Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

AXISYMMETRIC SOLUTIONS TO THE CAUCHY PROBLEM FOR TIME-FRACTIONAL DIFFUSION EQUATION IN A CIRCLE

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Abstract. The Cauchy problems for time-fractional diffusion equation with delta pulse initial value of a sought-for function is studied in a circle domain in the axisymmetric case under zero Dirichlet and Neumann boundary conditions, respectively. The Caputo fractional derivative is used. The Laplace and finite Hankel integral transforms are employed. The results are illustrated graphically.

1. Introduction

The time-fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\Delta u, \qquad 0 < \alpha \leq 2, \tag{1}$$

is a mathematical model of a wide range of important physical phenomena in amorphous and porous materials, fractals, disordered media, dielectrics and semiconductors, geophysical and geological processes, medicine and biological systems [1–8].

In Eq. (1), we use the Caputo fractional derivative [9]

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n u(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, \qquad n-1 < \alpha < n, \tag{2}$$

where $\Gamma(x)$ is the gamma function. The Laplace transform rule for the Caputo derivative has the following form:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha \mathcal{L}\left\{u(t)\right\} - \sum_{k=0}^{n-1} u^{(k)}(0^+) s^{\alpha-1-k}, \qquad n-1 < \alpha < n, \tag{3}$$

with s being the transform variable.

Several problems for time-fractional diffusion equation in a cylinder were considered in [10–14]. In this paper we investigate the Cauchy problems with delta function initial value of a sought-for function in a circle domain under zero Dirichlet and Neumann boundary conditions, respectively, and compare the obtained results with the corresponding solution in an infinite domain.

2. The Cauchy problem in an infinite domain

In order to gain a better insight of the considered problem in a circle, we recall the corresponding result for the infinite domain [15]. Let us study the Cauchy problem for time-fractional diffusion equation under delta-function initial condition for a sought-for function:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \qquad 0 < t < \infty, \quad 0 \le r < \infty, \tag{4}$$

$$t = 0: \quad u = \frac{p}{2\pi r}\,\delta_+(r), \quad 0 < \alpha \le 2, \tag{5}$$

$$t = 0: \quad \frac{\partial u}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{6}$$

As usually, we impose the zero condition at infinity:

$$\lim_{r\to\infty} u(r,t) = 0. \tag{7}$$

Using the Laplace transform with respect to time t and the Hankel transform with respect to the spatial coordinate r, we obtain

$$u^* = \frac{p}{2\pi}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}, \tag{8}$$

where the asterisk denotes the transforms.

Inversion of the Laplace transform is carried out in terms of the Mittag-Leffler functions

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0, \;\; z \in C, \tag{9}$$

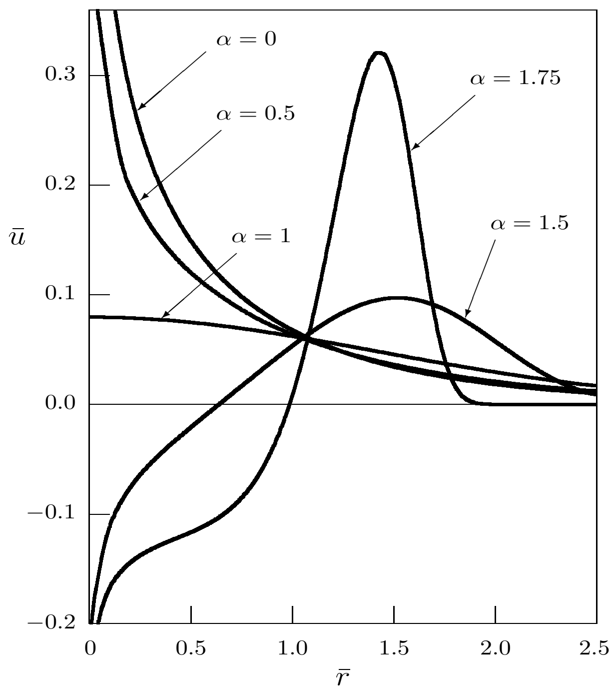

Fig. 1. Dependence of solution on the similarity variable
(the Cauchy problem with the delta pulse initial condition)

due to the following formula [9]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}\right\} = E_\alpha(-a\xi^2 t^\alpha). \tag{10}$$

Thus, we get

$$u = \frac{p}{2\pi}\int_0^\infty E_\alpha(-a\xi^2 t^\alpha)\, J_0(r\xi)\, \xi\, d\xi. \tag{11}$$

The similarity variable $\bar{r}$, new integration variable η and nondimensional solution $\bar{u}$ are defined as

$$\bar{r} = \frac{r}{\sqrt{a}t^{\alpha/2}}, \qquad \eta = \sqrt{a}t^{\alpha/2}\xi, \qquad \bar{u} = \frac{at^\alpha}{p}\, u. \tag{12}$$

Hence,

$$\bar{u} = \frac{1}{2\pi}\int_0^\infty E_\alpha(-\eta^2)\, J_0(\bar{r}\eta)\, \eta\, d\eta. \tag{13}$$

The behavior of the solution at the origin was analyzed in [15], where it was shown that only the fundamental solution to the classical diffusion equation

($\alpha = 1$) has no singularity at the origin. For $0 \le \alpha < 1$ and $1 < \alpha < 2$ the solution has the logarithmic singularity at the origin:

$$\bar{u} \sim -\frac{1}{2\pi\Gamma(1-\alpha)} \ln \bar{r}. \tag{14}$$

Dependence of nondimensional solution $\bar{u}$ on nondimensional distance $\bar{r}$ is shown in Fig. 1.

3. The Cauchy problem in a circle with zero Dirichlet boundary condition

Consider the following initial-boundary value problem for time-fractional diffusion equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad 0 < t < \infty, \quad 0 \le r < R, \tag{15}$$

$$t = 0: \quad u = \frac{p}{2\pi r} \, \delta_+(r), \quad 0 < \alpha \le 2, \tag{16}$$

$$t = 0: \quad \frac{\partial u}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{17}$$

$$r = R: \quad u = 0. \tag{18}$$

The finite Hankel transforms are used in cylindrical coordinates in the domain $0 \le r \le R$. The form of the finite Hankel transform depends on the type of boundary conditions at $r = R$. We restrict ourselves to the finite Hankel transform of the zeroth order. For Dirichlet boundary conditions with the given boundary value of a function at $r = R$ we have [16]

$$\mathcal{H}^{(\mathrm{D})}\{f(r)\} = f^*(\xi_n) = \int_0^R f(r) \, J_0(\xi_n r) \, r \, \mathrm{d}r \tag{19}$$

with the inverse transform

$$\mathcal{H}^{-1(\mathrm{D})}\{f^*(\xi_n)\} = f(r) = \frac{2}{R^2} \sum_{n=1}^{\infty} f^*(\xi_n) \frac{J_0(\xi_n r)}{J_1^2(\xi_n R)}, \tag{20}$$

where ξ_n are positive zeros of the transcendental equation

$$J_0(R\xi_n) = 0. \tag{21}$$

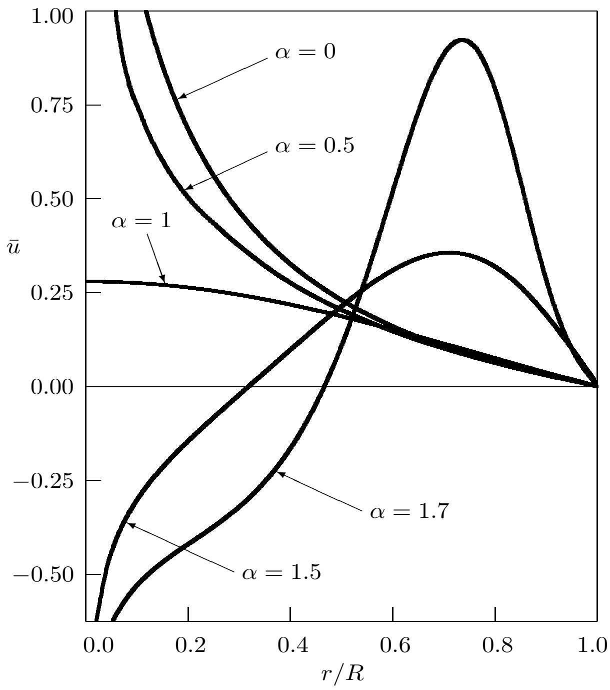

Fig. 2. Dependence of solution on distance
(the zero Dirichlet boundary condition; $\kappa = 0.5$).

The following formula plays important role in applications of the finite Hankel transform:

$$\mathcal{H}^{(\mathrm{D})}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi_n^2 f^*(\xi_n) + R\xi_n J_1(\xi_n R) f(R). \tag{22}$$

The integral transform technique allows us to get the solution in the transform domain:

$$u^* = \frac{p}{2\pi}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi_n^2}, \tag{23}$$

and after inversion we arrive at the series representation of the solution:

$$u = \frac{p}{\pi R^2}\sum_{n=1}^{\infty} E_\alpha(-a\xi^2 t^\alpha)\,\frac{J_0(r\xi_n)}{J_1^2(R\xi_n)}. \tag{24}$$

Introdusing nondimensional quantities

$$\eta_n = R\xi_n, \quad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}, \quad \bar{r} = \frac{r}{R}, \quad \bar{u} = \frac{R^2}{p}\,u, \tag{25}$$

we have

$$\bar{u} = \frac{1}{\pi} \sum_{n=1}^{\infty} E_\alpha(-\kappa^2 \eta_n^2) \, \frac{J_0(\bar{r}\eta_n)}{J_1^2(\eta_n)}. \tag{26}$$

Figure 2 shows the dependence of the solution (26) on distance for $\kappa = 0.5$.

4. The Cauchy problem in a circle with zero Neumann boundary condition

Now we study the time-fractional diffusion equation in a circle under delta pulse initial condition and zero Neumann boundary condition:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad 0 < t < \infty, \quad 0 \le r < R, \tag{27}$$

$$t = 0: \quad u = \frac{p}{2\pi r} \, \delta_+(r), \quad 0 < \alpha \le 2, \tag{28}$$

$$t = 0: \quad \frac{\partial u}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{29}$$

$$r = R: \quad \frac{\partial u}{\partial r} = 0. \tag{30}$$

For the Neumann boundary condition with the given value of normal derivative of a function, the corresponding finite Hankel transform is defined as [16]:

$$\mathcal{H}^{(\mathrm{N})}\{f(r)\} = f^*(\xi_n) = \int_0^R r f(r) J_0(r\xi_n) \, \mathrm{d}r, \tag{31}$$

having the inverse

$$\mathcal{H}^{-1(\mathrm{N})}\{f^*(\xi_n)\} = f(r) = \frac{2}{R^2} \sum_{n=0}^{\infty} f^*(\xi_n) \frac{J_0(r\xi_n)}{[J_0(R\xi_n)]^2}, \tag{32}$$

where ξ_n are nonnegative roots of the transcendental equation

$$J_1(R\xi_n) = 0. \tag{33}$$

To obtain the correct results, it should be emphasized that Eq. (33) also has the root $\xi_0 = 0$ which should be taken into consideration.

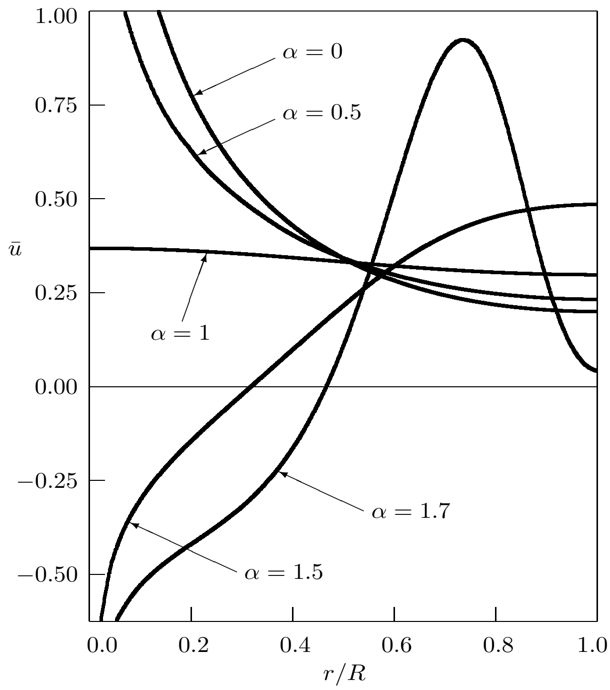

Fig. 3. Dependence of solution on distance
(the zero Neumann boundary condition; $\kappa = 0.5$).

The following formula explains importance of the finite Hankel transform of such a type for Neumann boundary value problems:

$$\mathcal{H}^{(\mathrm{N})}\left\{\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\right\} = -\xi_n^2 f^*(\xi_n) + R J_0(R\xi_n)\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)_{r=R}. \tag{34}$$

Thus, we obtain

$$u^* = \frac{p}{2\pi}\,\frac{s^{\alpha-1}}{s^\alpha + a\xi_n^2} \tag{35}$$

and

$$u = \frac{p}{\pi R^2}\sum_{n=0}^{\infty} E_\alpha(-a\xi^2 t^\alpha)\,\frac{J_0(r\xi_n)}{J_0^2(R\xi_n)} \tag{36}$$

or in terms of nondimensional quantities (25)

$$\bar{u} = \frac{1}{\pi}\sum_{n=0}^{\infty} E_\alpha(-\kappa^2\eta_n^2)\,\frac{J_0(\bar{r}\eta_n)}{J_0^2(\eta_n)}. \tag{37}$$

Dependence of the solution (37) on distance for $\kappa = 0.5$ is depicted in Fig. 3.

5. Concluding remarks

The results given by Eqs. (26) and (37) and displayed in Figures 2 and 3 are the primary results of this paper. The parameter κ describes nondimensional time and in the case of the wave equation ($\alpha = 2$) the values $0 < \kappa < 1$ and $\kappa = 1$ correspond to two characteristic cases: the wave front does not yet arrive at the boundary, and the wave front arrives at the boundary. For $0 \leq \alpha < 1$ and $1 < \alpha < 2$ in the case $\kappa = 0.5$ the solution does not "feel" the type of the boundary condition: the curves in Figs. 2 and 3 are very similar and do not differ essentially from the corresponding curves obtained for unbounded domain (see Fig. 1), including the logarithmic singularity at the origin. But for $\kappa = 1$ the situation changes substantially.

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