DETERMINATION OF AN OPTIMAL SHAPE OF DOMAIN USING THE TOPOLOGICAL DERIVATIVE AND THE BOUNDARY ELEMENT METHOD

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Abstract. In the paper, the topological derivative for the Laplace equation is taken into account. The governing equation is solved by means of the Boundary Element Method. The topological-shape sensitivity method is used to determine the points showing the lowest sensitivities. On the selected points, material is eliminated by opening a hole, using the appropriate iterative process. This one is halted when a given amount of material is removed. The objective of this work is to obtain an optimal topology of the domain considered. In the final part of the paper, the example of computations is shown.

Keywords: *topological derivative, topological sensitivity, topology optimization, heat transfer, Laplace equation, boundary element method*

1. Topological derivative

A topological derivative for the Laplace equation is considered in this work. The idea of the topological derivative D_T^* is based on the evaluation of the sensitivity of a given cost function (total potential energy) when the topology of the original domain Ω is changed by the creation of a small hole of radius ε inside the domain. In this way the new perturbed domain Ω_ε is obtained. The local value of the D_T^* is defined by the following limit [1-3]

$$D_T^*(x) = \lim_{\varepsilon \to 0} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)} \tag{1}$$

where $\psi(\Omega)$ and $\psi(\Omega_\varepsilon)$ are the cost functions calculated for the original and the perturbed domain, respectively. The function *f* is problem-dependent and $f(\varepsilon) \to 0$ where $\varepsilon \to 0$. Since it is impossible to establish a homeomorphism between domains with and without the hole, the authors in [1] proposed an alternative definition of the topological derivative.

The D_T^* is calculated using a new approach called the topological-shape sensitivity method, which is based on classical shape sensitivity analysis. They start from Ω_ε, where the hole already exists, causing a small perturbation on the radius of the hole $\delta\varepsilon$ (see Fig. 1).

$$D_T(x) = \lim_{\substack{\varepsilon\to 0 \\ \delta\varepsilon\to 0}} \frac{\psi(\Omega_{\varepsilon+\delta\varepsilon}) - \psi(\Omega_\varepsilon)}{f(\varepsilon+\delta\varepsilon) - f(\varepsilon)} \tag{2}$$

Both definitions 1 and 2 are equivalent, as shown in [1].

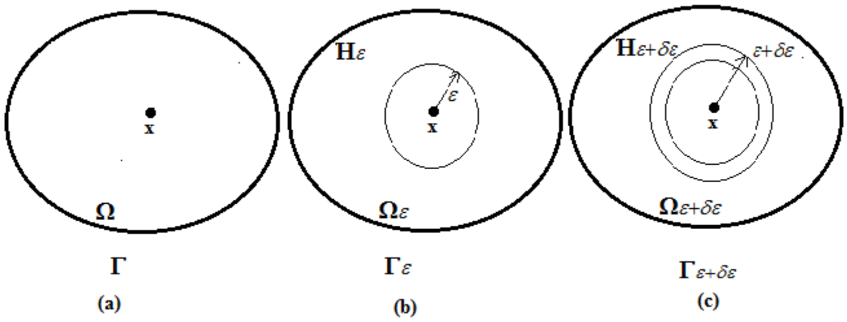

Fig. 1. Definition of the topological derivative: a), b) - original concept, c) - modified concept

In this paper the steady-state heat diffusion problem is considered. The temperature distribution in the domain is described by the Laplace equation supplemented by the boundary conditions [1]

$$\begin{cases} x \in \Omega_\varepsilon: & \lambda\nabla^2 T_\varepsilon(x) = 0 \\ x \in \Gamma_D: & T_\varepsilon(x) = T_b \\ x \in \Gamma_N: & -\lambda \dfrac{\partial T_\varepsilon(x)}{\partial n} = q_b \\ x \in \Gamma_R: & -\lambda \dfrac{\partial T_\varepsilon(x)}{\partial n} = \alpha\left(T_\varepsilon(x) - T_\infty\right) \\ x \in \mathrm{H}_\varepsilon: & h(a_1, a_2, a_3) = 0 \end{cases} \tag{3}$$

where $x = (x_1, x_2)$ are the spatial coordinates, λ is the thermal conductivity, $T_\varepsilon(x)$ is the temperature, $\partial T_\varepsilon/\partial n$ denotes the normal derivative and $n = [\cos\alpha_1, \cos\alpha_2]$ is the normal outward vector, T_b denotes the known temperature and q_b is the prescribed heat flux, α is the heat transfer coefficient and T_∞ is the ambient temperature.

The function

$$h(a_1,a_2,a_3) = a_1(T_\varepsilon - T_b^\varepsilon) + a_2\left(\lambda\frac{\partial T_\varepsilon}{\partial n} + q_b^\varepsilon\right) + a_3\left(\lambda\frac{\partial T_\varepsilon}{\partial n} + \alpha^\varepsilon(T_\varepsilon - T_\infty^\varepsilon)\right) = 0 \tag{4}$$

defines the kind of boundary condition on the hole Hε, meaning that

- if $a_1 = 1,\ a_2 = a_3 = 0$ (*Dirichlet condition*), then $h(a_1, a_2, a_3) = T_\varepsilon - T_b^\varepsilon$
- if $a_2 = 1,\ a_1 = a_3 = 0$ (*Neumann condition*), then $h(a_1, a_2, a_3) = \lambda \dfrac{\partial T_\varepsilon}{\partial n} + q_b^\varepsilon$
- if $a_3 = 1,\ a_1 = a_2 = 0$ (*Robin condition*), then $h(a_1, a_2, a_3) = \lambda \dfrac{\partial T_\varepsilon}{\partial n} + \alpha^\varepsilon (T_\varepsilon - T_\infty^\varepsilon)$

where T_b^ε, q_b^ε are the temperature and the heat flux on the hole boundary Hε, while α^ε and T_∞^ε are the hole's internal convection parameters.

Details of the calculation of D_T are described in [1]. The final expressions for the topological derivative using the total potential energy as the cost function are the following:

Table 1

Topological derivatives in the Laplace problem

Boundary condition on the hole	Topological derivative
Neumann condition and $q_b^\varepsilon = 0$	$D_T(x) = \lambda\ \Delta T \cdot \Delta T$
Neumann condition and $q_b^\varepsilon \neq 0$	$D_T(x) = -\, q_b^\varepsilon T$
Robin condition	$D_T(x) = -\frac{1}{2}\alpha^\varepsilon T(T - 2T_\infty^\varepsilon)$
Dirichlet condition	$D_T(x) = -\frac{1}{2}\lambda(T - T_b^\varepsilon)^2$

It should be pointed out that T is the solution of the original problem (without a hole). In this work, the boundary element method was used to ensure the numerical solution.

2. The boundary element method

To determine the temperature field in the domain of interest the boundary element method is used. The boundary integral equation for the Laplace equation is the following [4, 5]:

$$\xi \in \Gamma: \quad B(\xi)T(\xi) + \int_\Gamma q(x)T^*(\xi, x)\mathrm{d}\Gamma = \int_\Gamma T(x)q^*(\xi, x)\mathrm{d}\Gamma \tag{5}$$

where $B(\xi) \in (0,1)$ is the coefficient connected with the local shape of a boundary, ξ is the observation point and $q(x) = -\lambda\ \mathbf{n} \cdot \nabla T(x)$ is the heat flux. The fundamental solution T^* and heat flux q^* resulting from the fundamental solution are given by

$$T^*(\xi,x)=\frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{6}$$

$$q^*(\xi,x)=\frac{d}{2\pi r^2} \tag{7}$$

where r denotes the distance between $\xi = (\xi_1, \xi_2)$ and $x = (x_1, x_2)$

$$r=\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2} \tag{8}$$

while

$$d=(x_1-\xi_1)n_x+(x_2-\xi_2)n_y \tag{9}$$

n_x, n_y are the directional cosines of the normal outward vector **n**.

In numerical realization of the BEM, the boundary is divided into N linear boundary elements. The integrals in equation (5) are substituted by the sums of integrals over these elements

$$B(\xi_i)T(\xi_i)+\sum_{j=1}^{N}\int_{\Gamma_j} q(x)T^*(\xi_i,x)\mathrm{d}\Gamma_j=\sum_{j=1}^{N}\int_{\Gamma_j} T(x)q^*(\xi_i,x)\,\mathrm{d}\Gamma_j \tag{10}$$

Finally, one obtains the following system of algebraic equations

$$\sum_{r=1}^{R}G_{ir}q_r=\sum_{r=1}^{R}H_{ir}T_r,\quad i=1,2,...,R \tag{11}$$

The system of equations (11) can be written in the form

$$\mathbf{Gq}=\mathbf{HT} \tag{12}$$

Taking into account the known boundary conditions, equation (12) can be reordered

$$\mathbf{AX}=\mathbf{B} \tag{13}$$

where **A** is the main matrix, **X** is the unknown vector and **B** is the free terms vector. Equation (13) provides the determination of the missing boundary values.

Knowledge of nodal boundary temperatures and heat fluxes allows one to calculate the internal temperatures using the following integral equation

$$\xi\in\Omega:\qquad T(\xi)=\int_{\Gamma}T(x)q^*(\xi,x)\,\mathrm{d}\Gamma-\int_{\Gamma}q(x)T^*(\xi,x)\mathrm{d}\Gamma \tag{14}$$

or

$$T(\xi^i) = \sum_{r=1}^{R} H_{ir} T_r - \sum_{r=1}^{R} G_{ir} q_r \tag{15}$$

Further details about the BEM can be found in [5].

The topological derivative depends on the $T(x)$ or $\nabla T(x)$ and the boundary conditions on $H\varepsilon$ (see Table 1). The solution $T(x)$ is calculated using equation (14). Similarly, the gradient $\nabla T(x)$ can be obtained at the internal points by differentiating equation (14) with respect to the internal points.

$$\xi \in \Omega: \qquad \frac{\partial T(\xi)}{\partial x_i} = \int_{\Gamma} T(x) \frac{\partial q^*(\xi, x)}{\partial x_i} \, \mathrm{d}\Gamma - \int_{\Gamma} q(x) \frac{\partial T^*(\xi, x)}{\partial x_i} \, \mathrm{d}\Gamma \tag{16}$$

3. Numerical example

The rectangular domain of dimensions 0.05×0.1 m has been considered. Thermal conductivity equals $\lambda = 1$ W/(mK). The boundary conditions have been marked in Figure 2. The initial boundary has been divided into 60 linear boundary elements. The grid of 285 internal nodes has been used.

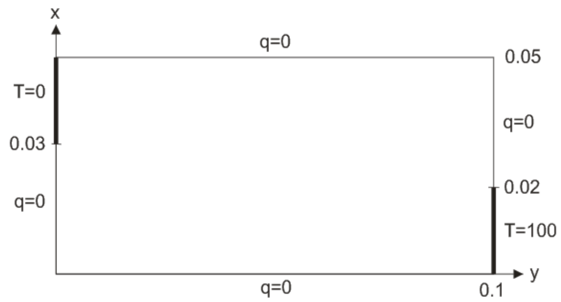

Fig. 2. Domain considered

In the holes created via topological derivative, the Neumann condition ($q_b^\varepsilon = 0$) is prescribed, hence D_T will be evaluated using the formula (see Table 1)

$$D_T(x) = \lambda \nabla T \cdot \nabla T \tag{17}$$

In order to obtain an optimal topology of the domain the following iterative process is taken into account [2]:

1. Provide the initial domain
2. Solve the problem using the boundary element method
3. Calculate D_T at internal points
4. Select the points with the lowest values of D_T

5. On the selected points create the holes by *punching out* disks of material
6. Check the stop criterion
7. Rebuild the mesh
8. Repeat the procedure until a given stopping criteria is obtained.

Holes with $r = 0.0025$ were used and the iterative procedure was stopped when 50% of material from the initial domain was eliminated.

Figure 3 illustrates the temperature distribution in the domain considered, while Figure 4 shows the topological derivative obtained in the first iteration ($i = 0$) taking into account the Neumann boundary condition on the holes. The boundary of the holes has been divided into 6 linear boundary elements (see Fig. 5). For instance, Figure 6 presents the removal of the holes and reconstruction of the boundary after the first iteration. During each iteration, 2% of material was eliminated, consisting of about six holes. The final result was obtained at iteration $i = 20$, as can be seen in Figure 7. This result was compared with a similar problem in [1] that was calculated by means of FEM. In both cases the final topologies are close.

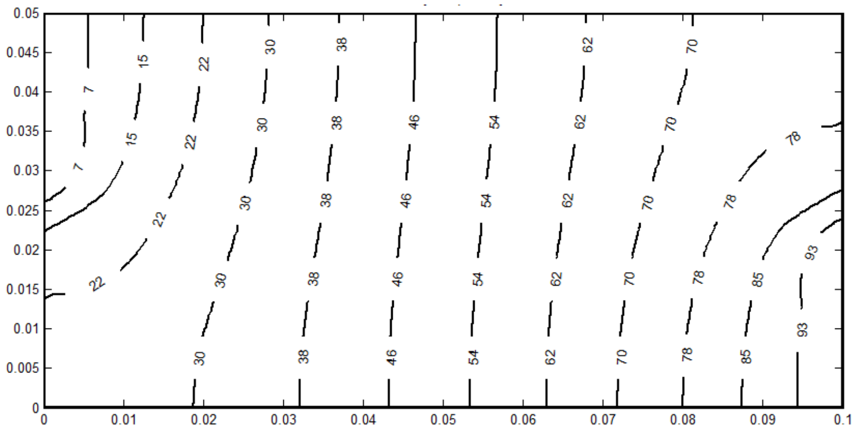

Fig. 3. Temperature distribution

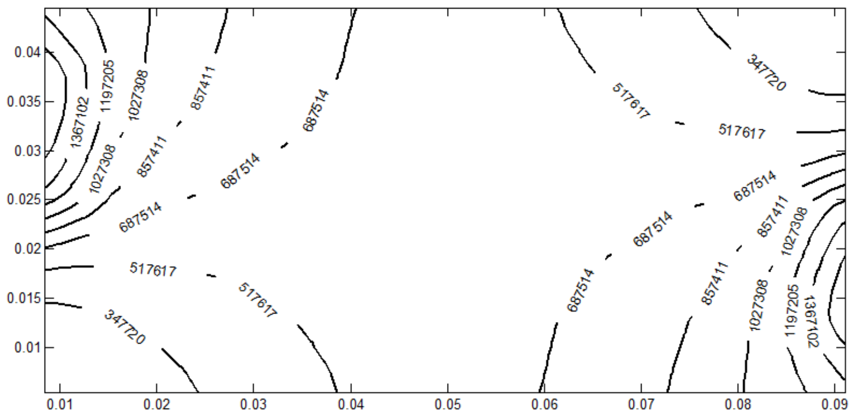

Fig. 4. Topological derivative at $i = 0$

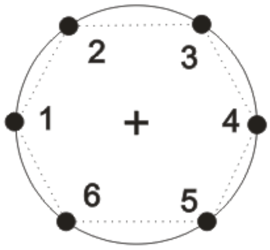

Fig. 5. Hexagonal hole

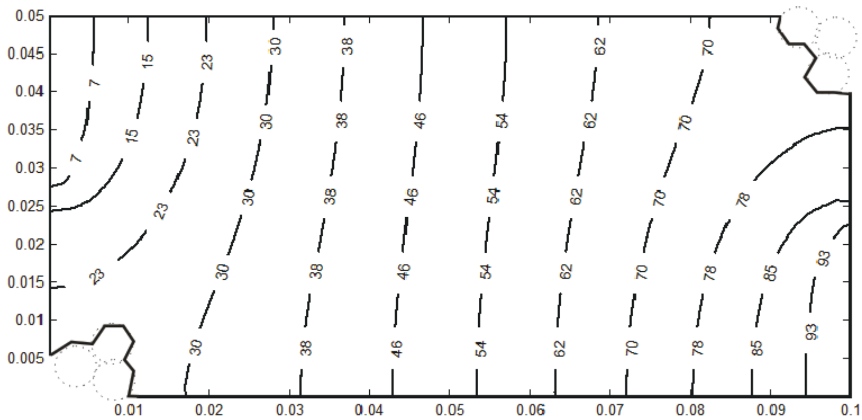

Fig. 6. Holes removal

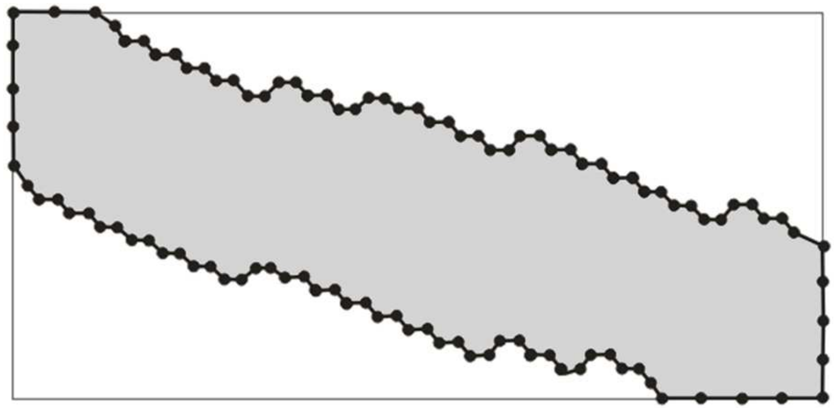

Fig. 7. Final result

Conclusions

In the present work the topological derivative is used to obtain the optimal shape of domain for a heat transfer problem. The topological-shape sensitivity method gives essential information concerning the positions where the holes must be created. Wherever the sensitivity is low enough, the material is progressively removed. The boundary element method was used to provide the numerical solution. Taking into account the given example, it can be stated that the iterative process is an effective approach to obtain the optimal topology of the domain. It was complicated to create the algorithm reconstructing the boundary of the domain after each iteration. This algorithm will be presented in the future.

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