive descriptor fractional discrete-time linear**  system with two different fractional orders by decentralized controller Łukasz SAJEWSKI

L. SAJEWSKI\*

Faculty of Electrical Engineering, Bialystok University of Technology, 45D Wiejska St., 15-351 Bialystok Poland orders. Method for finding the decentralized controller for the class of positive systems is given and its effectiveness is demonstrated on numerical

**Abstract.** Positive descriptor fractional discrete-time linear systems with fractional different orders are addressed in the paper. The decomposition of the regular pencil is used to extend necessary and sufficient conditions for positivity of the descriptor fractional discrete-time linear extends the decentralized stabilization problem in the alternative stateme system with different fractional orders. A method for finding the decentralized controller for the class of positive systems is proposed and its effectiveness is demonstrated on a numerical example effectiveness is demonstrated on a numerical example. **1. Introduction** 

Key words: positive, fractional, different order, noncomensurate, descriptor, stabilization, decentralized controller.

## controllable systems. This problem has been considered in **1. Introduction**

1- different fractional order will be formulated and solved. le The paper is organized as follows. In Section 2 basic inford mation on the positive fractional discrete-time linear systems ls with different fractional orders is recalled. Descriptor frace- tional discrete-time linear systems with different fractional orders are addressed in Section 3, where the decomposition and Decentralized (state-feedback) controller for linear time-invariant systems allows stabilization of unstable but controllable systems. This problem has been considered in many papers and books  $[1–5]$ . Linear time-invariant (LTI) system theory deals with numerous types of such systems e.g. positive [6–9], descriptor  $[5, 10-13]$  and/or fractional  $[4, 14-16]$ .

ts positivity conditions are presented. The main idea of the paper e is presented in Section 4, where the solution to decentralized stabilization of positive descriptor fractional discrete-time real numbers,  $\frac{1}{2}$  and section 5. LTI systems for which inputs, state variables and outputs take only non-negative values are called (internally) positive<br>take only non-negative values are called (internally) positive systems. A variety of models having positive finear systems behavior can be found in engineering, management science, behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of art in the positive systems theory is given in the monographs  $[8, 9]$ . systems. A variety of models having positive linear systems

Ine following notation will be used:  $\mathcal{H}$  – the set of real num-<br>hers,  $\mathcal{R}^{n \times m}$  – the set of  $n \times m$  real matrices,  $Z_+$  – the set of nonnegative integers,  $M_n$  – the set of  $n \times n$  Metzler matrices (with nonnegative integers,  $M_n$  – the set of  $n \times n$  Metzler matrices - nonnegative off-diagonal entries),  $I_n$  – the  $n \times n$  identity matrix. **2. Positive fractional different orders discrete-2. Positive fractional different orders**  Recently the fractional systems have drawn more attention since the fractional differential equations were used by engineers for modelling different processes [17, 18]. From the mathematical point of view, the fractional calculus is well-known [4, 14–16, 19], yet there are still areas in this field which have not been comprehensively addressed, e.g. the descriptor systems, systems with delays or systems with different fractional orders  $(non-commensurate)$  [20–22].

]. different fractional orders *α* and  $β$  of the form A solution to the state equation of descriptor fractional linear systems with regular pencils has been given in  $[12, 13, 23]$ .<br>A comparison of three different methods for finding the solu-13, 23]. A comparison of three different methods for tion of descriptor fractional discrete-time linear systems can the solution of descriptor matricial discrete-time initial systems can<br>be found in [24] and the solution to the descriptor fractional discrete-time linear systems with two different fractional orders has been introduced in [25]. Stability of positive fractional discrete-time linear systems have been addressed in [26–28] and the decentralized stabilization of fractional positive descriptor discrete-time linear systems in [3]. or where  $k \in \mathbb{Z}_+$ ,  $x_1(k) \in \mathbb{R}^n$  and  $x_2(k) \in \mathbb{R}^n$  are the state vectors,  $u(k) \in \mathbb{R}^m$  is the input vector and  $A_{ij} \in \mathbb{R}^{n_i \times m}$ ,  $B_i \in \mathbb{R}^{n_i \times m}$ ; systems with regular pencils has been given in [12, 13, 23].

*i*, *j* = 1, 2, *n* = *n*<sub>1</sub> + *n*<sub>2</sub>.<br>The fractional difference of  $\alpha$  ( $\beta$ ) order is defined by [4]

 $i, j = 1, 2, n = n_1 + n_2.$ 

2. *CONDUCTRACIONAL MITELER* 

In this paper the decentralized stabilization problem for positive descriptor fractional discrete-time linear systems with two

linear systems with different fractional orders is given and illustrated by numerical example. Concluding remarks are illustrated by numerical example. Concluding remarks are

The following notation will be used:  $\mathcal{R}$  – the set of real num-

Consider the fractional discrete-time linear system with two Consider the fractional discrete-time linear system

 $\Delta^{\beta} x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k),$  $\Delta^{\alpha} x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k),$ 

where  $k \in \mathbb{Z}_+$ ,  $x_1(k) \in \mathbb{R}^{n_1}$  and  $x_2(k) \in \mathbb{R}^{n_2}$  are the state vec-

$$
\Delta^{\alpha} x(k) = \sum_{j=0}^{k} (-1)^j {\binom{\alpha}{j}} x(k-j), \tag{2a}
$$

\*e-mail: l.sajewski@pb.edu.pl

linear systems in [3].

(1) (1)

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$$
\begin{pmatrix} \alpha \\ j \end{pmatrix} = \begin{cases} \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for } j=0 \\ \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for } j=1,2,... \end{cases}
$$
 Further, we will consider the des-  
Using (2), we can write the equation (1) in the matrix form  
and

Using  $(2)$ , we can write the equation  $(1)$  in the matrix form  $\mathbf{f}$  $\overline{\phantom{a}}$  $\frac{1}{2}$ Ĭ,  $(1)$  in t form form form form  $\frac{1}{2}$ ) in the matrix is  $(2)$ , we can write the equation  $(1)$  in the matrix  $\log(2)$ , we can write the equation (1) in the ma  $\sin g$  (2), we can write the equation (1) in the matrix fo  $\ddot{\phantom{0}}$ we can write the equation  $(1)$  in the matrix *B x k j*  $rac{1}{2}$  $\alpha$  can write the equation (1) in the mat  $\sin g$  (2), we can write the equation (1) *nj* atriv *nj* ing  $(2)$ , we can write the equation  $(1)$  in the Using (2), we can write the equation (1) in the matrix form  $\lim_{x \to 0} (2)$ , we can write the equation (1) in the Using (2), we can write the equation (1) in the matrix form  $\sin g$  (2), we can write the equation (1) in th *b* **i**  $\alpha$  *kx*  $\beta$  *lsing (2), we can write the equation (1) in the matrix form and*  $\beta$  $\epsilon$  can write the equation (1) Using (2), we can write the equation (1) i  $rm$ 

Using (2), we can write the equation (1) in the matrix form  
\n
$$
\begin{bmatrix} x_1(k+1) \ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{1\alpha} & A_{12} \ A_{21} & A_{2\beta} \end{bmatrix} \begin{bmatrix} x_1(k) \ x_2(k) \end{bmatrix} +
$$
\n
$$
+ \sum_{j=2}^{k+1} \begin{bmatrix} c_{\alpha,j}I_{n_1} & 0 \ 0 & c_{\beta,j}I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k-j+1) \ x_2(k-j+1) \end{bmatrix} + \begin{bmatrix} B_1 \ B_2 \end{bmatrix} u(k),
$$
\nfor some  $z \in C$  (the field of complex numbers), where matrix  $B_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ ,  
\n**Definition 1.** [4] The fractional system (1) is called (internally) under pencil can be decomposed e.g. by the use of the We positive if  $x_1(k) \in \Re^{n_1}$ ,  $x_2(k) \in \Re^{n_2}$ ,  $x_3(k) \in \Re^{n_3}$ ,  $x_4(k) \in \Re^{n_4}$ ,  $x_5(k) \in \Re^{n_5}$ ,  $x_6(k) \in \Re^{n_6}$ ,  $x_7(k) \in \Re^{n_7}$ ,  $x_7(k) \in \Re^{n_8}$ ,  $x_8 \in Z_{11}$ , for all  $x_1(0) \in \Re^{n_1}$ ,  $x_9(k) \in \Re^{n_1}$ ,  $x_9(k) \in \Re^{n_2}$ ,  $x_1(k) \in \Re^{n_3}$ ,  $x_1(k) \in \Re^{n_4}$ ,  $x_1(k) \in \Re^{n_5}$ ,  $x_1(k) \in \Re^{n_6}$ ,  $x_1(k) \in \Re^{n_7}$ ,  $x_2(k) \in \Re^{n_8}$ ,  $x_3(k) \in \Re^{n_9}$ ,  $x_4(k) \in \Re^{n_1}$ ,  $x_5(k) \in \Re^{n_2}$ ,  $x_6(k) \in \Re^{n_3}$ ,  $x_7$ 

 $A_{1a} = A_{11} + I_{n_1}a$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{a,0} = 1$ , *f***e**  $A_{1a} = A_{11} + I_{n_1}a$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{a,0} =$  $\overline{\phantom{a}}$ F  $\mathcal{C}_t$  $\mu_B = A_{22} + I_{n_2}I$  $\beta$  $\mathbf{I}_{1}$ ere de la derembre de<br>Derembre de la derembre de la décembre de la déce where  $A_{1\alpha} = A_{11} + I_{n_1} \alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2} \beta$ ,  $c_{\alpha,0} = 1$ , specuvely.<br>It is well-know  $I_n, \beta, c_{\alpha,0} = 1,$ where  $A_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ ,  $E_1, E_2$  cont<br>spectively.<br>It is well where  $A_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ , where  $A_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ , where  $A_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ , where  $A_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ , where  $A_{1\alpha} = A_{11} + I_{n_1}\alpha$ ,  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ ,

**Definition 1.** [4] The fractional system (1) is called (internally<br>positive if  $x_1(k) \in \mathbb{R}^{n_1}$ ,  $x_2(k) \in \mathbb{R}^{n_2}$ ,  $k \in \mathbb{Z}_+$ , for all  $x_1(0) \in \mathbb{R}^{n_1}$ <br> $\ldots$  (0)  $\in \mathbb{S}^{n_2}$  and suppose (b)  $\in \mathbb{S}^{m$ and every  $u(k) \in \mathbb{R}_+^{m}$ ,  $k \in \mathbb{Z}^{n^2}$ 1. [4] The fractional syst  $\mathcal{L}(1)$  The fraction  $\mathcal{L}(2)$  is called the fractional system (1) is called the fractional system (1) is called the fraction of  $\mathcal{L}(2)$ **Definition 1. Definition** 1. **Construction** 1. **Construction Definition 1. Definition** 1. **Construction** (1) is called the fraction of  $\mathbf{r}$ **Definition 1.** [4] The fractional system (1) is call positive if  $x_1(k) \in \mathbb{R}_+^{n_1}$ ,  $x_2(k) \in \mathbb{R}_+^{n_2}$   $k \in \mathbb{Z}_+$ . for  $x_2(0) \in \mathbb{R}^{n_2}$  and every  $u(k) \in \mathbb{R}^{m_2}$   $k \in \mathbb{Z}$ **Definition 1. Definition** 1. **Construction** (1) is called the fraction of  $\mathbf{A}$ **Definition 1.** [4] The<br>positive if  $x_1(k) \in \mathcal{F}$ <br> $x_2(0) \in \mathbb{R}^{n_2}$  and even **Definition 1.** [4] The fractional system (1) is called (internally)<br>
positive if  $x_1(k) \in \mathbb{R}_+^{n_1}$ ,  $x_2(k) \in \mathbb{R}_+^{n_2}$   $k \in Z_+$ , for all  $x_1(0) \in \mathbb{R}_+^{n_1}$ ,<br>  $x_2(0) \in \mathbb{R}^{n_2}$  and every  $u(k) \in \mathbb{R}^{m_1}$  **Definition 1.** [4] The fractional system (1) is called (internally) lie ular pencil can be decomposed<br>in  $(1) \in \mathbb{R}^{n_1}$  (1)  $\in \mathbb{R}^{n_2}$  1.  $\in \mathbb{Z}$  of  $\in \mathbb{R}^{n_1}$  (2)  $\in \mathbb{R}^{n_2}$  (1)  $\in \mathbb{R}^{n_2}$  1.  $\frac{1}{x_1}$ **Definition 1.** [4] The fractional system (1) is called (internally) ular p<br>positive if  $x_1(k) \in \mathbb{R}^{n_1}$ ,  $x_2(k) \in \mathbb{R}^{n_2}$   $k \in \mathbb{Z}_+$ . for all  $x_1(0) \in \mathbb{R}^{n_1}$ , strass-<br> $x_2(0) \in \mathbb{R}^{n_2}$  and every  $u(k) \$  $f, h \in \mathbb{Z}_+^2$ . *<i>x*  $\alpha$  is  $\alpha$  i  $x_2(0) \in \mathbb{R}_+^{n_2}$  and every  $u(k) \in \mathbb{R}_+^{m_1}$ ,  $k \in \mathbb{Z}_+$ . **Definition 1. Construction 1. Construction 1. Construction 1. Construction 1. Construction 1. Construction**  $\mathcal{L}(1)$  is called  $\mathcal{L}(2)$  is called  $\mathcal{L}(3)$  is called  $\mathcal{L}(4)$  is called  $\mathcal{L}(4)$ **Definition 1.** [4] The fractional system (1) is called (internally)<br>positive if  $x_1(k) \in \mathbb{R}_+^{n_1}$ ,  $x_2(k) \in \mathbb{R}_+^{n_2}$   $k \in Z_+$ . for all  $x_1(0) \in \mathbb{R}_+^{n_1}$ ,<br> $x_2(0) \in \mathbb{R}^{n_2}$  and every  $u(k) \in \mathbb{R}^{m_1}$   $k$  $\lambda_2(0) \in \mathcal{R}_+$  and every  $u(\kappa) \in \mathcal{R}_+$ ,  $\kappa \in Z_+$ . It is well-known [4] In<br>Definition 1. [4] The fractional system (1) is called (internally) ular pencil can be decomp

 $x_2(0) \in \mathfrak{R}_+$  and every  $u(x) \in \mathfrak{R}_+$ ,  $x \in \mathbb{Z}_+$ .<br> **Comma 1.**<br> **Comma 1.** positive if and only if (iff) *<sup>n</sup> x* ∈ℜ+ and every *<sup>m</sup> ku* ℜ∈ <sup>+</sup> )( , *<sup>n</sup> x* ∈ℜ+ and every *<sup>m</sup> ku* ℜ∈ <sup>+</sup> )( , *<sup>n</sup> x* ∈ℜ+ and every *<sup>m</sup> ku* ℜ∈ <sup>+</sup> )( , *<sup>n</sup>* ∈ℜ+ and every *<sup>m</sup> ku* ℜ∈+ )( , *x* ∞ ∞ *x* ∈ ∞ *x* ∞ *<sup>n</sup> x* ∈ℜ+ , 2 )0( <sup>2</sup> *n x* ∈ − 1 *x* ∞ ∈− ∋ *n x* ∈ → *x* → 2 )0( *n x* ∈ ∴ and every *m* ∴ and  $\sum_{n=1}^{\infty}$ *n x*  $\frac{1}{2}$  $T_{\text{T}}$  Theorem 1. **Theorem 1. For 0**  $\alpha$ *<sup>n</sup> x* ∈ℜ+ and every *<sup>m</sup> ku*ℜ∈ <sup>+</sup> )( , *<sup>n</sup> x* ∈ℜ+ and every *<sup>m</sup> ku* ℜ∈ <sup>+</sup> )( , *<sup>n</sup> x* ∈ℜ+ , 2 )0(2 *n x* ∈ − 1 matrix<br>1980 *nn* **Theorem 1.** [4] The fractional system (1) for  $0 < \alpha, \beta < 1$  is  $\overline{\phantom{a}}$ Î

$$
A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \mathfrak{R}^{n \times n}_+, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathfrak{R}^{n \times m}_+, \qquad (4) \qquad \begin{array}{c} P = \text{blockdiag } (P_1, Q_2) \\ Q = \text{blockdiag } (Q_1, Q_2) \end{array}
$$
such that

#### **3.** Positive descriptor fractional different orders discrete-time linear systems<br>Consider the descriptor fractional discrete-time linear  $\mathbf{B}$   $\mathbf{$  $\sum_{i=1}^{n} a_i$  $\sum_{i=1}^{n}$  $\sum_{i=1}^{n}$  $\mathbf{s}$ 1 systems **12 and 12 minutes** 1 12 *m*<sub>n</sub>  $\sum_{i=1}^{n}$  $\overline{P}$ **B**  $\frac{1}{2}$ **B**  $\frac{1}{2}$ *A A A* <sup>×</sup> ℜ∈ <sup>+</sup>  $\mathbf{u}$  $\left| \frac{1}{2} \right|$  $\frac{1}{2}$   $\$ **3. Positive descriptor fractional different**  3. Positive descriptor fractional different orders<br>discrete-time linear systems<br> $\begin{bmatrix} E_z & 0 \end{bmatrix}$ **discrete-time linear systems** 3. Positive descriptor fractional different orders discrete-time linear systems

with two different fractional orders  $\frac{1}{2}$ **B**  $\mathbf{B}$  **B**  $\mathbf{B}$   $\mathbf{B}$  $\overline{\phantom{a}}$ The descriptor fractional discrete-time linear system<br>  $P\begin{bmatrix} P\end{bmatrix}$   $\begin{bmatrix} P\end{bmatrix}$   $\begin{b$ **orders orders discrete-time linear system** Consider the descriptor fractional discrete-time linear system  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 

$$
E_1 \Delta^{\alpha} x_1(k+1) = A_{11} x_1(k) + A_{12} x_2(k) + B_1 u(k),
$$
  
\n
$$
E_2 \Delta^{\beta} x_2(k+1) = A_{21} x_1(k) + A_{22} x_2(k) + B_2 u(k),
$$
 (5) and

where  $k \in Z_+, x_1(k) \in \mathbb{R}_+^{n_1}$  and  $x_2(k) \in \mathbb{R}_+^{n_2}$  are the state vec-<br> $\overline{E}_+ = P_+ E_+ O_+$ tors,  $u(k) \in \mathbb{R}^m$  is the input vector and  $E_i \in \mathbb{R}^{n_i \times n_i}$   $A_{ij} \in \mathbb{R}^{n_i \times n_j}$  $B_i \in \mathbb{R}^{n_i \times m}$ ;  $i, j = 1, 2$ .  $\mathcal{H}^{\text{mean}}$ ;  $i, j = 1, 2$ .<br>The state gross solution to this class of fractions  $\mathbb{R}^n$   $\mathbb{R}^n$  ;  $i, j = 1, 2$ .<br>The state space solution to this close of fractions where  $k \in Z_+$ ,  $x_1(k) \in \mathbb{R}_+^{n_1}$  and  $x_2(k) \in \mathbb{R}_+^{n_2}$  are the state vectors,  $u(k) \in \mathbb{R}^m$  is the input vector and  $E_i \in \mathbb{R}^{n_i \times n_i}$ ,  $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ ,  $B_i \in \mathbb{R}^{n_i \times m_j}$ ;  $i, j = 1, 2$ .<br>The state-spa *n*, 2.<br>∴eolytion to this class of fractional systems.  $\frac{1}{2}$   $\frac{1}{2}$  where  $k \in \mathbb{Z}_+$ ,  $x_1(k) \in \mathfrak{R}_+$  and  $x_2(k) \in \mathfrak{R}_+$  are the state vector<br>tors,  $u(k) \in \mathfrak{R}^m$  is the input vector and  $E_i \in \mathfrak{R}^{n_i \times n_i}$ ,  $A_{ij} \in \mathfrak{R}^{n_i \times n_j}$ ,  $\overline{E}_1 = P_1 E_1 Q_1 = \begin{bmatrix} n_i \\ 0 \end{bmatrix}$ <br> $B_i \in \math$  $\frac{1}{2}$   $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \in \mathbb{Z}$  (1)  $\in \mathfrak{M}_{n}^{n}$  and (1)  $\in \mathfrak{M}_{n}^{n}$  and a term of tors,  $u(k) \in \mathbb{R}^m$  is the input vector and  $E_i \in \mathbb{R}^{n_i \times n_i}$ ,  $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ ,

can be found in [25].<br>Using the definition of fractional derivative (2), the fraction  $\mathcal{H}^{\text{max}}$ ;  $i, j = 1, 2$ .<br>The state-space solution to this class of fractional system systems system  $\mathcal{H}^{\text{max}}$ ;  $i, j = 1, 2$ .<br>he state-space solution to this class of fractional system  $(1)$ 2 2 121 222 2 2 2 121 222 2 **F**  $\frac{1}{2}$  *k* $\frac{1}{2}$  **<b>***k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *<i>k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *<i>k*  $\frac{1}{2}$  *k*  $\frac{1}{2}$  *<i>k*  $\frac{1}{2}$   $A_{1\alpha}$  –  $A_{1\alpha}$  $\mathcal{R}^{n+m}$ ;  $i, j = 1, 2$ .<br>The state-space solution to this class of fractional systems  $\overline{A}_{1\alpha} = I$ <br>be found in [25]. The state-space solution to this class of fractional system:<br>can be found in [25].<br>Using the definition of fractional derivative (2), the fractional<br>system (5) can be written as the descriptor system with delays  $A_{1a} = I_1$  $A_{1\alpha} = B$  $\frac{1}{2}$  $A_{1\alpha} = I_1 A_{1\alpha} Q_1$ his class of fractional systems  $\overline{A}_{1\alpha} = P_1 A_{1\alpha} Q_1$  $\kappa^{m}$ ; *i*, *j* = 1, 2.<br>
i.e state-space solution to this class of fractional<br>  $\epsilon$  found in [25]. stems  $\begin{aligned} \text{[}x_1, \ldots, x_n, t, j = 1, 2. \text{[}x_n \text{] } \text{[}x_n \text$ stems  $A_{1a} - I_1A_{1a}$  $A_{1a} = I_1A_1$  $A_{1\alpha}$ 2.<br>solution to this class of fractional systems  $\overline{A}$ . *Ex k AkxA x k kuB*  $B_i \in \mathfrak{R}^{n_i \times m}$ ; *i*, *j* = 1, 2.<br>The state-space solution to this class of fractional systems<br>can be found in [25].  $B_i \in \mathbb{R}^n$ ;  $i, j = 1, 2$ .<br>The state-space solution to this class of fractional systems  $\overline{A}$   $\overline{$ *ji nn Aij* to be found in [25].<br>Using the definition of fractional derivative (2) the fractional can be found to found in the found in  $\frac{1}{2}$  $B_i \in \mathcal{R}^{\perp}$ ;  $i, j = 1, 2$ .<br>The state-space solution to this class of fractic

system (5) can be written as the descriptor system with delays  $\overline{A}_{12} = P_1 A_{12}$ *E*  $\frac{1}{2}$   $\frac{1}{2$  $\frac{1}{2}$ acuona<br>+ + can be found in [25].<br>Using the definition of fractional derivative (2), the fractional<br>system (5) can be written as the descriptor system with delays system (5) can be written as the descriptor system with delays  $\overline{A}_{12} = P_1 A_{12} Q_2 = \begin{bmatrix} \overline{A}_{11}^{12} \\ \overline{A}_{12}^{12} \end{bmatrix}$ *ji nn Aij*  $T_{\rm eff}$  state-space solution to this class of fraction to this class of fractional systems of fractional systems of  $T_{\rm eff}$ *ji nn Aij*  $T_{\rm eff}$  state-space solution to this class of fraction to this class of fractional systems of fractional systems of  $T_{\rm eff}$ system with delays

$$
A_{12} = P_1 A_{12} Q_2 =
$$
\n
$$
E\left[\frac{x_1(k+1)}{x_2(k+1)}\right] = A\left[\frac{x_1(k)}{x_2(k)}\right] + \sum_{j=2}^{k+1} c_j \left[\frac{x_1(k-j+1)}{x_2(k-j+1)}\right] + Bu(k) \quad (6a)
$$
\n
$$
\overline{A}_{21} = P_2 A_{21} Q_1 =
$$
\nwhere\n
$$
\overline{A}_{22} = P_3 A_{22} Q_2 =
$$

where  $\frac{1}{\sqrt{2}}$  $where$ can be found in fact that the found in fact that the found in  $\mathcal{Z}_2$  $where$ can be found in fact that the found in  $\mathcal{Z}_2$  $where$ can be found in fact that the found in  $\mathcal{Z}_2$  $where$ can be found in fact that the found in  $\mathcal{Z}_2$  $where$ can be found in found in found in form  $where$ can be found in found in found in form

$$
E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix},
$$
\n
$$
B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad c_j = \begin{bmatrix} c_{\alpha,j} E_1 & 0 \\ 0 & c_{\beta,j} E_2 \end{bmatrix}
$$
\n(6b)

\n
$$
\overline{B}_1 = P_1 B_1 = \begin{bmatrix} \overline{B}_1^1 \\ \overline{B}_2^1 \end{bmatrix},
$$
\nwhere rank  $E_1 = n_1^1$ ,  $\text{rad } A_{1\alpha} = A_{11} + E_1\alpha, A_{2\beta} = A_{22} + E_2\beta.$ 

and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup><sup>β</sup> = *A*<sup>22</sup> + *E*2β . and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup><sup>β</sup> = *A*<sup>22</sup> + *E*2β .

ptor system consider the descriptor system v 0 *A A A A* es *E ill* consider the descriptor system  $\mathcal{L}(\mathcal{$ consider the descriptor system w  $\overline{1}$  $\frac{1}{2}$  $\mathbf{G}(\mathbf{G})$ l consider the descriptor system v  $\mathbf{G}(\mathbf{G})$ l consider the descriptor system v *B<sup>A</sup> <sup>A</sup> Ejjj* Further, we will consider the descriptor system with regular<br>pencil<br> $det E = 0$  (7a) == Further, we will consider the descriptor system with regular pencil = **Evaluation** Eventure with regular consider the descriptor system with re *B<sup>A</sup> <sup>A</sup> Ej* α Further, we will consider the descriptor system with regular pencil<br>  $\det E = 0$  (7a) re will consider the descriptor system with regular  $U$  can write the equation (1) in the matrix  $U$  in the matrix  $U$  in the matrix  $U$ onsider the descriptor system with iptor system with regular onsider the descriptor system with<br>  $\frac{d}{dt}$  $\frac{1}{2}$  consider the descriptor system *j* Further, we will consider the descriptor system with regular (2b) pencil ular<br>1988-yilda ===== == iptor system with regular *c i*ptor system with *c* Further, we will consider the descriptor system with regular  $(2b)$  pencil Further, we will cons<br>
(2b) **pencil**  $\sum_{k=1}^{\infty}$  denoted the description of  $\sum_{k=1}^{\infty}$  $\mathbb{R}^n$  bencil consider the descriptor system with  $\mathbb{R}^n$ Further, we will consider the descriptor system with re (2b) pencil  $\det F = 0$ 

$$
\det E = 0 \tag{7a}
$$

and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup><sup>β</sup> = *A*<sup>22</sup> + *E*2β .  $\frac{1}{\sqrt{2}}$  and  $\frac{1}{\sqrt{2}}$  and and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup><sup>β</sup> = *A*<sup>22</sup> + *E*2β .  $\mathbf{F}$  further, we will consider the descriptor system with  $\mathbf{F}$  and  $\math$ and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup><sup>β</sup> = *A*<sup>22</sup> + *E*2β .  $\mathbf{F}$  and  $\mathbf{F}$  and  $\mathbf{F}$  are description system with  $\mathbf{F}$  and  $\mathbf{F}$  are description system with  $\mathbf{F}$  $\mathbf{r}$ 2 *x k* and and and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup>β= *A*<sup>22</sup> + *E*2β .  $\mathbf{F}$  and description system with description system with  $\mathbf{F}$  and description system with  $\mathbf{F}$ and *A*<sup>1</sup><sup>α</sup> = *A*<sup>11</sup> + *E*1<sup>α</sup>, *A*<sup>2</sup><sup>β</sup> = *A*<sup>22</sup> + *E*2β . Further, we will consider the descriptor system with the descriptor system with  $\alpha$ 

(3) 
$$
\det \begin{bmatrix} E_1 z_1 & 0 \ 0 & E_2 z_2 \end{bmatrix} - \begin{bmatrix} A_{1\alpha} & A_{12} \ A_{21} & A_{2\beta} \end{bmatrix} \neq 0 \qquad (7b)
$$

 $u(k)$ ,<br>for some  $z \in C$  (the field of complex numbers), where matrices  $E_1$ ,  $E_2$  contain only  $n_1$ ,  $n_2$  linearly independent colum<br>spectively.  $A_{2\beta} = A_{22} + I_{n_2}\beta$ ,  $c_{\alpha,0} = 1$ ,<br>  $B_{1}$ ,  $D_{2}$  contain only  $n_1$ ,  $n_2$  inearly independent columns, respectively.<br>
It is well-known [4] that every descriptor system with reg- $\frac{1}{1}$ in only  $n_1$ ,  $n_2$  linearly independent con  $\mathbf c$  $\frac{1}{2}$ independent aly  $n_1^1$ ,  $n_2^1$  linearly independ  $\ldots$ ,  $\ldots$ ıe a only  $n_1$ ,  $n_2$  linearly independent colu $x_1$   $y_1$ ,  $y_2$  linearly independent co  $\frac{1}{\lambda}$ ndependent c<br>escriptor syste y  $n_1^1$ ,  $n_2^1$  linearly independent  $\ldots$  $\mathbf{n}$ n only  $n_1$ ,  $n_2$  linearly independent column  $\overline{\mathbf{c}}$ n independent o ly  $n_1^1$ ,  $n_2^1$  linearly independe for some  $z \in C$  (the field of complex numbers), where m<br>  $E_1$ ,  $E_2$  contain only  $n_1^1$ ,  $n_2^1$  linearly independent colum<br>
spectively. in only  $n_1$ ,  $n_2$  linearly independent column to the very description system.  $\overline{c}$ n independent − *A*<sub>1</sub>,  $n_1^1$ , *A*<sub>2</sub> linearly independently independently in  $\mu_1$ , *A*<sub>2</sub> linearly descriptors nearly independent columns, rerly independent columns, rely independent columns, rely independent columns, re- $\frac{1}{2}$  $n_1^1$ ,  $n_2^1$  linearly independent columns, rea only  $n_1$ ,  $n_2$  linearly independent column  $\lceil 4 \rceil$  that every descriptor system is  $\overline{c}$ ป *H*<sub>1</sub>,  $n_1^1$  linearly independent c<br> *A*  $(n \text{ [4] that every descriptor system}$ for some  $z \in C$  (the field of c dent colu<br>a matema ndependent columns, rependent columns, re- $\alpha$  *A*  $\alpha$  *Columns*, *re*  $\frac{1}{2}$  and  $\frac{1}{2}$  $221$  $z$ <sub>E</sub><br>*z*<sub>E</sub> <sup>2</sup> only  $n_1$ ,  $n_2$  linearly independent colu lι  $\overline{a}$ V dependent co  $n_1^1$ ,  $n_2^1$  linearly independent ar bony  $n_1$ ,  $n_2$  inearly independent columnation of  $\frac{1}{2}$  and  $\frac{1}{2}$  that every descriptor system is u  $n_1^1$ ,  $n_2^1$  linearly independent col<br>  $\Box$  [4] that every descriptor system  $n_1^1$ ,  $n_2^1$  linearly independent columns  $\begin{bmatrix} B_2 \ B_2 \end{bmatrix} u(k)$ , for some  $z \in C$  (the field of complex numbers), where matrices pendent columns, red *z* dent columns, restrictions. for some  $z \in C$  (the field of complex numbers), where matrices  $E_1$ ,  $E_2$  contain only  $n_1^1$ ,  $n_2^1$  linearly independent columns, respectively. .<br>س spectively.  $E_1, E_2$  contain only  $n_1, n_2$  incarry independent columns, spectively. for some  $z \in C$  (the field of complex numbers), where matrice  $E_1, E_2$  contain only  $n_1, n_2$  iniciarly independent columns,  $F$  spectively.  $(z)$ ,<br>for some  $z \in C$  (the field of complex numbers), where matrices for some  $z \in C$  (the field of complex numbers), where matrices  $E_1$ ,  $E_2$  contain only  $n_1^1$ ,  $n_2^1$  linearly independent columns, re-

known [4] that every descriptor system with reg-<br>in be decomposed e.g. by the use of the Weier $x_1^{y}$  diar penetrican be decomposed e.g. by the use of the weiel-<br> $x_1^{u_1}$ , strass-Kronecker decomposition theorem. wn [4] that every descriptor syster spectively.<br>It is well-known [4] that every descriptor system with reg-<br>ular pencil can be decomposed e.g. by the use of the Weier-<br>strass-Kronecker decomposition theorem.<br>Lemma 1. If (7a) and (7b) hold for the system with **for some** *z* ∈<sup>*C*</sup> (*t*)  $\leq$   $\$ It is well-known [4] that every descriptor system with reg-<br>nally) ular pencil can be decomposed e.g. by the use of the Weier- $\sum_{n=1}^{\infty}$  $\in \mathbb{R}_+^{n_1}$ , strass-Kronecker decomposition theorem.

 $\alpha, \beta < 1$  is ferent fractional orders (6), then there exist nonsingular matrices **E**<sub>2</sub> contain only 1 *n*<sup>1</sup> , <sup>1</sup> and every  $u(k) \in \mathbb{R}^m_+$ ,  $k \in Z_+$ .<br>
Lemma 1. If (7a) and (7b) hold for the system with two dif-<br>
1. [4] The fractional system (1) for  $0 < \alpha, \beta < 1$  is ferent fractional orders (6), then there exist nonsingular ma-It is well-known with the system with the syst It is well-known  $\mathcal{A}$  that every description system with  $\mathcal{A}$  that every description  $\mathcal{A}$  is a system with  $\mathcal{A}$ It is well-known with the event of the system with the system with the system with  $\alpha$ *n x* ∈ *x* ∈ + *x* + *x* + *(*  $\infty$  + )  $\sum_{i=1}^{n}$  that every description  $\sum_{i=1}^{n}$  that every description  $\sum_{i=1}^{n}$  that every description  $\sum_{i=1}^{n}$  and  $\sum_{i=1}^{n}$  and  $\sum_{i=1}^{n}$  and  $\sum_{i=1}^{n}$  and  $\sum_{i=1}^{n}$  and  $\sum_{i=1}^{n}$  and  $\sum_{i=1}^{n}$ It is well-known  $\mathcal{L}$  that every description system with  $\mathcal{L}$  that every description  $\mathcal{L}$  $\frac{1}{2}$  and  $\frac{1}{2}$ trices **Lemma 1.** If  $(\lambda a)$  and  $(\lambda b)$  hold for the system with two dif- $\mathcal{F} = \mathcal{F} \mathcal{F}(\mathcal{F}) = \mathcal{F}(\mathcal{F}|\mathcal{F})$ **Lemma 1.** If (7a) and (7b) hold for the system with two  $W = 4.58 \times 1.711118...$ **Lemma 1.** If (7a) and (7b) hold for the system with two dif $m_{\rm H}$ *B* **a 1.** If (7a) and (7b) hold for the system with two dif-**Example 1.** If (7a) and (7b) hold for the system with tw<br>  $<$  1 is ferent fractional orders (6), then there exist nonsingular *Q Q Q*  $\mathbf{e}$ **Lemma 1.** If  $(7a)$  and  $(7b)$  hold for the system with two

trices  
\n
$$
P = \text{blockdiag}(P_1, P_2) \in \mathfrak{R}^{n \times n},
$$
\n
$$
Q = \text{blockdiag}(Q_1, Q_2) \in \mathfrak{R}^{n \times n}
$$
\nsuch that

 $\mathbb{R}^n$  that **Lemma 1.** If  $\alpha$  is the system with two systems of the system of the system with two systems of different fractional orders (6), then the exist nonsingular  $\mathcal{O}_\mathbf{C}$ such that different fractional orders (6), then there exist nonsingular different fractional orders (6), then the exist nonsingular  $\mathcal{O}_\mathbf{C}$ is positive if and only if  $\alpha$  if  $\alpha$  if  $\alpha$ such that  $\frac{1}{2}$  $\mathbf{S}^{\dagger}$ different fractional orders (6), then the exist nonsingular  $\mathcal{O}_\mathbf{C}$ different fractional orders (6), then the exist nonsingular  $\mathcal{O}(1)$ 

e linear systems  
\ne linear systems  
\n
$$
P\left[\begin{bmatrix} E_1z_1 & 0 \ 0 & E_2z_2 \end{bmatrix} - \begin{bmatrix} A_{1\alpha} & A_{12} \ A_{21} & A_{2\beta} \end{bmatrix}\right]Q =
$$
\nfractional orders  
\n
$$
= A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k),
$$
\n
$$
= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k),
$$
\n(5) and

1. Sajewski  
\n
$$
\begin{pmatrix}\na \\
j\n\end{pmatrix} = \begin{cases}\na(\alpha-1) \dots (\alpha-j+1) & \text{for } j=0 \\
\frac{\alpha(\alpha-1) \dots (\alpha-j+1)}{j!} & \text{for } j=1,2,... \\
\frac{\alpha}{j!} & \text{for } j=1,2,...\n\end{cases}
$$
\n2. Sajewski  
\nUsing (2), we can write the equation (1) in the matrix form  
\n
$$
\begin{bmatrix}\nx_1(k+1) \\
x_2(k+1) \\
x_3(k+1)\n\end{bmatrix} = \begin{bmatrix}\nA_{ij} & A_{ij} \\
A_{ij} & A_{ij} \\
A_{ij} & A_{ij}\n\end{bmatrix} \begin{bmatrix}\nx_1(k) \\
x_2(k) \\
x_3(k) \\
x_4(k) \\
x_5(k) \\
x_6(k) \\
x_7(k) \\
x_8(k) \\
x_9(k) \\
x_1(k) \\
x_2(k) \\
x_3(k) \\
x_4(k) \\
x_5(k) \\
x_6(k) \\
x_7(k) \\
x_8(k) \\
x_9(k) \\
x_1(k) \\
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x_5(k) \\
x_6(k) \\
x_7(k) \\
x_8(k) \\
x_9(k) \\
x_1(k) \\
x_2(k
$$

where  $\text{rank } E_1 = n_1^1$ ,  $\text{rank } E_2 = n_2^1$ ,  $n_1 = n_1^1 + n_2^1$ ,  $n_2 = n_2^1 + n_2^2$ ,  $n = n_1 + n_2.$ 

The matrices *P* and *Q*, which decompose the system (6), can The matrices P and Q, which decompose the system (6), can<br>be found by the use of many different method, see e.g. [1, 29].<br>Demonstrative that distribute the system ((c) has the matrix e the system  $\infty$ matrices P and Q, which decompose by the use of many different met  $P$  and  $Q$ , which decompose the Ļ, 1<br>1  $P$  and  $Q$ , which decompose the ! !<br>: .  $\epsilon$  and  $Q$ , which decompose to<br>the use of many different method and  $\sum_{i=1}^{n} x_i$  *x x*  $x_i$  *x*  $x_i$  atrices  $P$  and  $Q$ , which decompose the s

be found by the dise of many director method, see e.g. [1, 25].<br>Premultiplying the state equation (6a) by the matrix  $P \in \mathfrak{R}^{n \times n}$  and introducing the new state vector *P*  $\in \mathbb{R}^{n \times n}$  and introducing the new state vector )<br>r dependence to the state equation (6a) by the r oducing the new state vector If  $\mu$  is the state equation (6a) by the state equation (6a) by the state equation (6a) by the state of  $\mu$  $\frac{1}{2}$  *k*  $\frac{0}{1}$ e use of many unferent method<br>lying the state equation (6a) *x*  $\frac{d}{dx}$  **k k x x x** *x x*  $\frac{d}{dx}$  *x*  $\frac{d}{dx}$  *x* iuli (ua)<br>te vector

$$
Q^{-1}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \overline{x}_1(k) \\ \overline{x}_2(k) \end{bmatrix} = \begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_2^2(k) \\ \overline{x}_2^1(k) \end{bmatrix}, \quad \overline{x}_1^1(k) \in \mathfrak{R}^{n_1^1},
$$
\n
$$
\overline{x}_1^2(k) \in \mathfrak{R}^{n_1^2}, \quad \overline{x}_2^1(k) \in \mathfrak{R}^{n_2^1} \quad \overline{x}_2^2(k) \in \mathfrak{R}^{n_2^2}
$$
\nThe  
\nfor  $k \in \mathbb{Z}$ , we obtain

Ī

for  $k \in Z_+$  we obtain  $\frac{1}{2}$  *k*  $\in$  *Z* we obtain *x PEQQUAL k*  $\in Z_+$  we obtain

For 
$$
k \in \mathbb{Z}_+
$$
 we obtain

\n
$$
PEQQ^{-1} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = PAQQ^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{cases} \text{agor} \\ + \sum_{j=2}^{k+1} c_j PEQQ^{-1} \begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + PBu(k). \end{cases}
$$
\n(12)

\n
$$
E = \frac{1}{2} \begin{bmatrix} 1 & \text{if } k \in \mathbb{Z} \end{bmatrix}
$$

Ī

 $\frac{1}{2}$   $\mathbf{1}$  $J=2$   $L=2$  + Taking into consideration (10), (10) Taking into consideration  $(10, 11)$  and recombining the state  $\overline{\mathbf{k}}$  $\overline{\phantom{a}}$  − + = ∑  $T_a$  is considered the consideration (10),  $T_a$ Taking into consideration  $(10, 11)$  and recombining the state equation (12) we obtain two subsystems – the standard dynam-  $\beta$  < 1 is p ical subsystem  $\mathfrak{u}$ ع<br>. Ĭ ֦֦֦֦֦֦֡֕֕֕֡֡֡֜֡֡֜  $\mathbf{c}$  $\frac{1}{2}$  $\overline{a}$  $\frac{1}{\pi}$  $\frac{1}{2}$ ,<br>,լ  $\mathfrak{r}$ נכ<br>+י  $\mathbf{r}$ !'<br>.  $\overline{\phantom{a}}$ ب<br>۱  $\frac{1}{2}$  $\overline{\phantom{a}}$ ical subsy Faxing fino consideration  $(10, 11)$  and recombining the<br>equation  $(12)$  we obtain two subsystems – the standard d<br>ical subsystem ć. ical subsyste  $\overline{a}$  $\overline{\phantom{a}}$ .<br>21 t,  $\mathbf{l}$  $\cdot$ d subsyste ์<br>1  $s - \text{inc } \text{su}$  $\frac{1}{2}$  $\mathsf{u}$  $\mathfrak{p}$ tain tw  $\overline{1}$  $\mathsf{I}$ |
|
| *kx A A kx A A* + = α  $\cdot$  $\epsilon$ 

$$
\begin{aligned}\n\left[\overline{x}_{1}^{1}(k+1)\right] &= \begin{bmatrix} \overline{A}_{1\alpha}^{11} & \overline{A}_{11}^{12} \\ \overline{A}_{21}^{21} & \overline{A}_{22}^{22} \end{bmatrix} \begin{bmatrix} \overline{x}_{1}^{1}(k) \\ \overline{x}_{2}^{2}(k) \end{bmatrix} + \begin{bmatrix} \overline{A}_{12}^{11} & \overline{A}_{12}^{12} \\ \overline{A}_{12}^{21} & \overline{A}_{22}^{22} \end{bmatrix} \begin{bmatrix} \overline{x}_{1}^{2}(k) \\ \overline{x}_{2}^{2}(k) \end{bmatrix} \\
&\quad + \sum_{j=2}^{k+1} c_{j} \begin{bmatrix} \overline{x}_{1}^{1}(k-j+1) \\ \overline{x}_{2}^{1}(k-j+1) \end{bmatrix} + \begin{bmatrix} \overline{B}_{1}^{1} \\ \overline{B}_{2}^{1} \end{bmatrix} u(k) \\
\text{Proof. It is stable Metz} \\
\text{stable Metz}\n\end{aligned}
$$

static (algebraic) subs 1  $\mathcal{C}_{\mathcal{C}}$ *h* the static (algebraic) sub J.  $\int$ <sub>s</sub>eb  $\mathsf{h}$ and the static (algebraic) su r,  $\frac{1}{2}$ and the static (algebraic) s  $\ddot{\phantom{a}}$  $\mathsf{e}$  $\overline{a}$ alg  $\overline{a}$  $\mathfrak{p}$ and the static (algebraic) subsystem<br>  $\begin{bmatrix} 1 & -12 & -12 \end{bmatrix} \begin{bmatrix} -11 & -12 \end{bmatrix}$ Ĭ

$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{A}_{21}^{11} & \overline{A}_{21}^{12} \\ \overline{A}_{21}^{21} & \overline{A}_{21}^{22} \end{bmatrix} \begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_2^1(k) \end{bmatrix} + \begin{bmatrix} \overline{A}_{21}^{11} & \overline{A}_{22}^{12} \\ \overline{A}_{22}^{21} & \overline{A}_{22}^{22} \end{bmatrix} \begin{bmatrix} \overline{x}_1^2(k) \\ \overline{x}_2^2(k) \end{bmatrix} + \begin{bmatrix} \overline{B}_1^2 \\ \overline{B}_2^2 \end{bmatrix} u(k).
$$
\n(14)

The subsystems (13) and (14) can be written as  $\frac{1}{2}$  (13) and (14) can be writ

$$
\tilde{x}_1(k+1) = \tilde{A}_{11}\tilde{x}_1(k) + \tilde{A}_{12}\tilde{x}_2(k)
$$
  
+ 
$$
\sum_{j=2}^{i+1} c_j \tilde{x}_1(k-j+1) + \tilde{B}_1 u(k),
$$
 (15a)

$$
0 = \widetilde{A}_{21}\widetilde{x}_1(k) + \widetilde{A}_{22}\widetilde{x}_2(k) + \widetilde{B}_2u(k), \tag{15b}
$$

where where where

$$
\begin{aligned} \widetilde{A}_{11}=&\begin{bmatrix} \overline{A}_{1\alpha}^{11} & \overline{A}_{11}^{12} \\ \overline{A}_{11}^{21} & \overline{A}_{1\beta}^{22} \end{bmatrix} \in \mathfrak{R}^{\widetilde{n_1} \times \widetilde{n_1}}, \ \ \widetilde{A}_{12}=\begin{bmatrix} \overline{A}_{11}^{11} & \overline{A}_{12}^{12} \\ \overline{A}_{12}^{21} & \overline{A}_{12}^{22} \end{bmatrix} \in \mathfrak{R}^{\widetilde{n_1} \times \widetilde{n_2}},\\ \widetilde{A}_{21}=&\begin{bmatrix} \overline{A}_{21}^{11} & \overline{A}_{21}^{12} \\ \overline{A}_{21}^{21} & \overline{A}_{21}^{22} \end{bmatrix} \in \mathfrak{R}^{\widetilde{n_2} \times \widetilde{n_1}}, \ \ \widetilde{A}_{22}=\begin{bmatrix} \overline{A}_{2\alpha}^{11} & \overline{A}_{22}^{12} \\ \overline{A}_{22}^{21} & \overline{A}_{2\beta}^{22} \end{bmatrix} \in \mathfrak{R}^{\widetilde{n_2} \times \widetilde{n_2}}, \end{aligned}
$$

 $\widetilde{x}_1(k) = \left| \frac{x_1(k)}{\tau^1(k)} \right| \in \mathfrak{R}^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{x_1(k)}{\tau^2(k)} \right| \in \mathfrak{R}^{\widetilde{n}_2},$  $B_1 = \left[\frac{-1}{B_2}\right] \in \mathfrak{R}^{n_1 \wedge n_2}, \ \ B_2 = \left[\frac{-1}{B_2}\right] \in \mathfrak{R}^{n_2 \wedge n_2},$  $\overline{x}_{2}^{2}(k)$  $\widetilde{x}_2(k) = \begin{cases} \overline{x}_1^2(k) \\ \overline{x}_2^2(k) \end{cases}$  $|_{\subseteq}$ 22 21  $\overline{1}$  $\widetilde{x}_1(k) = \left| \frac{\overline{x}_1^1(k)}{1} \right| \in \mathfrak{R}^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{\overline{x}_1^2(k)}{2} \right| \in \mathfrak{R}^{\widetilde{n}_2},$  $\overrightarrow{B}_2 = \left[ \frac{B_1}{\overrightarrow{B}_2} \right] \in \mathfrak{R}$  $\mathfrak{R}$  $\mathsf{L}$  $\widetilde{B}_1 = \left| \frac{B_1}{B_2^1} \right| \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \ \ \widetilde{B}_2 = \left| \frac{B_1}{B_2^2} \right| \in \mathfrak{R}^{\widetilde{n}_2 \times m},$  $\lfloor x_2 \rfloor$  $\overline{x}_2(k) = \left| \frac{x_1}{x_2} \right|$  $\widetilde{x}_1(k) = \left| \frac{x_1(k)}{\overline{x}^1(k)} \right| \in \mathfrak{R}^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{x_1(k)}{\overline{x}^2(k)} \right| \in \mathfrak{R}^{\widetilde{n}_2},$ *A A*  $\overline{a}$  $\overline{\phantom{a}}$  $\overline{a}$  $\overline{\phantom{a}}$  $\zeta$  $\frac{1}{2}$  $\overline{x}_2^1(k)$   $\left[\in \mathfrak{R}^+, x_2(k)\right]$  $\ddot{\phantom{0}}$  $\widetilde{x}_1(k) = \left| \frac{\overline{x}_1^{\epsilon}(k)}{\overline{x}_1^{\epsilon}(k)} \right| \in \mathfrak{R}^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{\overline{x}_1^{\epsilon}(k)}{\overline{x}_2^{\epsilon}(k)} \right| \in \mathfrak{R}^{\widetilde{n}_2},$  $\cdot$ <sup>r</sup>  $\left| \frac{B_1}{B_2} \right| \in \Re^{n_1 \times m}, \quad B_2 =$ 29].  $\qquad \qquad \widetilde{B}_1 = \left| \frac{B_1}{B_2^1} \right| \in \mathfrak{R}^{\widetilde{n_1} \times m}, \quad \widetilde{B}_2 = \left| \frac{B_1}{B_2^2} \right| \in \mathfrak{R}^{\widetilde{n_2} \times m},$ β  $\widetilde{n}_1 = n_1^1 + n_2^1, \quad \widetilde{n}_2 = n_1^2 + n_2^2, \quad n = \widetilde{n}$ *<sup>B</sup> <sup>B</sup>*  $\widetilde{n}_1 = n_1^1 + n_2^1, \quad \widetilde{n}_2 = n_1^2 + n_2^2, \quad n =$  $\overline{x}_{2}^{2}$  ( 2  $\tilde{x}_1, \ \tilde{x}_2(k) = \begin{vmatrix} \overline{x}_1^2 \\ \overline{x}_2^2 \end{vmatrix}$  $^{(1)}$  $\widetilde{x}_1(k) = \left| \frac{\overline{x}_1^1(k)}{1} \right| \in \mathfrak{R}^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{\overline{x}_1^2(k)}{1} \right| \in \mathfrak{R}^{\widetilde{n}_2}$  $L^{2}$  2  $\perp$ <sup>n</sup>,  $\overrightarrow{B}_2 = \left[ \frac{B_1}{B_2^2} \right] \in$ |∈ `  $\widetilde{B}_1 = \left| \frac{B_1}{B_2^1} \right| \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \ \ \widetilde{B}_2 = \left| \frac{B_1}{B_2^2} \right| \in \mathfrak{R}^{\widetilde{n}_2 \times m},$  $\lfloor x \rfloor$  $\lambda^{\tilde{n}_1}$ ,  $\tilde{x}_2(k) = \left| \frac{x_1}{\overline{x}} \right|$  $\widetilde{x}_1(k) = \left| \frac{x_1(k)}{\tau^2(k)} \right| \in \Re^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{x_1(k)}{\tau^2(k)} \right| \in \Re^{\widetilde{n}_1}$ *A A*  $\overline{ }$ ı Ĭ  $\mathbf{k}$  $\overline{c}$  $\left[\overline{x}_2^1(k)\right] \in \mathcal{R}^1$ ,  $\lambda$ ľ  $\begin{vmatrix} \frac{\overline{x}_1^1(k)}{\overline{x}_2^1(k)} \end{vmatrix} \in \mathfrak{R}^{\widetilde{n}_1}, \ \ \widetilde{x}_2(k) =$  $\zeta$  $=\left|\frac{B_1}{B_2}\right| \in \Re^{n_1 \times m}, \ \ \widetilde{B}_2$  $=\left|\frac{D_1}{B_2}\right|\in \mathfrak{R}^{n_1\times m}$ β (11)  $\tilde{n}_1 = n_1^1 + n_2^1, \quad \tilde{n}_2 = n_1^2 + n_2^2, \quad n =$  $\tilde{n}$  $k)$ m  $\frac{1}{2}$  $\mathbf{r}$ Ļ,  $\mathbf{r}$  $\mathbf{k}$  $\vert$  ,  $\frac{1}{1}$  $\overline{\lambda}$  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  $\overline{a}$  $\lceil \bar{x}_1^1(k) \rceil$  $\tilde{\mathcal{R}}^{\widetilde{n}_{1}\times}$  $\lfloor B_2 \rfloor$  $\mathsf{L}$  $\mathbf l$ 1  $\overline{x}_1^1(k)$  $\overline{a}$ 1  $\left| \begin{matrix} 1 \\ 1 \end{matrix} \right|$  $\left[\overline{x}_1^1(k)\right]$   $\infty$ <sup>*n*</sup>  $=$   $\mathfrak{R}^{\widetilde{n}}$  $\times m$ ,  $\times m$ ,  $\tilde{h}$  (10) 6  $\tilde{n}_2$  (1) *B*  $)$  $\overline{a}$  $(16)$  $\overline{\phantom{a}}$  (16) (16)  $\frac{1}{2}$  $\left[\vec{x}_1(k)\right]_{\infty} \mathfrak{R}^{\tilde{n}_1} \quad \left[\vec{x}_2(k)\right]_{\infty} \mathfrak{R}^{\tilde{n}_2}$  (b)  $\left[\vec{x}_1^2(k)\right]_{\infty} \mathfrak{R}^{\tilde{n}_2}$  (16)  $\widetilde{B}_1 = \begin{bmatrix} \overline{B}_1^1 \\ -1 \end{bmatrix} \in \mathfrak{R}^{\widetilde{n}_1 \times m}, ~\widetilde{B}_2 = \begin{bmatrix} \overline{B}_1^2 \\ -2 \end{bmatrix} \in \mathfrak{R}^{\widetilde{n}_2 \times m},$  $\left[\overline{x}_{1}^{1}(k)\right]_{\in\mathfrak{R}^{\widetilde{n}_{1}}}$   $\left[\overline{\widetilde{x}}_{2}(k)\right]_{\infty}$  $\left[\overline{x}_1^1(k)\right]_{\alpha} \propto \tilde{u} \quad \sim (1) \quad \left[\overline{x}_1^2(k)\right]_{\alpha} \propto$ 11  $\overline{B}_1^2$   $\overline{\phantom{a}}$  $\left|\begin{array}{c} B_1 \\ -1\end{array}\right| \in \mathfrak{R}$  $\sim \left[\overline{B}_{1}^{1}\right]$   $\sim \tilde{v} \sim m$   $\sim \left[\overline{B}_{1}^{2}\right]$   $\sim \tilde{v} \sim m$ *nn nn*  $\overline{x}_1^1(k)$ *<sup>A</sup> <sup>A</sup> <sup>A</sup>*  $\mathfrak{R}^{\widetilde{n}_2 \times m}$ .  $\left[\overline{R}^2\right]$   $\sim$  $=$  $\begin{array}{c|c} | & b_1 \\ \hline \end{array}$   $\in$ *A*  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  $(k)\Big|_{\epsilon} \mathfrak{R}^{\tilde{n}_2}$ l.  $\Re^{\tilde{n}_1}$ ,  $\tilde{x}_2(k) = \left[\overline{x}_1^2\right]$  $=\left|x_1^{\mathrm{T}}(k)\right|_{\infty}$  $\overline{a}$ ≡  $\equiv$  $\sqrt{R^2}$ ì  $\overline{\phantom{a}}$  $=\left|\frac{D_1}{D_1}\right|\in\mathfrak{R}'$  $\overline{a}$ =  $\overline{a}$ β  $\widetilde{B}_1 = \begin{bmatrix} \overline{B}_1^1 \ \overline{\phantom{A}}_1^1 \end{bmatrix} \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \ \ \widetilde{B}_2 = \begin{bmatrix} \overline{B}_1^2 \ \overline{\phantom{A}}_2^1 \end{bmatrix} \in \mathfrak{R}^{\widetilde{n}_2 \times m},$  $\lfloor x_2(x) \rfloor$   $\lfloor x_2(x) \rfloor$ <br>  $\lfloor x_2(x) \rfloor$ <br>  $\lfloor x_2(x) \rfloor$  $\lfloor x_2^-(k) \rfloor$   $\lfloor x_2^-(k) \rfloor$ 1  $\sum_{i=1}^{n} \tilde{n}_{i} = n_{1}^{2} +$  $= n_1^1 + n_2^1$   $\tilde{n}_2 = n_1^2 + n_2^2$  r *<sup>B</sup> <sup>B</sup> B*  $\left[ \begin{matrix} x_2(\kappa) \end{matrix} \right]$ *mn mn*  $=$  $n =$ [iiiiiiiiiiiiiiiiiiiiiiii<sub>2</sub>,  $\tilde{n}_2 = n_1^2 + n_2^2$ ,  $\frac{1}{1} + n$ l. 1<br>1  $\left[\vec{x}_1(k)\right] \in \Re^{\tilde{n}_1} \quad \tilde{x}_2(k) = \left[\vec{x}_1^2(k)\right] \in \Re^{\tilde{n}_2} \quad (16)$  $=\left[\overline{x}_1^1(k)\right]_{\infty}$   $\widetilde{x}_1 \widetilde{x}_2(k)$ 12  $\left[\overline{x}_1^1(k)\right]_{\alpha} \infty \tilde{u} \sim d \lambda \left[\overline{x}_1^2(k)\right]_{\alpha}$ 11  $\lceil \overline{B}^2 \rceil$   $\ldots \tilde{r}$   $\vee m$  $=\left|\begin{array}{c} B_1 \\ -1\end{array}\right|\in$ 11 1 *nn nn nn nn*  $\lambda = \frac{\overline{x}_1^1}{\overline{x}_1^1}$ *<sup>A</sup> <sup>A</sup> <sup>A</sup>* ℜ∈  $\in \mathfrak{R}^{\widetilde{n}_2 \times d}$  $\lceil \overline{R^2} \rceil$   $\sim$  $\mathbf{A}_{\mathbf{A}} = \begin{pmatrix} \mathbf{B}_1 \end{pmatrix}$ *A A* × ×  $\tilde{n}$  $\left. \overline{x}_1^2(k) \right| \in \Re^{\tilde{n}}$  $\overline{a}$  $\left[\cos \widetilde{n}_1, \widetilde{x}_2(k)\right] = \sqrt{\overline{x}}$  $\overline{a}$  $(k) = \left| \overline{x}_1^1(k) \right|$ k  $\overline{a}$  $\frac{1}{2}$  $\overline{a}$  $\bar{R}$ ≈  $\bar{R}^2$  $\widetilde{B}_1 = \left| \frac{B_1}{B_1} \right| \in$  $\overline{a}$ *b*  $\widetilde{B}_1 = \begin{bmatrix} \overline{B}_1^1 \\ -1 \end{bmatrix} \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \; \; \widetilde{B}_2 = \begin{bmatrix} \overline{B}_1^2 \\ -1 \end{bmatrix} \in \mathfrak{R}^{\widetilde{n}_2 \times m},$ β  $\begin{bmatrix} x_2 & x_1 \\ y_2 & y_1 \end{bmatrix}$ <br>  $\tilde{n}_1 = n_1^1 + n_2^1, \quad \tilde{n}_2 = n_1^2 + n_2^2, \quad n = \tilde{n}_1 + \tilde{n}_2^1$  $\lfloor x_2^*(k) \rfloor$   $\lfloor x_2^*(k) \rfloor$  $\tilde{n}_1 = n_1^1 + n_2^1$   $\tilde{n}_2 = n_1^2 + n_2^2$  $\begin{bmatrix} x_1 \\ y_2 \\ z_1 \end{bmatrix}$  $\mathbf{p}$ *mn mn*  $\overline{r}$  $n_2^2$ , r  $\left[\bar{x}(k)\right]$ <br>+  $n_2$ ,  $\tilde{n}_2 = n_1^2 + n_2^2$  $= n_1^1 +$  $[n_2(n)]$ <br>  $=n_1^1 + n_2^1$ ,  $\tilde{n}_2 = n_1^2 + n_2^2$ ,  $n = \tilde{n}_1 + \tilde{n}_2$ . ~~  $\mathfrak{R}^{n_1\times m}$ ,  $\left[\overline{B}_1^1\right]_{\sub{\infty}} \mathfrak{R}^{\widetilde{n}_1 \times m}$   $\qquad \widetilde{p}$   $\left[\overline{B}_1^2\right]_{\sub{\infty}} \mathfrak{R}^{\widetilde{n}_2 \times m}$ *A A*  $\in \mathfrak{R}^n$ <sup>*A*</sup>  $\begin{bmatrix} \overline{B}_1^1 \end{bmatrix}$   $\alpha \tilde{n} \times m$   $\tilde{\alpha}$   $\begin{bmatrix} \overline{B}_1^2 \end{bmatrix}$   $\alpha \tilde{n} \times m$  $\dddot{\ }$  $\overline{n}$  $\widetilde{B}_2 = \left| \frac{\overline{B}_1^2}{\overline{B}_2^2} \right| \in \Re^{\widetilde{n}_{2} \times \widetilde{n}_{2}}$  $\overline{a}$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\left[\overline{B}_1^1\right]$   $\infty \tilde{n} \times m$   $\approx$   $\left[\overline{B}_1^2\right]$   $\infty \tilde{n} \times m$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$   $\begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$  $\left[ \begin{array}{c} \lambda_2 \end{array} \right]$  $\frac{1}{2}$   $\approx$   $\frac{2}{2}$   $\approx$  $\overline{\phantom{a}}$  $\mathfrak{R}^{\widetilde{n}_1}$ ,  $\widetilde{x}_2(k) = \begin{bmatrix} \overline{x}_1^2(k) \end{bmatrix}$  $\begin{bmatrix} 1 \end{bmatrix}$   $\begin{bmatrix} 1 \end{bmatrix}$   $\begin{bmatrix} 1 \end{bmatrix}$   $\begin{bmatrix} 1 \end{bmatrix}$  $n = n$ *x*  $\binom{k}{k}$ *B*  $\int \frac{1}{2}$ *B*  $\begin{bmatrix} 1(t) \end{bmatrix}$   $\begin{bmatrix} -2(t) \end{bmatrix}$ *n n*  $, \, \, \cdot$  $\lfloor x_2(k) \rfloor$ <br>+  $n_2^2$ ,  $n = \tilde{n}_1 + \tilde{n}_2$ .  $i_2 = n_1^2$ .  $\mathfrak{i}_2$  $\begin{bmatrix} x_2(\kappa) \end{bmatrix}$   $\begin{bmatrix} x_2(\kappa) \end{bmatrix}$ <br>  $\tilde{n}_1 = n_1^1 + n_2^1$ ,  $\tilde{n}_2 = n_1^2 + n_2^2$ ,  $n = \tilde{n}_1 + \tilde{n}_2$ .  $\overline{a}$  $\left| K \right| \in$  $\overline{\phantom{a}}$  $\widetilde{\mathcal{X}}_{1}^{\widetilde{n}_{1}}, \widetilde{x}_{2}(k) = \begin{bmatrix} \overline{x}_{1}^{2} \end{bmatrix}$  $\in \Re$  $\mathbb{R}$  $\overline{\phantom{a}}$  $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$  $\widetilde{x}_1(k) = \left| \frac{\overline{x}_1^1(k)}{\overline{x}_2^1(k)} \right| \in \Re^{\widetilde{n}_1}, \ \ \widetilde{x}_2(k) = \left| \frac{\overline{x}_1^2(k)}{\overline{x}_2^2(k)} \right| \in \Re^{\widetilde{n}_2},$  $\overline{B}_1 = \left| \frac{B_1}{B_2} \right| \in \mathfrak{R}^{n_1 \times m}, \ \overline{B}_2 = \left| \frac{B_1}{B_2} \right| \in \mathfrak{R}^{n_2}.$  $\frac{1}{2}$  $\bar{x}_2^2(k) = \begin{vmatrix} \bar{x}_1^2 \\ -2 \end{vmatrix}$  $\tilde{n}$ 1 2  $\mathcal{F}_1(k) = \begin{vmatrix} \overline{x}_1^1(k) \\ -1 \end{vmatrix} \in \mathfrak{R}^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \begin{vmatrix} \overline{x}_1^2(k) \\ -2 \end{vmatrix} \in \mathfrak{R}^{\widetilde{n}_2}$ 2  $\widetilde{B}_2 =$ ~ 1 2  $\widetilde{B}_1 = \left| \frac{B_1}{B_2^1} \right| \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \ \ \widetilde{B}_2 = \left| \frac{B_1}{B_2^2} \right| \in \mathfrak{R}$  $\overline{x}_2^2(k)$  $\widetilde{x}_1(k) = \left| \frac{\overline{x}_1^1(k)}{\overline{x}_2^1(k)} \right| \in \Re^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \left| \frac{\overline{x}_1^2(k)}{\overline{x}_2^2(k)} \right| \in \Re^{\widetilde{n}_1}$ <sup>*l*</sup>,  $\dot{B_2} =$  $\overline{B}_1 = \left| \frac{B_1}{B_2} \right| \in \mathfrak{R}^{n_1 \times m}, \ \overline{B}_2 = \left| \frac{B_1}{B_2} \right| \in \mathfrak{R}^n$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\frac{1}{2}$  $\overline{\phantom{a}}$ L  $\mathsf{L}$  $\left[\in \Re^{\tilde{n}_1}, \ \tilde{x}_2(k)=\right]$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$ L I L  $=$  $\overline{a}$  $\overline{b}$ b  $\left(\frac{B_1}{B_2}\right) \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \ \ \widetilde{B}_2$  $\bar{t}$  $\overline{r}$  $\widetilde{B}_1=\left|\frac{B_1}{B_2^1}\right|\in\mathfrak{R}^{\widetilde{n}_1\times m},\;\;\widetilde{B}_2=\left|\frac{B_1}{B_2^2}\right|\in\mathfrak{R}$  $\begin{bmatrix} \overline{p}1 \\ \overline{p}2 \end{bmatrix}$  $(k) = \left[\frac{x_1(\kappa)}{\bar{x}_2^1(k)}\right] \in \mathfrak{R}^{n_1}, \quad \tilde{x}_2(k) = \left[\frac{x_1(\kappa)}{\bar{x}_2^2(k)}\right] \in \mathfrak{R}^{n_2},$  $B_1 = \left[\frac{1}{B_2}\right] \in \mathfrak{R}^{n_1 \cdots n_r}, \quad B_2 = \left[\frac{1}{B_2}\right] \in \mathfrak{R}^{n_2 \cdots n_r},$ 2  $\mathcal{L}(x) = \frac{x_1(x)}{x_2(x)}$ : J  $\hat{y} = \begin{cases} x_1(k) \\ \frac{1}{x_2(k)} \end{cases}$ 2 2022 - 2022 - 2022 - 2022 - 2022  $B_1 = \left[\frac{1}{B_2}\right] \in \mathfrak{R}^{m_1 \cdot \cdot \cdot \cdot}$ ,  $B_2 = \left[\frac{1}{B_2^2}\right] \in \mathfrak{R}^{m_2 \cdot \cdot \cdot \cdot \cdot}$ i,  $x_2(x) = \frac{1}{2}a_1 + \frac{1}{2}a_2$ *x k*  $\begin{bmatrix} k \end{bmatrix} = \begin{bmatrix} \frac{x_1}{x_2^2(k)} \end{bmatrix}$ *x k*  $\chi$  =  $\begin{pmatrix} x_1(\kappa) \\ \overline{x}_2(\kappa) \end{pmatrix} \in \Re^{n_1}, \quad \tilde{x}_2(k) = \begin{pmatrix} x_1(\kappa) \\ \overline{x}_2(\kappa) \end{pmatrix} \in \Re^{n_2},$ Ĵ  $\ddot{\phantom{0}}$ ŗ,  $\frac{2}{1}$ ŗ.  $\mathfrak{R}^{\tilde{n}_1}$ ,  $\tilde{x}_2(k) = \begin{bmatrix} x_1^{-1} \\ \overline{x}_2^{-1} \end{bmatrix}$ Ĵ  $\ddot{\phantom{0}}$ Ľ  $\frac{x_1}{x_2}$ ŋ  $\mathbf{r} = \mathbf{r} \cdot \mathbf{r}$  $\overline{\phantom{a}}$  $\overline{a}$  $\left[\in \mathfrak{R}^{\mathbb{Z}_{p^2}}$ ,  $B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $=\left[\frac{1}{B_2}\right] \in \mathfrak{R}^{\mathfrak{m}\cdot\mathfrak{m}}, \ \ B_2=\left[\frac{1}{B_2}\right] \in \mathfrak{R}^{\mathfrak{m}\cdot\mathfrak{m}}.$ . ~ ~ , ~ , ~ 1 2 2 2 1 2 1 1 *n n n n n n n n n* = + = + = +  $\mathbf{k}$ ) =  $\begin{bmatrix} \overline{x}_1^1(k) \\ \overline{x}_1^1(k) \end{bmatrix} \in \Re^{\widetilde{n}_1}, \quad \widetilde{x}_2(k) = \begin{bmatrix} \overline{x}_1^2(k) \\ \overline{x}_2^2(k) \end{bmatrix} \in \Re^{\widetilde{n}_2},$  $\widetilde{B}_1=\left|\frac{B_1^+}{\_1}\right|\in\mathfrak{R}^{\widetilde{n}_1\times m},~~\widetilde{B}_2=\left|\frac{B_1^+}{\_2}\right|\in\mathfrak{R}^{\widetilde{n}_2\times m},$ .<br>lz  $\sum_{k=1}^{\infty} \overline{x}_1^2(k)$ .<br>2  $\bar{x}_1 = \left[\bar{x}_1^1(k)\right] \in \mathfrak{R}^{\tilde{n}_1}$   $\tilde{x}_2(k) = \left[\bar{x}_1^2(k)\right] \in \mathfrak{R}^{\tilde{n}_2}$ 2 2  $\frac{1}{2} = \frac{\overline{B_1}^2}{\overline{B_2}^2}$  $\tilde{n}$ 1 2  $\tilde{B}_1 = \left| \frac{\overline{B}_1^1}{\overline{B}_1^1} \right| \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \ \ \widetilde{B}_2 = \left| \frac{\overline{B}_1^2}{\overline{B}_2^2} \right| \in \mathfrak{R}^{\widetilde{n}_2}.$  $\frac{1}{\sqrt{k}}$  $\left[\overline{x}_1^2(k)\right]$  $\frac{1}{\sqrt{k}}$  $\left[\overline{x}_1^1(k)\right] \in \mathfrak{R}^{\tilde{n}_1}$   $\approx (k) - \left[\overline{x}_1^2(k)\right] \in \mathfrak{R}^{\tilde{n}_2}$ *B*  $\widetilde{B}_2 = \begin{vmatrix} B \\ B \end{vmatrix}$ *B*  $\widetilde{B}_1 = \left[ \frac{B_1^1}{B_2^1} \right] \in \mathfrak{R}^{\widetilde{n}_1 \times m}, \quad \widetilde{B}_2 = \left[ \frac{B_1^2}{B_2^2} \right] \in \mathfrak{R}^{\widetilde{n}_2 \times m},$ J.  $\overline{c}$  $\frac{2}{\pi}$  $\Re^{\tilde{n}_1}, \ \tilde{x}_2(k) = \begin{bmatrix} \overline{x}_1^2 \\ \overline{x}_2^2 \end{bmatrix}$ J  $\mathbf{i}$  $\overline{\phantom{a}}$  $\mathbf{I}$  $\overline{x}_1^1$  $\overline{\phantom{a}}$ J  $\frac{1}{2}$ J  $\mathsf{I}$  $\mathsf{I}$  $\left[\in \Re^{\widetilde{n}_1 \times m}, \widetilde{B}_2 = \right]$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\overline{\phantom{a}}$  $\mathbb{E}=\left|\begin{array}{c} \overline{B}_1^1 \ \overline{-1}\end{array}\right|\in \mathfrak{R}^{\widetilde{n}_1\!\times\!m},\;\; \widetilde{B}_2=\left|\begin{array}{c} \overline{B}_1^{\,2} \ \overline{-2}\end{array}\right|\in \mathfrak{R}^{\widetilde{n}_2\!\times\!m}.$ (16) (16)

trices (monomial matrices) and  $Q^+ \in \mathbb{R}_+^{n \times n}$ , matrices  $E_1, E_2$  contains respectively<br>pendent columns and the rest are zero col<br>rices  $Q_1 \in \mathfrak{R}_+^{n_1 \times n_1}, Q_2 \in \mathfrak{R}_+^{n_2 \times n_2}$  are a pern The fact, that matrices  $E_1$ ,  $E_2$  contains respectively only  $n_1$ ,  $n_2$ <br>linearly independent columns and the rest are zero columns, imply  $\lbrack \mathbf{r}(k) \rbrack$ t, that matrices  $E_1$ ,  $E_2$  contains respectively only  $n_1^1$ ,  $n_2^2$ trices  $Q_1 \in \mathbb{R}_+^{n_1 \times n_1}$ The fact, that matrices  $E_1$ ,  $E_2$  contains respectively only  $n_1^1$ ,  $n_2^2$ <br>linearly independent columns and the rest are zero columns, imply<br>that the matrices  $Q_1 \in \mathbb{R}_+^{n_1 \times n_1}$ ,  $Q_2 \in \mathbb{R}_+^{n_2 \times n_2}$  are a (12)  $k \in Z_+$  and  $\tilde{x}_1(k) \in \mathbb{R}_+^{\tilde{n}_1}$ ,  $\tilde{x}_2(k) \in \mathbb{R}_+^{\tilde{n}_2}$  for  $k \in Z_+$ , since  $\bar{x}_1(k) \in \mathbb{R}_+^{\tilde{n}_2}$  $\bar{x}_2(k) \in \mathbb{R}_+^{n_2}$  for  $k \in Z_+$ . a<br>e<br>9 <sup>1</sup>  $\left[\mathbf{x}\right]$ Inat the matrices  $Q_1 \in \mathcal{R}_+$ <br>trices (monomial matrices) 2 *PBu k* and  $\tilde{x}_1(k)$ *kx j*  $l \in \mathbb{Z}$  and  $\tilde{z}$  (*k*)  $\in \mathfrak{S}^{n}$ *x k x k* 000 nomial *kx*  $\tilde{\mathbf{x}}_1(k) \in \mathfrak{R}_+^{\tilde{n}_1}, \tilde{\mathbf{x}}_2(k) \in \mathfrak{R}_+^{\tilde{n}_2} \text{ for } k \in \mathbb{Z}_+, \text{ since } \bar{\mathbf{x}}_1(k) \in \mathfrak{R}_+^{n_1},$  $\frac{1}{2}$  +  $\frac{1$ that the matrices  $Q_1 \in \mathcal{R}_+$ ,  $Q_2 \in \mathcal{R}_+$  are a permutation matrices (monomial matrices) and  $Q^{-1} \in \mathcal{R}_+^{n \times n}$ , since Q is block di- $\begin{bmatrix} x_1(k) \\ y_2(k) \end{bmatrix}$  $\mathcal{R}^n_+, k \in Z$  $\sum_{n=1}^{\infty}$  ,  $\frac{1}{n^2}$  $\frac{1}{x}$   $\frac{1}{x}$  is block digonal (8). Therefore, if  $\begin{vmatrix} x_1(k) \\ x_2(k) \end{vmatrix} \in \mathfrak{R}_+^n$ ,  $k \in Z_+$  then  $Q^{-1} \begin{vmatrix} x_1(k) \\ x_2(k) \end{vmatrix} \in \mathfrak{R}_+^n$ , ices  $E_1$ ,  $E_2$  contains respect that matrices  $F_1$ ,  $F_2$  contains respectively only  $\mu$ *x k x k x k <sup>x</sup> <sup>k</sup> kx* ℜ∈ y<br>.  $\overline{r}$  $\epsilon$  $\overline{a}$  $E_1, E_2$  contains respe r<br>E  $\mathfrak{p}$ matrices  $E_1$ ,  $E_2$  contains respectively only  $n_1^1$ ,  $n_2^2$  $[A2(h)]$   $[A2(h)]$   $[A2(h)]$ *nn Q*  $\leq$  ≠  $\leq$  +  $\leq$ atrices  $E_1$ ,  $E_2$  contains resp ct that matrices  $F_1$ .  $F_2$  contains respectively only *x k*  $arctan$ *x k <sup>x</sup> <sup>k</sup> kx* ℜ∈  $\frac{76}{2}$  $\frac{1}{2}$  $\mathbf{e}$  $\overline{a}$  $E_1$ ,  $E_2$  contains re  $\overline{a}$  $\mathbf{I}$ The fact, that matrices  $E_1$ ,  $E_2$  contains respectively only  $n_1^1$ ,  $n_2^2$ agonal (8). Therefore, if  $\begin{bmatrix} x_1(k) \\ y_1 \end{bmatrix} \in \mathbb{R}^n_+$ ,  $k \in Z_+$  then  $Q^{-1} \begin{bmatrix} x_1(k) \\ y_1 \end{bmatrix} \in \mathbb{R}^n_+$  $[A_2(h)]$   $[A_2(h)]$   $[A_2(h)]$  $\mathcal{L}_1(v) \subseteq V_+$ , agonal (8). Therefore, if  $\begin{bmatrix} x_1(k) \\ x(k) \end{bmatrix} \in \mathfrak{R}_+^n$ ,  $k \in Z_+$  then  $Q^{-1} \begin{bmatrix} x_1(k) \\ x(k) \end{bmatrix} \in \mathfrak{R}_+^n$ ,  $A \tilde{x}_1(k) \in \mathbb{R}^{\tilde{n}_1} \tilde{x}_2(k) \in \mathbb{R}^{\tilde{n}_2}$  for  $k \in \mathbb{Z}$  since  $\tilde{x}_1(k) \in \mathbb{R}^{n_1}$ <sup>−</sup> ℜ∈<sup>1</sup> , since *Q* is block diagonal (8).  $\lfloor x_2(\kappa) \rfloor$   $\lfloor x_2(\kappa) \rfloor$ *kx* )(  $\frac{1}{2}$  $\ddot{\phantom{1}}$ agonal (8). Therefore, if  $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in \mathbb{R}^n_+$ ,  $k \in \mathbb{Z}_+$  then  $Q^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in \mathbb{R}^n_+$ ,  $k \in Z_+$  and  $\tilde{x}_1(k) \in \mathfrak{R}_+^{\tilde{n}_1}, \tilde{x}_2(k) \in \mathfrak{R}_+^{\tilde{n}_2}$  for  $k \in Z_+$ , since  $\bar{x}_1(k) \in \mathfrak{R}_+^{n_1},$ <br> $\bar{x}_2(k) \in \mathfrak{R}_+^{n_2}$  for  $k \in Z_+$ . The fact, that matrices *E*<sup>1</sup> , *E*2 contains respectively only The fact, that matrices  $E_1$ ,  $E_2$  contains respectively only  $n_1^1$ ,  $n_2^2$ agonal (6). Therefore,  $\Pi\left[x_2(k)\right] \in \mathcal{P}_+, \kappa \in \mathbb{Z}_+$  then  $\mathcal{Q}\left[\chi_2(k)\right] \in \mathcal{P}_+,$  $\overline{x}_2(k) \in \mathfrak{R}_+^n$  for  $k \in \mathbb{Z}_+$ . agonal (8). Therefore, if  $\begin{bmatrix} x_1(k) \\ y_2(k) \end{bmatrix}$ *x*2(*k*) ı  $\in \mathfrak{R}^n_+$ , *k* ∈ *Z*<sub>+</sub> then  $Q^{-1}$  $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$ *x*2(*k*) ı,  $\in \Re_{+}^n,$ 

ecombining the state **Theorem 2.** The descriptor fractional system (5) for  $0 < a$ , Figure 1. The description fractional system (3) for  $0 <$ <br>ynam-<br>matrix and<br>matrix and  $P \leq P$  is positive if  $n_2$   $\leq m_{n_2}$  is asymptotomatrix and <sup>1</sup> , <sup>∈</sup> *<sup>Z</sup>*<sup>+</sup> *<sup>k</sup>* then 21 , <sup>∈</sup> *<sup>Z</sup>*<sup>+</sup> *<sup>k</sup>* then  $\overline{y}$  $\frac{1}{2}$ tional s **Theorem 2.** The descriptor fractional system (5) for  $0 < \alpha$ ,  $\beta \leq 1$  is a conventionable to the Matcheson  $\alpha$  +  $\alpha$ <sup>1</sup> <sup>1</sup> , <sup>∈</sup> *<sup>Z</sup>*<sup>+</sup> *<sup>k</sup>* and 1 <sup>1</sup> )( <sup>~</sup> *<sup>n</sup> kx* ℜ∈ <sup>+</sup> , 2  $\ddot{\phantom{0}}$ **Theorem 2.** The descriptor fractional system (5) for  $0 < \alpha$ ,  $\beta < 1$  is positive iff  $\tilde{A}_{22} \in M_{n_2}$  is asymptotically stable Metzler **Theorem 2.** The descriptor fractional system (5) for  $0 < a$ , *<sup>n</sup> kx* ℜ∈ <sup>+</sup> , 2 )( <sup>2</sup>

$$
\tilde{A}_{11} \in \mathfrak{R}_{+}^{\tilde{n}_{1}\times\tilde{n}_{1}}, \ \tilde{A}_{12} \in \mathfrak{R}_{+}^{\tilde{n}_{1}\times\tilde{n}_{2}}, \ \tilde{A}_{21} \in \mathfrak{R}_{+}^{\tilde{n}_{2}\times\tilde{n}_{1}},
$$
\n
$$
\tilde{B}_{1} \in \mathfrak{R}_{+}^{\tilde{n}_{1}\times m}, \ \tilde{B}_{2} \in \mathfrak{R}_{+}^{\tilde{n}_{2}\times m}.
$$
\n(17)

**Proof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_{n_2}$  is asymptoted to the initial  $\tilde{A}^{-1} \in \mathbb{S}^{n_2 \times n_2}$ . Let the initial  $\tilde{A}^{-1} \in \mathbb{S}^{n_2 \times n_2}$ . stable include matrix then  $-2/2 \in \mathcal{N}_+$ . In this case if (15b) and definition of positive **Proof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_{n_2}$  is asym stable interest matrix then  $-2/2 \leq v_+$ . In this case static intervalse **roof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_{n_2}$  is asymptotically able Metzler matrix then  $-\tilde{A}_{22}^{-1} \in \mathbb{R}_{+}^{n_2 \times n_2}$ . In this case from<br> **Sh**) and definition of positive system we have **Proof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_n$ , is asymptotically stable Metzler matrix then  $-\tilde{A}_{22}^{\text{1}} \in \mathbb{R}_{+}^{n_2 \times n_2}$ . In this case from (15b) and definition of positive system we have t is well-known [4] that if  $\tilde{A}_2$  $\lim_{x \to 2} \cos x + \sin x$ <br>ion of positive system we have  $m_{\text{cm}}$   $m_{\text{cm}}$   $m_{\text{cm}}$   $m_{\text{cm}}$   $m_{\text{cm}}$   $m_{\text{cm}}$   $m_{\text{cm}}$ is well-known [4] that if  $\tilde{A}_{22} \in M_{n_2}$  is asymptotically<br>letzler matrix then  $-\tilde{A}_{22}^{-1} \in \mathbb{R}^{n_2 \times n_2}_{+}$ . In this case from<br>d definition of positive system we have well-known [4] that if  $A_{22} \in M_{n_2}$  is asymptotically the matrix then  $-\tilde{A}_{22}^{-1} \in \mathfrak{R}_{+}^{n_2 \times n_2}$ . In this case from  $\frac{1}{22} \leq x_+$   $\leq$   $m_+$  $~ v_{+}$  . In this case from  $~$  $\text{well-known [4] that if } \tilde{A}_{22} \in M_n$ , is asymptotice  $\lim_{n \to \infty} \frac{1}{4} \sum_{i=1}^{n} \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2$ (15b) and definition of positive system we have **Proof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_{n_2}$  is asymptotically stable Metzler matrix then  $-\tilde{A}_{22}^{-1} \in \mathbb{R}^{n_2 \times n_2}$ . In this case from asymptotically stable Metzler matrix then <sup>1</sup> <sup>22</sup> 22 <sup>~</sup> *nn <sup>A</sup>* <sup>×</sup> + <sup>−</sup> − ℜ∈ . **IT TOOL.** It IS WELL-KNOWN [4] that IT  $A_2^0 \n\in M_{n_2}$  is asymptotically **nof**. It is well-known [4] that if  $\tilde{A}_{22} \in M$ , is asymptotically uble Metz **roof**. It is well-known [4] that if  $\tilde{A}_{22} \in M$  is asymptotically able Metz **Proof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_{n_2}$  is asymptotically stable Metzler matrix then  $-\tilde{A}_{22}^{-1} \in \mathbb{R}^{n_2 \times n_2}_{+}$ . In this case from (15b) and definition of positive system we have Statute Metzlei matrix then  $-A_2^0 \in \mathcal{V}_+$ . In this case from  $(150)$ 

(13)  
\n
$$
\tilde{x}_2(k) = -\tilde{A}_{22}^{-1}(\tilde{A}_{21}\tilde{x}_1(k) + \tilde{B}_{2}u(k)) \in \mathfrak{R}_+^{\tilde{n}_2}, k \in Z_+
$$
\n(18)  
\n
$$
\text{iff } \tilde{A}_{21} \in \mathfrak{R}_+^{\tilde{n}_2 \times \tilde{n}_1} \text{ and } \tilde{B}_2 \in \mathfrak{R}_+^{\tilde{n}_2 \times m}.
$$
\n(14)  
\nNext, the substitution of (18) into (15a) yields

iff  $A_{21} \in \mathbb{R}^{n_2 \times n_1}$  and  $B_2 \in \mathbb{R}^{n_2 \times m}$ .<br>Next, the substitution of (18) into (15a) yields <sup>~</sup> , *<sup>n</sup> <sup>m</sup> <sup>B</sup>* <sup>×</sup> ℜ∈ <sup>+</sup>  $\ddot{ }$  . (17)  $\frac{1}{2}$  **a**  $\frac{1}{2}$  **b**  $\frac{1}{2}$   $\sim$   $\sim$   $\sim$   $\sim$   $\sim$ 

$$
\widetilde{x}_1(k+1) = \widehat{A}_{11}\widetilde{x}_1(k) + \sum_{j=2}^{k+1} c_j \widetilde{x}_1(k-j+1) + \widehat{B}_1 u(k),
$$
\n
$$
k \in Z_+
$$
\n(19)

where where

here  

$$
\hat{A}_{11} = \tilde{A}_{11} - \tilde{A}_{12} \tilde{A}_{22}^{-1} \tilde{A}_{21}, \ \hat{B}_1 = \tilde{B}_1 - \tilde{A}_{12} \tilde{A}_{22}^{-1} \tilde{B}_2.
$$
 (20)

It is well-known [4] that  $C_j > 0$  for  $j = 2, 3, ...$  and  $0 < \alpha, \beta < 1$ . In this case, from (19) and definition of positive system we have  $\tilde{x}_1(k) \in \mathfrak{R}_+^{n_1}, k \in \mathbb{Z}_+$  for  $\tilde{x}_1(0) \in \mathfrak{R}_+^{n_1}, u(k) \in \mathfrak{R}_+^m, k \in \mathbb{Z}_+$  iff  $\hat{A}_1 \in \mathfrak{R}^{\tilde{n}_1 \times \tilde{n}_1}$  and  $\hat{B}_1 \in \mathfrak{R}^{\tilde{n}_1 \times m}$ . Further, from (20) we have that the In this case, from (19) and definition of positive system we<br>have  $\tilde{x}_1(k) \in \mathbb{R}^{n_1}$ ,  $k \in Z_+$  for  $\tilde{x}_1(0) \in \mathbb{R}^{n_1}$ ,  $u(k) \in \mathbb{R}^m$ ,  $k \in Z_+$  iff  $\hat{A}_{11}$  and  $\hat{B}_1$  are positive if  $-\tilde{A}_{22}^{-1} \in \mathfrak{R}_{+}^{\tilde{n}_2 \times \tilde{n}_2}, \tilde{A}_{11} \in \mathfrak{R}_{+}^{\tilde{n}_1 \times \tilde{n}_1},$ <br>  $\tilde{\lambda}_{12} = \Omega \tilde{n}_1 \times \tilde{n}_2 \quad \tilde{\Omega}_{13} \times \tilde{n}_1 \quad \tilde{\Omega}_{24} \in \mathfrak{R}_{+}^{\tilde{n}_1 \times \tilde{n}_2}, \tilde{\Omega}_{34} \in \mathfrak{R}_{$  $\theta \in \mathbb{R}^{\bar{n}_1 \times \bar{n}_1}$  and  $\hat{B}_1 \in \mathbb{R}^{\bar{n}_1 \times \bar{m}}$ . Further, from (20) we have that the trices  $\hat{A}_u$  and  $\hat{B}_v$  are positive if  $-\tilde{A}_v^{-1} \in \mathbb{R}^{\bar{n}_2 \times \bar{n}_2}$   $\tilde{A}_u \in \mathbb{R}^{\bar{n}_1 \times \bar{n}_1}$  $\in \mathfrak{R}_+^{\tilde{n}_1 \times \tilde{n}_2}, \tilde{A}_2$  $\tilde{P}\in\mathfrak{R}_{+}^{\tilde{n}_{1}\times\tilde{n}_{2}},$   $\tilde{A}_{21}\in\mathfrak{R}_{+}^{\tilde{n}_{2}\times\tilde{\tilde{n}}_{1}},$   $\tilde{B}_{1}\in\mathfrak{R}_{+}^{\tilde{n}_{1}\times m},$   $\tilde{B}_{2}\in\mathfrak{R}_{+}^{\tilde{n}_{2}\times m}.$  $\hat{H}_1 \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}$  and  $\hat{B}_1 \in \mathbb{R}^{\tilde{n}_1 \times m}$ . Further, from (20) we have that the strices  $\hat{A}_\nu$  and  $\hat{B}_\nu$  are positive if  $-\tilde{A}_\nu^{-1} \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_2}$   $\tilde{A}_\nu \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}$  $\in \mathfrak{R}_{+}^{\tilde{n}_1 \times \tilde{n}_2}, \tilde{A}_1$  $\in \mathfrak{R}^{\tilde{n}_1\times \tilde{n}_2}_{+}, \tilde{A}_{21}\in \mathfrak{R}^{\tilde{n}_2\times \tilde{n}_1}_{+}, \tilde{B}_1\in \mathfrak{R}^{\tilde{n}_1\times m}_{+}, \tilde{B}_2\in \mathfrak{R}^{\tilde{n}_2\times m}_{+}.$  $\tilde{B}_1 \in \mathfrak{R}_{+}^{n_{[N} \times m}$ ,  $\tilde{B}_2 \in \mathfrak{R}_{+}^{n_{[N} \times m}$ .<br> **oof.** It is well-known [4] that if  $\tilde{A}_{22} \in M_{n_0}$  is asymptotically<br>
ble Metzler matrix then  $-\tilde{A}_{22}^{-1} \in \mathfrak{R}_{+}^{m_{[N} \times n}$ . In this case from<br>
5 It is well-known [4] that  $C_j > 0$  for  $j = 2, 3, ...$  and  $0 < \alpha, \beta < 1$ .<br>In this case, from (19) and definition of positive system we have  $\tilde{x}_1(k) \in \mathbb{R}^{n_1}, k \in \mathbb{Z}$  for  $\tilde{x}_1(0) \in \mathbb{R}^{n_1}, u(k) \in \mathbb{R}^{m}, k \in \mathbb{Z}$  if  $\in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}$  and  $\hat{B}_1 \in \mathbb{R}^{\tilde{n}_1 \times \tilde{m}_1}$ . Further, from (20) we have that the trices  $\hat{A}_u$  and  $\hat{B}_v$  are positive if  $-\tilde{A}_v^{-1} \in \mathbb{R}^{\tilde{n}_2 \times \tilde{n}_2}$   $\tilde{A}_u \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}$  $\tilde{B}_1 \in \mathfrak{R}_+^{n_1 \times m}$ ,  $\tilde{B}_2 \in \mathfrak{R}_+^{n_2 \times m}$ .<br> **(a)**<br> **Moof.** It is well-known [4] that if  $\tilde{A}_{22} \subseteq M_{n_1}$  is asymptotically<br>
ble Metzler matrix then  $-\tilde{A}_{22}^{-1}(\tilde{A}_{21}\tilde{X}_1(k) + \tilde{B}_{2}u(k)) \in \mathfrak{R}_+^{\$  $A_{11} \in \mathbb{R}_+^{n_1 \times n_1}$  and  $B_1 \in \mathbb{R}_+^{n_1 \times m}$ . Further, from (20) we have that the matrices  $\hat{A}_{11}$  and  $\hat{B}_1$  are positive if  $-\tilde{A}_{22}^{-1} \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_2}$ ,  $\tilde{A}_{11} \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}$ .  $\tilde{A}_{12} \in \mathfrak{R}_{+}^{\tilde{n}_{1} \times \tilde{n}_{2}}, \tilde{A}_{21} \in \mathfrak{R}_{+}^{\tilde{n}_{2} \times \tilde{n}_{1}}, \tilde{B}_{1} \in \mathfrak{R}_{+}^{\tilde{n}_{1} \times m}, \tilde{B}_{2} \in \mathfrak{R}_{+}^{\tilde{n}_{2} \times m},$  $\hat{A}_{11} \in \mathfrak{R}_{+}^{\hat{n}_1 \times \hat{n}_1}$  and  $\hat{B}_1 \in \mathfrak{R}_{+}^{\hat{n}_1 \times \hat{m}_1}$ . Further, from (20) we have that the matrices  $\hat{A}_{11}$  and  $\hat{B}_1$  are positive if  $-\tilde{A}_{22}^{-1} \in \mathfrak{R}_{+}^{\tilde{n}_1 \times \tilde{n}_2}$ ,  $\tilde{A}_{11} \in \mathfrak{R}_{+}^{\tilde{n}_1 \times \tilde{n}_1}$ ,  $\widetilde A_{12}\in \mathfrak R_+^{\tilde n_1\times \tilde n_2},\,\widetilde A_{21}\in \mathfrak R_+^{\tilde n_2\times \tilde n_1},\,\widetilde B_1\in \mathfrak R_+^{\tilde n_1\times m},\,\widetilde B_2\in\mathfrak R_+^{\tilde n_2\times m}.$ 

 $A_{12} \in \mathfrak{R}^{m_1}_{+}$ <br>From  $A_{12} \in \mathcal{N}_+$ ,  $A_{21} \in \mathcal{N}_+$ ,  $D_1 \in \mathcal{N}_+$ ,  $D_2 \in \mathcal{N}_+$ .<br>
From (18) it follows that  $\tilde{x}_2(k) \in \mathfrak{R}_+^{\tilde{n}_2}$ ,  $k \in Z_+$  iff  $\tilde{x}_1(k) \in \mathfrak{R}_+^{\tilde{n}_1}$ <br>
and  $\tilde{B}_2 u(k) \in \mathfrak{R}_+^m$  for  $k \in Z_+$ . and  $\tilde{B}_2 u(k) \in \mathfrak{R}_+^m$  for  $k \in Z_+$ .  $\square$ <sup>~</sup> *nn <sup>A</sup>* <sup>×</sup>  $\Box$ <sup>~</sup> *nn <sup>A</sup>* <sup>×</sup> ℜ∈ <sup>+</sup> ,  $\overline{\phantom{a}}$  *n* ∈  $\overline{\phantom{a}}$   $\overline{\phantom{a}}$   $\overline{\phantom{a}}$   $\overline{\phantom{a}}$ *n n* ∈  $Z_+$ . □  $\theta \in \mathbb{R}_+^{\bar{n}_1 \times \bar{n}_2}, \tilde{A}_{21} \in \mathbb{R}_+^{\bar{n}_2 \times \bar{n}_1}, \tilde{B}_1 \in \mathbb{R}_+^{\bar{n}_1 \times \bar{m}}, \tilde{B}_2 \in \mathbb{R}_+^{\bar{n}_2 \times \bar{m}}.$ <br>From (18) it follows that  $\tilde{x}_2(k) \in \mathbb{R}_+^{\bar{n}_2}, k \in \mathbb{Z}_+$  iff  $\tilde{x}_1(k) \in \tilde{B}_2^m$  for  $k \in \mathbb{Z$  $\leq x_{+}$  for  $k \in \mathbb{Z}_{+}$ . and  $\tilde{B}_2 u(k) \in \mathbb{R}^m_+$  for  $k \in Z_+$ . □ <sup>11</sup> *<sup>A</sup>*ˆ and 1 *<sup>B</sup>*ˆ are positive if 22 <sup>~</sup> <sup>~</sup> <sup>1</sup>

#### **4. Decentralized stabilization of positive descriptor fractional linear systems descriptor fractional linear systems**

For positive system (5) we are looking for a gain matrix  $\mathbf{F}_{\mathbf{p}}$  we are looking for a gain for  $F(0)$ 

$$
K = NG^{-1}, G = diag[g_1, ..., g_n],
$$
  
\n
$$
g_k > 0, k = 1, ..., n, N \in \mathfrak{R}^{m \times n}
$$
\n(21)

such that the close-loop system matrix  $\mathbf{r}$ such that the close-loop system matrix

$$
A_c = A + BK \in M_n \tag{22}
$$

*Ac* = *A*+ *BK* ∈*Mn* (22) *is* asymptotically stable. Matrices *G* and *N* can be computed by the use of Linear <sup>~</sup> *nn <sup>A</sup>* <sup>×</sup> Matrices *G* and *N* can be computed by the use of Linear  $\frac{1}{2}$  is Matrices *G* and *N* can be computed by the use of Linear

Is asymptotically stable.<br>We choose matrices  $N$  and  $G$  such that  $\forall x \in C_{11}$ <sup>−</sup> − ℜ∈ .  $\frac{1}{2}$  Matrix  $\frac{1}{2}$  and  $\frac{1}{2}$  $\frac{1}{2}$  is asymptotically stable.<br>
We choose matrices *N* and *G* such that Now, we have asymptotically stable Metzler matrix then 1 222 stable matrix then 1 222 <sup>~</sup> *nn <sup>A</sup>* <sup>×</sup> se matrices

$$
AG + BN \in M_n \text{ and } (AG + BN)G^{-1} < 0. \tag{23}
$$

Metzler matrix (see. [30]). Then, if  $(23)$  holds then the matrix  $(22)$  is asymptotically stable **Detzler matrix (see.** [30]).  $Metz$  $Metzler$  $\leq$ 

matrix inequalities method or linear programming, see e.g. [29].<br>where  $\hat{A}$ Matrices *G* and *N* can be computed by the use of linear Matrices  $\hat{G}$  and  $\hat{N}$  can be computed by the use of linear  $\mathbf{r}$ 

**Definition 2.** [3] The positive system (3) (or equivalently the pair  $(A, B)$  defined by  $(4)$ ) is called stabilizable by the state feedback if there exists a gain matrix  $(21)$  such that the closedloop system matrix (22) is asymptotically stable.  $\frac{1}{2}$  e system (3) (or equivalently the  $\frac{1}{2}$   $\frac{1}{2$ equality  $\alpha$  (3) (or equivalently the Takin  $\alpha$  ratio  $\alpha$  *x* abilizable by the state  $\alpha$  *MD*  $\tilde{K}_2$  $\epsilon$ (or equivalently the Taking unit  $\overline{K}$  and  $\overline{K$ 

**Definition 3.** [3] The matrix  $C \in \mathbb{R}^{n_2 \times n_1}$  satisfying the equality  $C \in \mathbb{R}^{n_2 \times n_1}$  satisfying the equality  $\approx (L+1)$ 

$$
\tilde{x}_2 = C\tilde{x}_1,\tag{24}
$$

is called contracting matrix if  $\frac{1}{2}$  if

$$
\|\tilde{x}_2\| < \|\tilde{x}_1\|,\tag{25}
$$

where the norm of  $\tilde{x}_2$ ,  $(\tilde{x}_1)$  is defined as  $\frac{1}{2}$  when

$$
\|\tilde{x}_2\| = \sum_{j=1}^{\tilde{n}_2} \left|\tilde{x}_2^j\right|, \quad \tilde{x}_2 = \left[\tilde{x}_2^1 \quad \tilde{x}_2^2 \quad \dots \quad \tilde{x}_2^{n_2}\right]^T. \tag{26}
$$
\nIt is well-known [13] that  $\sum_{j=2}^{\tilde{n}_2} c_j = 0$  and  $\sum_{j=2}^{\tilde{n}_2} c_j = \left[\begin{array}{cc} 1 - \alpha & \alpha \\ 0 & 1 \end{array}\right]$ 

\nIn this case we compute the matrix  $\tilde{K}_1$  so that the matrix  $\tilde{K}_2$  is a function of  $\tilde{K}_1$ .

 $(A, B)$  is not sufficient for the stabilization of close-loop system  $\hat{A}_{11} + \begin{bmatrix} A_{11} \\ B_{11} \end{bmatrix}$  $(A, B)$  is not sufficient for the stabilization of close-loop system with Metzler matrix. The pair  $(A, B)$  should be stabilizable. Consider the fractional system ( $\overline{15}$ ) with decentralized controller **Remark 1.** For positive systems the controllability of the pair  $(A, B)$  is not sufficient for the atchilization of class leap systems  $\mathbf{Re}$  $\frac{1}{2}$   $\frac{1}{2}$  <sup>~</sup> , *mn <sup>B</sup>*<sup>×</sup> ℜ∈ <sup>+</sup>

$$
u(k) = \begin{bmatrix} \tilde{K}_1 & 0 \\ 0 & \tilde{K}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix}, \ \tilde{K}_1 \in \mathfrak{R}^{1 \times \tilde{n}_1}, \ \tilde{K}_2 \in \mathfrak{R}^{1 \times \tilde{n}_2}, \quad (27) \quad \text{If the co-and the closed-loop system has the form} \tag{1.1}
$$

and the closed-loop system has the form<br> $\begin{bmatrix} x & 0 \end{bmatrix} \begin{bmatrix} x & 0 \end{bmatrix} = (x + 1)^1 = [\begin{bmatrix} x & \tilde{p} & \tilde{r} \end{bmatrix}$  $\frac{1}{\sqrt{2}}$  $\frac{1}{\sqrt{2}}$ **4. Decentralized stabilization of positive**   $\mathbf{r}$ 

$$
\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1 \tilde{K}_1 & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} + \tilde{B}_2 \tilde{K}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{matrix} \text{This imply } \tilde{x}_2(k) \in \mathfrak{R}^2 \\ + \begin{bmatrix} \sum_{j=2}^{k+1} c_j \tilde{x}_1(k-j+1) \\ 0 \end{bmatrix}, \, k \in Z_+ . \end{matrix}
$$
\nTherefore, it was pro

The close-loop system  $(28)$  is called (internally) positive if

$$
\tilde{x}_i(k) \in \mathfrak{R}_+^{\tilde{n}_i}, \ i = 1, 2; \ k \in Z_+
$$
\n
$$
\text{for } \tilde{x}_i(0) \in \mathfrak{R}_+^{\tilde{n}_i}, \ i = 1, 2. \tag{29}
$$

close-loop sys  $\overline{\phantom{0}}$  $\frac{1}{2}$  $\mu$  The positive close-loop system (28) is called asymptotically stable if stable if  $\frac{d}{dx}$ 

$$
\lim_{k \to \infty} \tilde{x}_i(k) = 0
$$
  
for all  $\tilde{x}_i(0) \in \mathfrak{R}_+^{n_i}$ ,  $i = 1, 2$ . (30)

Now, we have to find  $K_1$  and  $K_2$ , which stabilize<br>system and do not violate its positivity.<br>System (28) we have *i x*  $\frac{1}{2}$   $\frac{$  $(\mathbf{y}, \mathbf{y})$ Now, we have to find  $\tilde{K}_1$  and  $\tilde{K}_2$ , which stabilize the descriptor  $\mathbf{M} = \mathbf{1} \times \mathcal{C} \mathbf{1} \widetilde{\mathcal{U}} = \mathbf{1} \widetilde{\mathcal{U}} = \mathbf{1} \mathbf{1} \times \mathbf{1} \mathbf{1} \times \mathbf{1} \times \mathbf{1}$ 

 $F$ From  $(28)$  we have  $20 \text{W} \times \text{A}$  *n*  $20 \text{W} \times \text{B}$  *n*  $20 \text{W} \times \text{B}$  $26$  we have

$$
\tilde{x}_2(k) = -\hat{A}_{22}^{-1}\tilde{A}_{21}\tilde{x}_1(k) \in \mathfrak{R}_+^{\tilde{n}_2}
$$
\n
$$
\text{for } -\hat{A}_{22}^{-1}\tilde{A}_{21} \in \mathfrak{R}_+^{\tilde{n}_2 \times \tilde{n}_2},
$$
\n(31)

where  $\hat{A}_{22} = \tilde{A}_{22} + \tilde{B}_2 \tilde{K}_2$ .  $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 &$  $T_{\rm eff}$  under consideration (21)-(23), we compute (using (using  $\alpha$ ), we compute (using  $\alpha$ ),  $\mathbb{R}$ .

Taking under consideration (21–23), we compute (using e.g.<br>Taking under consideration (21–23), we compute (using e.g. Taking under consideration (21–23), we compute (using e.g.<br>LMI)  $\tilde{K}_2$  so that  $\hat{A}_{22} \in M_{\tilde{n}_2}$  is asymptotically stable and  $-\hat{A}_{22}^{-1}\tilde{A}_{21}$ <br>is contracting matrix  $\lim_{z \to 2} R_2$  so that  $A_{22} \in$ 

Substituting (31) into (28) we obtain Substituting (31) into (28) we obtain Substituting (31) into (28) we obtain = 2 *j*

$$
\widetilde{x}_1(k+1) = (\hat{A}_{11} + \widetilde{B}_1 \widetilde{K}_1) \widetilde{x}_1(k) + \sum_{j=2}^{k+1} c_j \widetilde{x}_1(k-j+1), \quad (32)
$$

where where where

$$
\hat{A}_{11} = \tilde{A}_{11} - \tilde{A}_{12}\hat{A}_{22}^{-1}\tilde{A}_{21} \in \mathfrak{R}_+^{\tilde{n}_1 \times \tilde{n}_1} \text{ for } \tilde{A}_{11} \in \mathfrak{R}_+^{\tilde{n}_1 \times \tilde{n}_1}. \quad (33)
$$

It is well-known [13] that  $\sum_{j=2}^{\tilde{n}_2} c_j = 0$  and  $\sum_{j=2}^{\tilde{n}_2} c_j = \begin{bmatrix} 1 - \alpha \\ 0 \end{bmatrix}$ It is well-known [13] that  $\sum_{j=2} c_j = 0$  and  $\sum_{j=2} c_j = \begin{bmatrix} 1 - \alpha & 0 \\ 0 & 1 - \beta \end{bmatrix}$ . It is well-known [13] that  $\sum_{j=2}^{\infty} c_j = 0$  and  $\sum_{j=2}^{\infty} c_j = \begin{bmatrix} 1 - \alpha & 0 \\ 0 & 1 - \beta \end{bmatrix}$ .  $\sum_{j=2}^{\infty} c_j = 0$  and  $\sum_{j=1}^{\infty}$  $\sum_{j=2}^{\infty} c_j = \begin{bmatrix} 1 - \alpha & 0 \\ 0 & 1 - \end{bmatrix}$ 0  $1-\beta$  $\overline{1}$ 3] that  $\sum_{i=1}^{\infty} c_i = 0$  and  $\sum_{i=1}^{\infty} c_i = \begin{bmatrix} 1 - \alpha & 0 \\ 0 & 1 - \alpha \end{bmatrix}$ .  $\frac{1}{2}$  $\frac{1}{2}$ It is well-known It is well-known [13] that  $\sum_{i=0}^{\infty} c_i = 0$  and  $\sum_{i=0}^{\infty} c_i = \begin{bmatrix} 1 - a & 0 \\ 0 & 1 - a \end{bmatrix}$ 

$$
\hat{A}_{11} + \begin{bmatrix} I_{\tilde{n}_1} (1 - \alpha) & 0 \\ 0 & I_{\tilde{n}_2} (1 - \beta) \end{bmatrix} + \tilde{B}_1 \tilde{K}_1 \in \mathfrak{R}_+^{\tilde{n}_1 \times \tilde{n}_1} \qquad (34)
$$

is asymptotically stable. is asymptotically stable.

The condition (34) is satisfied then from (32) it follows<br>that  $\tilde{x}_1(k) \in \mathfrak{R}_+^{\tilde{n}_1}$ ,  $k \in Z_+$  and that  $\tilde{x}_1(k) \in \mathbb{R}_+^{n_1}$ ,  $k \in Z_+$  and  $\tilde{x}_2(k) = \tilde{x}_1(k)$  $t_1(t) \geq 1$ 

$$
\lim_{k \to \infty} \tilde{x}_1(k) = 0,\tag{35}
$$

This imply  $\tilde{x}_2(k) \in \mathfrak{R}_+^{\tilde{n}_2}$ ,  $k \in Z_+$  and If the condition (34) is satisfied then from (34) is satisfied then from (32) it follows the follows  $\mathcal{L}(\mathcal{L})$ If the condition (34) is satisfied then from (34) is satisfied then from (32) it follows the matrix  $\mathcal{L}$ 

$$
\lim_{k \to \infty} \tilde{x}_2(k) = 0.
$$
\n(36)

~ ~ <sup>2</sup> )( <sup>~</sup> *<sup>n</sup> <sup>x</sup> <sup>k</sup>* ℜ∈ <sup>+</sup> , ∈*Z*<sup>+</sup> *<sup>k</sup>* and <sup>2</sup> )( <sup>~</sup> *<sup>n</sup> <sup>x</sup> <sup>k</sup>* ℜ∈ <sup>+</sup> , ∈*Z*<sup>+</sup> *<sup>k</sup>* and  $\overline{h} \rightarrow \infty$ <br>Therefore, it was proven that:

**Theorem 3.** The positive descriptor fractional discrete-time linear system with two different fractional orders (6) can be stabilized by the decentralized controller (27) iff the pairs  $(\tilde{A}_{22}, \tilde{B}_2)$ ,  $(\tilde{A}_{11}, \tilde{B}_1)$  are stabilizable and  $-\tilde{A}_{22}^{-1}\tilde{A}_{21}$  is the contracting matrix.

**Example 1.** Find the solution of the descriptor fractional linear system (6) with the fractional orders  $\alpha = 0.5$ ,  $\beta = 0.6$  and the matrices

$$
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$
  
\n
$$
B_1 = \begin{bmatrix} 0.05 \\ 0 \\ -0.05 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.05 \\ 0 \\ 0.05 \end{bmatrix},
$$
  
\n
$$
A_{11} = \begin{bmatrix} -0.45 & 0.1 & 0.05 \\ 0.06 & -0.5 & 0 \\ -0.05 & 0 & -0.05 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.05 & 0.1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix},
$$
  
\n
$$
A_{21} = \begin{bmatrix} 0.22 & 0.07 & 0.25 \\ 0.46 & 0.08 & 0.15 \\ 0.1 & 0.35 & 0.85 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0.13 & -0.55 & 0.14 \\ -0.54 & 0.02 & 0.07 \\ 0.11 & 0.23 & 0.11 \end{bmatrix}.
$$

It is easy to check that the matrices (37) satisfies the assumptions (7a, 7b). In this case the matrices *P* and *Q* have the form

$$
P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$
  
\n
$$
Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$
 (38)

and the decomposition (10) is given by

$$
\overline{E}_1 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \ \overline{E}_2 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \ \overline{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.05 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0 \\ 0.05 \\ 0.05 \end{bmatrix},
$$

$$
\overline{A}_{1\alpha} = \begin{bmatrix} 0 & 0.06 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0.05 & 0.05 \end{bmatrix}, \qquad \overline{A}_{12} = \begin{bmatrix} 0.05 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \qquad (39)
$$

$$
\overline{A}_{21} = \begin{bmatrix} 0.08 & 0.46 & 0.15 \\ 0.07 & 0.22 & 0.25 \\ 0.35 & 0.1 & 0.85 \end{bmatrix}, \quad \overline{A}_{2\beta} = \begin{bmatrix} 0.06 & 0.02 & 0.07 \\ 0.13 & 0.05 & 0.14 \\ 0.11 & 0.23 & 0.11 \end{bmatrix}.
$$

Using (16) we have

$$
\tilde{A}_{11} = \begin{bmatrix}\n0 & 0.06 & 0.05 & 0.1 \\
0.1 & 0 & 0.2 & 0.1 \\
0.08 & 0.46 & 0.06 & 0.02 \\
0.08 & 0.22 & 0.13 & 0.05\n\end{bmatrix}, \quad\n\tilde{A}_{12} = \begin{bmatrix}\n0 & 0.1 \\
0 & 0 \\
0.15 & 0.07 \\
0.25 & 0.14\n\end{bmatrix},
$$
\n
$$
\tilde{A}_{21} = \begin{bmatrix}\n0 & 0.05 & 0 & 0 \\
0.35 & 0.1 & 0.11 & 0.23\n\end{bmatrix}, \quad\n\tilde{A}_{22} = \begin{bmatrix}\n0.05 & 0.4 \\
0.85 & 0.11\n\end{bmatrix}, \quad (40)
$$
\n
$$
\tilde{B}_1 = \begin{bmatrix}\n0 \\
0 \\
0 \\
0.05\n\end{bmatrix}, \quad\n\tilde{B}_2 = \begin{bmatrix}\n0.05 \\
0.05\n\end{bmatrix}.
$$

Now, taking under consideration  $(21-23)$  we obtain

$$
\hat{A}_{22} = \tilde{A}_{22} + \tilde{B}_2 \tilde{K}_2 = (\tilde{A}_{22} \tilde{G}_2 + \tilde{B}_2 \tilde{N}_2) \tilde{G}_2^{-1} \in M_2.
$$
 (41)

We assume, that our desired matrix should have the form

$$
\hat{A}_{22} = \begin{bmatrix} -0.75 & 0.05 \\ 0.05 & -0.24 \end{bmatrix} \in M_2.
$$
 (42)

since it is asymptotically stable Meltzer matrix and

$$
-\hat{A}_{22}^{-1}\tilde{A}_{21} = \begin{bmatrix} 0.1 & 0.1 & 0.03 & 0.06 \\ 1.48 & 0.44 & 0.46 & 0.97 \end{bmatrix}
$$
 (43)

is contracting matrix, e.g. by Definition 3 for vector  $\tilde{x}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ we have  $\|\tilde{x}_1\| = 4$ ,  $\|\tilde{x}_2\| = 3.64$ .

Lets take  $\tilde{G}_2 = \text{diag}[1, 1]$ . Using one of the well-known methods (in this case Symbolic Math Toolbox), we compute

$$
\tilde{K}_2 = \tilde{N}_2 \tilde{G}_2^{-1} = [-16 \quad -7] \tag{44}
$$

which satisfy  $(41)$ .

Unlike the matrix (41) which should be Metzler, the matrix (34) need to be positive. In this case, using similar approach, we compute  $\tilde{K}_1 = \begin{bmatrix} -4 & -4 & -2 & -5 \end{bmatrix}$  for which

$$
\hat{A}_{11} + \begin{bmatrix} I_{\tilde{n}_1}(1-\alpha) & 0 \\ 0 & I_{\tilde{n}_2}(1-\beta) \end{bmatrix} + \tilde{B}_1 \tilde{K}_1 =
$$
\n
$$
= \begin{bmatrix} 0.65 & 0.1 & 0.1 & 0.2 \\ 0.1 & 0.5 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.35 \end{bmatrix}.
$$
\n(45)

is positive and asymptotically stable matrix, since its eigenvalues are  $\lambda = [0.99 \ 0.54 \ 0.19 \ 0.28]$ .

Inversing recombination and decomposition on the matrix  $\tilde{K}$  = blockdiag( $\tilde{K}_1$ ,  $\tilde{K}_2$ ) we can find the gain matrix of decentralized controller for descriptor fractional discrete-time linear system with two different fractional orders (5) described by matrices (37) of the form

$$
K = \begin{bmatrix} -4 & -4 & -16 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -2 & -7 \end{bmatrix}.
$$
 (46)

### **5. Concluding remarks**

The positive fractional discrete-time linear systems with two different fractional orders were analyzed. Based on the decomposition of the regular pencil, necessary and sufficient conditions for the positivity were extended to the descriptor fractional discrete-time linear system with two different fractional orders. A method for finding the decentralized controller for the class of positive systems was proposed and its effectiveness demonstrated on a numerical example. An extension of these considerations to the systems consisting of *n* subsystems with different fractional orders is possible.

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