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Set-valued minimax fractional programming problems under ρ-cone arcwise connectedness*

by

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Abstract: In this paper, we consider a set-valued minimax fractional programming problem (MFP), where the objective as well as constraint maps are set-valued. We introduce the notion of ρ-cone arcwise connectedness of set-valued maps as a generalization of cone arcwise connected set-valued maps. We establish the sufficient Karush-Kuhn-Tucker (KKT) conditions for the existence of minimizers of the problem (MFP) under ρ-cone arcwise connectedness assumption. Further, we study the Mond-Weir (MWD), Wolfe (WD), and mixed (MD) types of duality models and prove the corresponding weak, strong, and converse duality theorems between the primal (MFP) and the corresponding dual problems under ρ-cone arcwise connectedness assumption.

Keywords: convex cone; set-valued map; contingent epiderivative; arcwise connectedness; duality

1. Introduction

The class of minimax fractional programming problems are mainly studied in various fields of mathematics, economics, and operational research. In 1990, Yadav and Mukherjee (1990) formulated two types of duality models and proved the duality theorems of differentiable fractional minimax programming problems. Later, in 1995, Chandra and Kumar (1995) introduced two types of modified duality models and derived the duality theorems of differentiable fractional minimax programming problems. Bector and Bhatia (1985) and Weir (1992) established the optimality conditions and studied weak and strong duality theorems of differentiable fractional minimax programming problems. Zamlai

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(1987) established the necessary and sufficient optimality conditions and derived duality theorems of differentiable fractional minimax programming problems under generalized invexity assumption. Mishra (1995, 1998, 2001) studied pseudolinear fractional minmax programming, generalized pseudo convex minimax programming, and pseudoconvex complex minimax programming problems. Mishra et al. (2003, 2004) studied nondifferentiable minimax fractional programming and complex minimax programming problems under generalized convexity. Liu and Wu (1998) established the sufficient optimality conditions and proved duality theorems of differentiable fractional minimax programming problems under (F, ρ)-convexity assumption. Ahmad (2003) established the sufficient optimality conditions and proved duality theorems of differentiable fractional minimax programming problems under ρ-invexity assumption. Liang and Shi (2003) established the sufficient optimality conditions and formulated the duality theorems of fractional minimax programming problems under (F, α, ρ, d)-convexity assumption. Lai et al. (1999) established the necessary and sufficient optimality conditions and studied the parametric duality theorems of nondifferentiable fractional minimax programming problems under generalized convexity assumption. Lai and Lee (2002) proved the parameter-free duality theorems of nondifferentiable minimax fractional problems under generalized convexity assumption. Ahmad and Husain (2006) established the sufficient optimality conditions and proved the duality theorems of nondifferentiable minimax fractional programming problems under (F, α, ρ, d)-convexity assumptions. Das and Nahak (2017) established the KKT sufficient optimality conditions of set-valued minimax programming problems via contingent epiderivative and generalized cone convexity assumptions. They also formulated the Mond-Weir, Wolfe, and mixed types duals and proved the corresponding duality theorems.

In 1976, Avriel (1976) introduced the concept of arcwise connectedness as a generalization of convexity by replacing the line segment joining two points by a continuous arc. Later, Fu and Wang (2003) and Lalitha et al. (2003) introduced the notion of cone arcwise connected set-valued maps as an extension of the class of convex set-valued maps. Lalitha et al. (2003) established the sufficient optimality condition for a constrained set-valued optimization problem via contingent epiderivative and cone arcwise connectedness assumptions. In 2013, Yu (2013) established the necessary and sufficient optimality conditions for global proper efficiency in vector optimization problem involving cone arcwise connected set-valued maps. Yihong and Min (2016) introduced the concept of α-order nearly cone arcwise connected set-valued maps and derived the necessary and sufficient optimality conditions of some set-valued optimization problems. Yu (2016) established the necessary and sufficient optimality conditions for global proper efficient element in vector optimization problem with cone arcwise connected set-valued maps. In 2018, Peng and Xu (2018) introduced the notion of cone subarcwise connected set-valued maps and established the second-order necessary optimality conditions for local global proper efficient elements of set-valued optimization problems.

In this paper, we establish the sufficient KKT conditions of a set-valued minimax fractional programming problem (MFP) under contingent epiderivative and generalized cone arcwise connectedness assumptions. Further, we formulate the different types of duality models and prove a variety of duality relationships between the primal problem (MFP) and the corresponding dual problems (MWD), (WD), and (MD).

This paper is organized as follows. Section 2 deals with some definitions and preliminary concepts of set-valued maps. A set-valued minimax fractional programming problem (MFP) is introduced and the sufficient KKT conditions are established for the problem (MFP) in Section 3. Various types of duality theorems are proved under generalized cone arcwise connectedness assumptions.

2. Definitions and preliminaries

Let V be a real normed space and Ω be a nonempty subset of V. Then Ω is called a cone if $\lambda v \in \Omega$, for all $v \in \Omega$ and $\lambda \geq 0$. Furthermore, Ω is called non-trivial if $\Omega \neq \{\theta_V\}$, proper if $\Omega \neq V$, pointed if $\Omega \cap (-\Omega) = \{\theta_V\}$, solid if $\mathrm{int}(\Omega) \neq \emptyset$, closed if $\overline{\Omega} = \Omega$, and convex if $\lambda\Omega + (1-\lambda)\Omega \subseteq \Omega$, for all $\lambda \in [0,1]$, where $\mathrm{int}(\Omega)$ and $\overline{\Omega}$ denote the interior and closure of Ω, respectively and θ_V is the zero element of V.

Aubin (1981) and Aubin and Frankowska (1990) introduced the notion of contingent cone to a nonempty subset of a real normed space. Also, Aubin and Frankowska (1990) as well as Cambini et al. (1999) introduced the notion of second-order contingent set to a nonempty subset of a real normed space.

Definition 1 (Aubin, 1981; Aubin and Frankowska, 1990) *Let V be a real normed space, $\emptyset \neq B \subseteq V$, and $v' \in \overline{B}$. The contingent cone to B at v' is denoted by $T(B, v')$ and is defined as follows: an element $v \in T(B, v')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{v_n\}$ in V, with $v_n \to v$, such that*

$$v' + \lambda_n v_n \in B, \quad \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{v'_n\}$ in B, with $v'_n \to v'$, such that

$$t_n(v'_n - v') \to v.$$

Let U, V be real normed spaces, 2^V be the set of all subsets of V, and Ω be a solid pointed convex cone in V. Let $\mathcal{F} : U \to 2^V$ be a set-valued map from U to V, i.e., $\mathcal{F}(u) \subseteq V$, for all $u \in U$. The effective domain, image, graph, and epigraph of $\mathcal{F}$ are defined respectively by

$$\mathrm{dom}(\mathcal{F}) = \{u \in U : \mathcal{F}(u) \neq \emptyset\},$$

$$\mathcal{F}(A) = \bigcup_{u \in A} \mathcal{F}(u), \text{ for any } \emptyset \neq A \subseteq U,$$

$$\mathrm{gr}(\mathcal{F}) = \{(u,v) \in U \times V : v \in \mathcal{F}(u)\},$$

and

$$\mathrm{epi}(\mathcal{F}) = \{(u,v) \in U \times V : v \in \mathcal{F}(u) + \Omega\}.$$

Jahn and Rauh (1997) introduced the notion of contingent epiderivative of set-valued maps, which plays a vital role in various aspects of set-valued optimization problems.

Definition 2 (Jahn and Rauh, 1997) *A single-valued map $D_{\uparrow}\mathcal{F}(u',v') : U \to V$, whose epigraph coincides with the contingent cone to the epigraph of $\mathcal{F}$ at (u',v'), i.e.,*

$$\mathrm{epi}(D_{\uparrow}\mathcal{F}(u',v')) = T(\mathrm{epi}(\mathcal{F}), (u',v')),$$

is said to be the contingent epiderivative of $\mathcal{F}$ at (u',v').

We now turn our attention to the notion of cone convexity of set-valued maps, introduced by Borwein (1977).

Definition 3 (Borwein, 1977) *Let A be a nonempty convex subset of a real normed space U. A set-valued map $\mathcal{F} : U \to 2^V$, with $A \subseteq \mathrm{dom}(\mathcal{F})$, is called Ω-convex on A if $\forall u_1, u_2 \in A$ and $\lambda \in [0,1]$,*

$$\lambda\mathcal{F}(u_1) + (1-\lambda)\mathcal{F}(u_2) \subseteq \mathcal{F}(\lambda u_1 + (1-\lambda)u_2) + \Omega.$$

Avriel (1976) introduced the notion of arcwise connectedness as a generalization of convexity by replacing the line segment joining two points by a continuous arc.

Definition 4 *A subset A of a real normed space U is said to be an arcwise connected set if for all $u_1, u_2 \in A$ there exists a continuous arc $\mathcal{H}_{u_1,u_2}(\lambda)$ defined on $[0,1]$, with a value in A, such that $\mathcal{H}_{u_1,u_2}(0) = u_1$ and $\mathcal{H}_{u_1,u_2}(1) = u_2$.*

Fu and Wang (2003) and Lalitha et al. (2003) introduced the notion of cone arcwise connected set-valued maps as an extension of the class of convex set-valued maps.

Definition 5 (Fu and Wang, 2003; Lalitha et al., 2003) *Let A be an arcwise connected subset of a real normed space U and $\mathcal{F} : U \to 2^V$ be a set-valued map, with $A \subseteq \mathrm{dom}(\mathcal{F})$. Then $\mathcal{F}$ is said to be Ω-arcwise connected on A if*

$$(1-\lambda)\mathcal{F}(u_1) + \lambda\mathcal{F}(u_2) \subseteq \mathcal{F}(\mathcal{H}_{u_1,u_2}(\lambda)) + \Omega, \quad \forall u_1, u_2 \in A \text{ and } \forall \lambda \in [0,1].$$

Peng and Xu (2018) introduced the notion of cone subarcwise connected set-valued maps.

Definition 6 (Peng and Xu, 2018) *Let A be an arcwise connected subset of a real normed space U, $e \in \operatorname{int}(\Omega)$, and $\mathcal{F} : U \to 2^V$ be a set-valued map, with $A \subseteq \operatorname{dom}(\mathcal{F})$. Then, $\mathcal{F}$ is said to be Ω-subarcwise connected on A if*

$$(1-\lambda)\mathcal{F}(u_1) + \lambda\mathcal{F}(u_2) + \epsilon e \subseteq \mathcal{F}(\mathcal{H}_{u_1,u_2}(\lambda)) + \Omega, \\ \forall u_1, u_2 \in A, \forall \epsilon > 0, \text{ and } \forall \lambda \in [0,1].$$

Definition 7 *A set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called upper semicontinuous if $\mathcal{F}^+(V) = \{x \in \mathbb{R}^n : F(x) \subseteq V\}$ is open in $\mathbb{R}^n$ for any open set V in $\mathbb{R}^m$.*

Definition 8 *Let B be a nonempty subset of $\mathbb{R}^m$. Then, B is said to be $\mathbb{R}^m_+$-semicompact if every open cover of complements of the form*

$$\{(y_i + \mathbb{R}^m_+)^c : y_i \in B, i \in I\}$$

has a finite subcover.

Definition 9 *A set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called $\mathbb{R}^m_+$-semicompact-valued if $\mathcal{F}(x)$ is $\mathbb{R}^m_+$-semicompact, for all $x \in \operatorname{dom}(\mathcal{F})$.*

Let U, V be real topological vector spaces, A be a nonempty subset of U, $\mathcal{F} : U \to 2^V$ be a set-valued map, and Ω be a pointed convex cone in V. Consider a set-valued optimization problem (P):

$$\max_{u \in A} \quad \mathcal{F}(u). \tag{P}$$

The maximizer of the problem (P) is defined in the following way.

Definition 10 *Let $u' \in A$ and $v' \in \mathcal{F}(u')$. Then (u', v') is called a maximizer of the problem (P) if there exist no $u \in A$ and $v \in \mathcal{F}(u)$ such that*

$$v' < v.$$

Corley (1987) derived the existence results of solutions of a set-valued maximization problem in real topological vector spaces, where the objective function is an upper semicontinuous and cone semicompact-valued set-valued map.

Theorem 1 (Corley, 1987) *Let U, V be real topological vector spaces, A be a nonempty compact subset of U, and Ω be an acute (i.e. $\overline{\Omega}$ is pointed) convex cone in V. Let $\mathcal{F} : U \to 2^V$ be Ω-semicompact-valued and upper semicontinuous. Then there exists a maximal point for the problem (P).*

3. ρ-cone arcwise connectedness

Das and Nahak (2014, 2016a,b, 2017a,b, 2020a,b), and Treanţă and Das (2021) introduced the notion of ρ-cone convexity of set-valued maps. They established the sufficient KKT optimality conditions and developed the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions. For $\rho = 0$, we have the usual notion of cone convex set-valued maps introduced by Borwein (1977).

We introduce the notion of ρ-cone arcwise connectedness of set-valued maps as a generalization of cone arcwise connected set-valued maps.

Definition 11 *Let A be an arcwise connected subset of a real normed space U, $u_1,\ u_2 \in A$, $e \in \operatorname{int}(\Omega)$, and $\mathcal{F} : U \to 2^V$ be a set-valued map, with $A \subseteq \operatorname{dom}(\mathcal{F})$. Then $\mathcal{F}$ is said to be ρ-Ω-arcwise connected with respect to e on A for $u_1,\ u_2$ if there exists $\rho \in \mathbb{R}$, such that*

$$(1-\lambda)\mathcal{F}(u_1) + \lambda\mathcal{F}(u_2) \subseteq \mathcal{F}(\mathcal{H}_{u_1,u_2}(\lambda)) + \rho\lambda(1-\lambda)\|u_1 - u_2\|^2 e + \Omega, \quad \forall \lambda \in [0,1]. \tag{1}$$

Definition 12 *Let A be an arcwise connected subset of a real normed space U $e \in \operatorname{int}(\Omega)$, and $\mathcal{F} : U \to 2^V$ be a set-valued map, with $A \subseteq \operatorname{dom}(\mathcal{F})$. Then $\mathcal{F}$ is said to be ρ-Ω-arcwise connected with respect to e on A if there exists $\rho \in \mathbb{R}$, such that* (1) *holds for all $u_1,\ u_2 \in A$.*

Remark 1 *If $\rho > 0$, then $\mathcal{F}$ is said to be strongly ρ-Ω-arcwise connected, if $\rho = 0$, we have the usual notion of Ω-arcwise connectedness, and if $\rho < 0$, then $\mathcal{F}$ is said to be weakly ρ-Ω-arcwise connected. Obviously, strongly ρ-Ω-arcwise connectedness $\Rightarrow$ Ω-arcwise connectedness $\Rightarrow$ weakly ρ-Ω-arcwise connectedness.*

Further, we construct an example of ρ-cone arcwise connected set-valued map, which is not cone arcwise connected.

Example 1 *Let $U = \mathbb{R}^2,\ V = \mathbb{R},\ \Omega = \mathbb{R}_+$, and*

$$A = \left\{ u = (u_1, u_2) \mid u_1 + u_2 \geq \frac{1}{3},\ u_1 \geq 0,\ u_2 \geq 0 \right\} \subseteq U.$$

Define

$$\mathcal{H}_{u,u'}(\lambda) = (1-\lambda^2)u + \lambda^2 u',$$

where $u = (u_1, u_2)$, $u' = (u'_1, u'_2)$, and $\lambda \in [0,1]$. Clearly, A is an arcwise connected set. Define a set-valued map $\mathcal{F} : \mathbb{R}^2 \to 2^{\mathbb{R}}$ as follows:

$$\mathcal{F}(u) = \begin{cases} [0,3], & \text{if } u_1 + u_2 \geq \frac{1}{3},\ u_1 \neq 3u_2, u = (u_1, u_2), \\ [4,7], & \text{otherwise.} \end{cases}$$

We choose $u = (1,0)$, $u' = (0,1)$, and $\lambda = \frac{1}{2}$. Then,

$$\mathcal{H}_{u,u'}\left(\frac{1}{2}\right) = \left(\frac{3}{4}, \frac{1}{4}\right)$$

and

$$\frac{1}{2}\mathcal{F}(1,0)+\frac{1}{2}\mathcal{F}(0,1)=\frac{1}{2}[0,3]+\frac{1}{2}[0,3]=[0,3]\nsubseteq[4,7]+\mathbb{R}_{+}=\mathcal{F}\Big(\frac{3}{4},\frac{1}{4}\Big)+\mathbb{R}_{+}.$$

Hence, $\mathcal{F}$ is not $\mathbb{R}_+$-arcwise connected. On the other hand, by considering $\rho=-2$ and $e=4$, we get that

$$(1-\lambda)\mathcal{F}(1,0)+\lambda\mathcal{F}(0,1)=(1-\lambda)[0,3]+\lambda[0,3]=[0,3]$$

and

$$\mathcal{F}(\mathcal{H}_{u,u'}(\lambda))+\rho\lambda(1-\lambda)\|u-u'\|^2e=\mathcal{F}(1-\lambda^2,\lambda^2)-16\lambda(1-\lambda).$$

For $\lambda\neq\frac{1}{2}$, we have

$$\mathcal{F}(1-\lambda^2,\lambda^2)=[0,3].$$

So,

$$(1-\lambda)\mathcal{F}(1,0)+\lambda\mathcal{F}(0,1)+16\lambda(1-\lambda)=[0,3]+16\lambda(1-\lambda)\subseteq[0,3]+\mathbb{R}_{+}=\mathbb{R}_{+}.$$

For $\lambda=\frac{1}{2}$, we have

$$\mathcal{F}(1-\lambda^2,\lambda^2)=\mathcal{F}\Big(\frac{3}{4},\frac{1}{4}\Big)=[4,7].$$

So,

$$(1-\lambda)\mathcal{F}(1,0)+\lambda\mathcal{F}(0,1)+12\lambda(1-\lambda)=[0,3]+4=[4,7]\subseteq[4,7]+\mathbb{R}_{+}.$$

Consequently, $\mathcal{F}$ is a (-2)-$\mathbb{R}_+$-arcwise connected set-valued map with respect to 4 on A for $(1,0)$, $(0,1)$.

THEOREM 1 *Let A be an arcwise connected subset of a real normed space U, $e\in\operatorname{int}(\Omega)$, and $\mathcal{F}:U\to 2^V$ be ρ-Ω-arcwise connected with respect to e on A. Let $u'\in A$ and $v'\in\mathcal{F}(u')$. Then,*

$$\mathcal{F}(u)-v'\subseteq D_{\uparrow}\mathcal{F}(u',v')(\mathcal{H}'_{u',u}(0+))+\rho\|u-u'\|^2e+\Omega,\quad\forall u\in A,$$

where

$$\mathcal{H}'_{u',u}(0+)=\lim_{\lambda\to0+}\frac{\mathcal{H}_{u',u}(\lambda)-\mathcal{H}_{u',u}(0)}{\lambda},$$

assuming that $\mathcal{H}'_{u',u}(0+)$ exists for all $u,u'\in A$.

PROOF Let $u\in A$. As $\mathcal{F}$ is ρ-Ω-arcwise connected with respect to e on A, we have

$$(1-\lambda)\mathcal{F}(u')+\lambda\mathcal{F}(u)\subseteq\mathcal{F}(\mathcal{H}_{u',u}(\lambda))+\rho\lambda(1-\lambda)\|u-u'\|^2e+\Omega,\quad\forall\lambda\in[0,1].$$

Let $v \in \mathcal{F}(u)$. Consider a real sequence $\{\lambda_n\}$, with $\lambda_n \in (0,1)$, $n \in \mathbb{N}$, such that $\lambda_n \to 0+$ when $n \to \infty$. Suppose

$$u_n = \mathcal{H}_{u',u}(\lambda_n)$$

and

$$v_n = (1-\lambda_n)v' + \lambda_n v - \rho\lambda_n(1-\lambda_n)\|u-u'\|^2 e.$$

Therefore,

$$v_n \in \mathcal{F}(u_n) + \Omega.$$

It is clear that

$$u_n = \mathcal{H}_{u',u}(\lambda_n) \to \mathcal{H}_{u',u}(0) = u', v_n \to v', \text{ when } n \to \infty,$$

$$\frac{u_n - u'}{\lambda_n} = \frac{\mathcal{H}_{u',u}(\lambda_n) - \mathcal{H}_{u',u}(0)}{\lambda_n} \to \mathcal{H}'_{u',u}(0+), \text{ when } n \to \infty,$$

and

$$\frac{v_n - v'}{\lambda_n} = v - v' - \rho(1-\lambda_n)\|u-u'\|^2 e \to v - v' - \rho\|u-u'\|^2 e, \text{ when } n \to \infty.$$

Therefore,

$$(\mathcal{H}'_{u',u}(0+), v - v' - \rho\|u-u'\|^2 e) \in T(\operatorname{epi}(\mathcal{F}), (u',v')) = \operatorname{epi}(D_{\uparrow}\mathcal{F}(u',v')).$$

Consequently,

$$v - v' - \rho\|u-u'\|^2 e \in D_{\uparrow}\mathcal{F}(u',v')(\mathcal{H}'_{u',u}(0+)) + \Omega,$$

which is true, for all $v \in \mathcal{F}(u)$. Hence,

$$\mathcal{F}(u) - v' \subseteq D_{\uparrow}\mathcal{F}(u',v')(\mathcal{H}'_{u',u}(0+)) + \rho\|u-u'\|^2 e + \Omega, \quad \forall u \in A.$$

Hence, the theorem follows. □

4. Formulation of the main problem

Let A be a nonempty subset of $\mathbb{R}^n$ and B be a nonempty compact subset of $\mathbb{R}^m$. Let D_1 and D_2 be $n \times n$ positive semidefinite matrices. Let $\mathcal{F}, \mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ and $\mathcal{G} : \mathbb{R}^n \to 2^{\mathbb{R}^p}$ be set-valued maps, with

$$A \times B \subseteq \operatorname{dom}(\mathcal{F}) \cap \operatorname{dom}(\mathcal{H}) \text{ and } A \subseteq \operatorname{dom}(\mathcal{G}).$$

Consider a set-valued minimax fractional programming problem

$$\begin{aligned} &\underset{x \in A}{\text{minimize}} \quad \max \bigcup_{y \in B} \frac{\mathcal{F}(x,y) + (x^T D_1 x)^{\frac{1}{2}}}{\mathcal{H}(x,y) - (x^T D_2 x)^{\frac{1}{2}}} \\ &\text{subject to} \quad \mathcal{G}(x) \cap (-\mathbb{R}^p_+) \neq \emptyset. \end{aligned} \tag{MFP}$$

Define a set-valued map $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ by

$$\Phi(x,y) = \frac{\mathcal{F}(x,y) + (x^T D_1 x)^{\frac{1}{2}}}{\mathcal{H}(x,y) - (x^T D_2 x)^{\frac{1}{2}}}, \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^m,$$

assuming that

$$\mathcal{F}(x,y) + (x^T D_1 x)^{\frac{1}{2}} \geq 0$$

and

$$\mathcal{H}(x,y) - (x^T D_2 x)^{\frac{1}{2}} > 0, \forall (x,y) \in A \times B.$$

We assume that the set-valued map $\Phi(x,.) : \mathbb{R}^m \to 2^{\mathbb{R}}$ is $\mathbb{R}_+$-semicompact-valued and upper semicontinuous on B, for all $x \in A$. Therefore, by Theorem 1, $\max \bigcup_{y \in B} \Phi(x,y)$ always exists, for all $x \in A$. As $\Phi(x,y) \subseteq \mathbb{R}$, for each $x \in A$ there exists only one maximal point of the problem $\max \bigcup_{y \in B} \Phi(x,y)$. The feasible set of the problem (MFP) is

$$S' = \{x \in A : \mathcal{G}(x) \cap (-\mathbb{R}^p_+) \neq \emptyset\}.$$

The minimizer of the problem (MFP) is defined in the following way.

Definition 13 *Let $x' \in S'$ be a feasible point of the problem* (MFP) *and $z' = \max \bigcup_{y \in B} \Phi(x',y)$. Then (x',z') is called a minimizer of the problem* (MFP) *if there exist no $x \in S'$ and $z = \max \bigcup_{y \in B} \Phi(x,y)$, with $x \neq x'$, such that*

$$z < z'.$$

For $x \in A$, define

$$I(x) = \{j : 0 \in \mathcal{G}_j(x), 1 \leq j \leq p\}$$

$$J(x) = \{1,...,p\} \setminus I(x),$$

$$B(x) = \Big\{ y \in B : \max \bigcup_{y \in B} \Phi(x,y) \subseteq \Phi(x,b) \Big\},$$

and

$$K(x) = \Big\{ (k, z^*, \widetilde{y}) \in \mathbb{N} \times \mathbb{R}^k_+ \times \mathbb{R}^{mk} : 1 \leq k \leq n+1, z^* = (z^*_1,...,z^*_k) \in \mathbb{R}^k_+,$$
$$\text{with} \sum_{i=1}^{k} z^*_i = 1, \widetilde{y} = (\overline{y_1},...,\overline{y_s}), \text{ with } \overline{y_i} \in B(x), i = 1,...,k \Big\}.$$

As $\overline{\Phi}(x,.)$ is $\mathbb{R}_+$-semicompact-valued and upper semicontinuous on $B, \forall x \in A$, we have

$$B(x') \neq \emptyset, \forall x' \in S'.$$

Let D be an $n \times n$ positive semidefinite matrix. Then, for all $x, w \in \mathbb{R}^n$,

$$x^T D w \leq (x^T D x)^{\frac{1}{2}} (w^T D w)^{\frac{1}{2}}.$$

Moreover, if $(w^T D w)^{\frac{1}{2}} \leq 1$, we have

$$x^T D w \leq (x^T D x)^{\frac{1}{2}}. \tag{2}$$

5. Sufficient optimality conditions

We establish the sufficient KKT conditions of the set-valued minimax programming problem (MFP) under ρ-cone arcwise connectedness assumption.

THEOREM 2 *(Sufficient optimality conditions) Let A be an arcwise connected subset of $\mathbb{R}^n$, x' be a feasible point of* (MFP)*, and $z' = \max \bigcup_{y \in B} \Phi(x', y)$. Suppose that there exist $k \in \mathbb{N}$, (where $1 \leq k \leq n+1$), $z^* = (z_1^*, ..., z_k^*) \in \mathbb{R}_+^k$, with $\sum_{i=1}^{k} z_i^* = 1$, $\overline{y_i} \in B(x')$, $(1 \leq i \leq k)$, $w, v \in \mathbb{R}^n$, $w^* = (w_1^*, ..., w_p^*) \in \mathbb{R}_+^p$, and $w_j' \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$, such that*

$$\sum_{i=1}^{k} z_i^* \Big(D_\uparrow \mathcal{F}(.,\overline{y_i})(x',\overline{y_i}) + D_1 w - z'(D_\uparrow(-\mathcal{H})(.,\overline{y_i})(x',\overline{y_i}) - D_2 v) \Big) (\mathcal{H}'_{x',x}(0+))$$
$$+ \sum_{j=1}^{p} w_j^* D_\uparrow \mathcal{G}_j(x', w_j')(\mathcal{H}'_{x',x}(0+)) \geq 0, \forall x \in A, \tag{3}$$

$$\sum_{j=1}^{p} w_j^* w_j' = 0, \tag{4}$$

$$w^T D_1 w \leq 1, v^T D_2 v \leq 1, \tag{5}$$

$$(x'^T D_1 x')^{\frac{1}{2}} = x'^T D_1 w, \tag{6}$$

and

$$(x'^T D_2 x')^{\frac{1}{2}} = x'^T D_2 v. \tag{7}$$

Assume that $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(., \overline{y_i})$ is ρ'_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, satisfying

$$\sum_{i=1}^{k} z_i^* \Big(\rho_i + \overline{\rho_i} - z'(\rho'_i + \overline{\rho'_i}) \Big) + \sum_{j=1}^{p} w_j^* \nu_j \geq 0. \tag{8}$$

Then (x', z') is a minimizer of the problem (MFP).

Proof Suppose that (x', z') is not a minimizer of the problem (MFP). Then, there exist $x \in S'$ and $z = \max \bigcup_{y \in B} \Phi(x, y)$, with $x \neq x'$, such that

$$z < z'.$$

Since $\overline{y_i} \in B(x')$, $i = 1, ..., k$, we have

$$\max \bigcup_{y \in B} \Phi(x', y) \in \Phi(x', \overline{y_i}).$$

As $z' = \max \bigcup_{y \in B} \Phi(x', y)$, we have

$$z' \in \Phi(x', \overline{y_i}), i = 1, ..., k.$$

Let $z_i \in \Phi(x, \overline{y_i})$. Again, as $z = \max \bigcup_{y \in B} \Phi(x, y)$ and $\overline{y_i} \in B(x') \subseteq B$, we have

$$z_i \leq z.$$

Hence,

$$z_i < z'.$$

As $z' \in \Phi(x', \overline{y_i})$, there exist $\overline{z'_i} \in \mathcal{F}(x', \overline{y_i})$ and $\overline{z''_i} \in \mathcal{H}(x', \overline{y_i})$ such that

$$z' = \frac{\overline{z'_i} + (x'^T D_1 x')^{\frac{1}{2}}}{\overline{z''_i} - (x'^T D_2 x')^{\frac{1}{2}}}.$$

So,

$$\overline{z'_i} + (x'^T D_1 x')^{\frac{1}{2}} - z'(\overline{z''_i} - (x'^T D_2 x')^{\frac{1}{2}}) = 0, \forall i = 1, ..., k. \tag{9}$$

Since $z_i \in \Phi(x, \overline{y_i})$, there exist $z'_i \in \mathcal{F}(x, \overline{y_i})$ and $z''_i \in \mathcal{H}(x, \overline{y_i})$ such that

$$z_i = \frac{z'_i + (x^T D_1 x)^{\frac{1}{2}}}{z''_i - (x^T D_2 x)^{\frac{1}{2}}}.$$

Therefore,

$$\frac{z'_i + (x^T D_1 x)^{\frac{1}{2}}}{z''_i - (x^T D_2 x)^{\frac{1}{2}}} < z'.$$

So,

$$z'_i + (x^T D_1 x)^{\frac{1}{2}} - z'(z''_i - (x^T D_2 x)^{\frac{1}{2}}) < 0, \forall i = 1, ..., k. \tag{10}$$

Hence, we have

$$\begin{aligned}
&\sum_{i=1}^{k} z_i^* \Big(z'_i + (x^T D_1 w) - z'(z''_i - (x^T D_2 v)) \Big) \\
&\leq \sum_{i=1}^{k} z_i^* \Big(z'_i + (x^T D_1 x)^{\frac{1}{2}} - z'(z''_i - (x^T D_2 x)^{\frac{1}{2}}) \Big), \text{(from (2) and (5))} \\
&< 0, \text{(from (10))} \\
&= \sum_{i=1}^{k} z_i^* \Big(\overline{z'_i} + (x'^T D_1 x')^{\frac{1}{2}} - z'(\overline{z''_i} - (x'^T D_2 x')^{\frac{1}{2}}) \Big), \text{(from (9))} \\
&= \sum_{i=1}^{k} z_i^* \Big(\overline{z'_i} + (x'^T D_1 w) - z'(\overline{z''_i} - (x'^T D_2 v)) \Big), \text{(from (6) and (7))}.
\end{aligned}$$

As $x \in S'$, there exists

$$w_j \in \mathcal{G}_j(x) \cap (-\mathbb{R}_+).$$

Since $w_j^* \geq 0$ $(1 \leq j \leq p)$, we have

$$w_j^* w_j \leq 0, \forall 1 \leq j \leq p.$$

So,

$$\sum_{j=1}^{p} w_j^* w_j \leq 0.$$

Again, from (4), we have

$$\sum_{j=1}^{p} w_j^* w'_j = 0.$$

Therefore,

$$\sum_{j=1}^{p} w_j^* w_j \leq \sum_{j=1}^{p} w_j^* w'_j.$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{k} z_i^* \Big(z_i' + (x^T D_1 w) - z'(z_i'' - (x^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j \\
&< \sum_{i=1}^{k} z_i^* \Big(\overline{z_i'} + (x'^T D_1 w) - z'(\overline{z_i''} - (x'^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j'.
\end{aligned} \tag{11}$$

As $\mathcal{F}(.,\overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $\overline{z_i'} \in \mathcal{F}(x',\overline{y_i})$, we have

$$\mathcal{F}(x,\overline{y_i}) - \overline{z_i'} \subseteq D_{\uparrow}\mathcal{F}(.,\overline{y_i})(x',\overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i\|x - x'\|^2 + \mathbb{R}_+.$$

Again, as $z_i' \in \mathcal{F}(x,\overline{y_i})$, we have

$$z_i' - \overline{z_i'} \in D_{\uparrow}\mathcal{F}(.,\overline{y_i})(x',\overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i\|x - x'\|^2 + \mathbb{R}_+. \tag{12}$$

Since $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, we have

$$x^T D_1 w - x'^T D_1 w \geq D_1 w(\mathcal{H}'_{x',x}(0+)) + \overline{\rho_i}\|x - x'\|^2 + \mathbb{R}_+. \tag{13}$$

As $-\mathcal{H}(.,\overline{y_i})$ is ρ_i'-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $\overline{z_i''} \in \mathcal{H}(x',\overline{y_i})$, we have

$$-\mathcal{H}(x,\overline{y_i}) + \overline{z_i''} \subseteq D_{\uparrow}(-\mathcal{H})(.,\overline{y_i})(x',\overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i'\|x - x'\|^2 + \mathbb{R}_+.$$

Again, as $z_i'' \in \mathcal{H}(x,\overline{y_i})$, we have

$$-z_i'' + \overline{z_i''} \in D_{\uparrow}(-\mathcal{H})(.,\overline{y_i})(x',\overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i'\|x - x'\|^2 + \mathbb{R}_+. \tag{14}$$

Since $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, we have

$$x^T D_2 v - x'^T D_1 v \geq D_2 v(\mathcal{H}'_{x',x}(0+)) + \overline{\rho_i'}\|x - x'\|^2 + \mathbb{R}_+. \tag{15}$$

As $\mathcal{G}_j$ is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $w_j' \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, we have

$$\mathcal{G}_j(x) - w_j' \subseteq D_{\uparrow}\mathcal{G}_j(x',w_j')(\mathcal{H}'_{x',x}(0+)) + \nu_j\|x - x'\|^2 + \mathbb{R}_+.$$

Since $w_j \in \mathcal{G}_j(x) \cap (-\mathbb{R}_+)$, we have

$$w_j - w_j' \in D_{\uparrow}\mathcal{G}_j(x',w_j')(\mathcal{H}'_{x',x}(0+)) + \nu_j\|x - x'\|^2 + \mathbb{R}_+. \tag{16}$$

From (2), (8), and (12) - (16), we have

$$\sum_{i=1}^{k} z_i^* \Big(z_i' + (x^T D_1 w) - z'(z_i'' - (x^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j$$
$$\geq \sum_{i=1}^{k} z_i^* \Big(\overline{z_i'} + (x'^T D_1 w) - z'(\overline{z_i''} - (x'^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j',$$

which contradicts (11). Consequently, (x', z') is a minimizer of the problem (MFP). □

6. Mond-Weir type dual

We consider a Mond-Weir type dual (MWD), correspoing to the primal problem (MFP), where the set-valued maps $\mathcal{F}(.,\overline{y_i})$, $-\mathcal{H}(.,\overline{y_i})$ and $\mathcal{G}_j$ are contingent epiderivable set-valued maps. Hence, we consider the following problem:

$$\text{maximize } z' \tag{MWD}$$

subject to

$$\sum_{i=1}^{k} z_i^* \Big(D_\uparrow \mathcal{F}(.,\overline{y_i})(x',\overline{y_i}) + D_1 w$$
$$- z'(D_\uparrow(-\mathcal{H})(.,\overline{y_i})(x',\overline{y_i}) - D_2 v) \Big) (\mathcal{H}'_{x',x}(0+))$$
$$+ \sum_{j=1}^{p} w_j^* D_\uparrow \mathcal{G}_j(x', w_j')(\mathcal{H}'_{x',x}(0+)) \geq 0, \forall x \in A,$$

for some $k \in \mathbb{N}, (1 \leq k \leq n+1)$ and $\overline{y_i} \in B(x')$,

$$\sum_{j=1}^{p} w_j^* w_j' \geq 0,$$
$$w^T D_1 w \leq 1, v^T D_2 v \leq 1, (x'^T D_1 x')^{\frac{1}{2}} = x'^T D_1 w,$$
$$(x'^T D_2 x')^{\frac{1}{2}} = x'^T D_2 v, \text{ for some } w, v \in \mathbb{R}^n,$$
$$x' \in A, z' = \max \bigcup_{y \in B} \Phi(x', y), w' = (w_1', ..., w_p'), w_j' \in \mathcal{G}_j(x'),$$
$$z^* = (z_1^*, ..., z_k^*), w^* = (w_1^*, ..., w_p^*), z_i^* \geq 0, w_j^* \geq 0, \sum_{i=1}^{k} z_i^* = 1,$$

where $1 \leq i \leq k$ and $1 \leq j \leq p$.

A point (x', z', w', z^*, w^*), satisfying all the constraints of (MWD) is called a feasible point of the problem (MWD).

DEFINITION 14 *A feasible point* (x', z', w', z^*, w^*) *of the problem* (MWD) *is called a maximizer of* (MWD) *if there exists no feasible point* (x, z, w, z_1^*, w_1^*) *of* (MWD) *such that*

$$z' < z.$$

THEOREM 3 *(Weak duality) Let A be an arcwise connected subset of $\mathbb{R}^n$, x_0 be a feasible point of* (MFP) *and* (x', z', w', z^*, w^*) *be a feasible point of the problem* (MWD). *Assume that $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(., \overline{y_i})$ is ρ_i'-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, satisfying*

$$\left(\rho_i + \overline{\rho_i} - z'(\rho_i' + \overline{\rho_i'})\right) + \sum_{j=1}^{p} w_j^* \nu_j \geq 0. \tag{17}$$

Then,

$$\max \bigcup_{y \in B} \Phi(x_0, y) \nless z'.$$

PROOF We prove the theorem by the method of contradiction. Suppose that for $z_0 = \max \bigcup_{y \in B} \Phi(x_0, y)$, $z_0 < z'$. Since $\overline{y_i} \in B(x')$, $i = 1, ..., k$, we have

$$\max \bigcup_{y \in B} \Phi(x', y) \in \Phi(x', \overline{y_i}).$$

As $z' = \max \bigcup_{y \in B} \Phi(x', y)$, we have

$$z' \in \Phi(x', \overline{y_i}), i = 1, ..., k.$$

Let $z_i \in \Phi(x_0, \overline{y_i})$. Again, as $z_0 = \max \bigcup_{y \in B} \Phi(x_0, y)$ and $\overline{y_i} \in B(x') \subseteq B$, we have

$$z_i \leq z_0.$$

Hence, $z_i < z'$. As $z' \in \Phi(x', \overline{y_i})$, there exist $\overline{z_i'} \in \mathcal{F}(x', \overline{y_i})$ and $\overline{z_i''} \in \mathcal{H}(x', \overline{y_i})$ such that

$$z' = \frac{\overline{z_i'} + (x'^T D_1 x')^{\frac{1}{2}}}{\overline{z_i''} - (x'^T D_2 x')^{\frac{1}{2}}}.$$

So,

$$\overline{z'_i} + (x'^T D_1 x')^{\frac{1}{2}} - z'(\overline{z''_i} - (x'^T D_2 x')^{\frac{1}{2}}) = 0, \forall i = 1, ..., k. \tag{18}$$

Since $z_i \in \Phi(x_0, \overline{y_i})$, there exist $z'_i \in \mathcal{F}(x_0, \overline{y_i})$ and $z''_i \in \mathcal{H}(x_0, \overline{y_i})$ such that

$$z_i = \frac{z'_i + (x_0^T D_1 x_0)^{\frac{1}{2}}}{z''_i - (x_0^T D_2 x_0)^{\frac{1}{2}}}.$$

Therefore,

$$\frac{z'_i + (x_0^T D_1 x_0)^{\frac{1}{2}}}{z''_i - (x_0^T D_2 x_0)^{\frac{1}{2}}} < z'.$$

So,

$$z'_i + (x_0^T D_1 x_0)^{\frac{1}{2}} - z'(z''_i - (x_0^T D_2 x_0)^{\frac{1}{2}}) < 0, \forall i = 1, ..., k. \tag{19}$$

We have

$$\sum_{i=1}^{k} z_i^* \Big(z'_i + (x_0^T D_1 w) - z'(z''_i - (x_0^T D_2 v)) \Big)$$
$$\leq \sum_{i=1}^{k} z_i^* \Big(z'_i + (x_0^T D_1 x_0)^{\frac{1}{2}} - z'(z''_i - (x_0^T D_2 x_0)^{\frac{1}{2}}) \Big), \text{(from (2) and the constraints of (MWD))}$$
$$< 0, \text{(from (19))}$$
$$= \sum_{i=1}^{k} z_i^* \Big(\overline{z'_i} + (x'^T D_1 x')^{\frac{1}{2}} - z'(\overline{z''_i} - (x'^T D_2 x')^{\frac{1}{2}}) \Big), \text{(from (18))}$$
$$= \sum_{i=1}^{k} z_i^* \Big(\overline{z'_i} + (x'^T D_1 w) - z'(\overline{z''_i} - (x'^T D_2 v)) \Big), \text{(from the constraints of (MWD))}.$$

As $x_0 \in S'$, there exists

$$w_j \in \mathcal{G}_j(x_0) \cap (-\mathbb{R}_+).$$

Since $w_j^* \geq 0$ $(1 \leq j \leq p)$, we have

$$w_j^* w_j \leq 0, \forall j, (1 \leq j \leq p).$$

So,

$$\sum_{j=1}^{p} w_j^* w_j \leq 0.$$

Again, from the constraints of (MWD), we have

$$\sum_{j=1}^{p} w_j^* w_j' \geq 0.$$

Therefore,

$$\sum_{j=1}^{p} w_j^* w_j \leq \sum_{j=1}^{p} w_j^* w_j'.$$

Hence,

$$\begin{aligned} &\sum_{i=1}^{k} z_i^* \Big(z_i' + (x_0^T D_1 w) - z'(z_i'' - (x_0^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j \\ &< \sum_{i=1}^{k} z_i^* \Big(\overline{z_i'} + (x'^T D_1 w) - z'(\overline{z_i''} - (x'^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j'. \end{aligned} \tag{20}$$

As $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $\overline{z_i'} \in \mathcal{F}(x', \overline{y_i})$, we have

$$\mathcal{F}(x_0, \overline{y_i}) - \overline{z_i'} \subseteq D_{\uparrow}\mathcal{F}(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x_0}(0+)) + \rho_i \|x_0 - x'\|^2 + \mathbb{R}_+.$$

Again, as $z_i' \in \mathcal{F}(x_0, \overline{y_i})$, we have

$$z_i' - \overline{z_i'} \in D_{\uparrow}\mathcal{F}(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x_0}(0+)) + \rho_i \|x_0 - x'\|^2 + \mathbb{R}_+. \tag{21}$$

Since $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, we have

$$x_0^T D_1 w - x'^T D_1 w \geq D_1 w(\mathcal{H}'_{x',x_0}(0+)) + \overline{\rho_i} \|x_0 - x'\|^2 + \mathbb{R}_+. \tag{22}$$

As $-\mathcal{H}(., \overline{y_i})$ is ρ_i'-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $\overline{z_i''} \in \mathcal{H}(x', \overline{y_i})$, we have

$$-\mathcal{H}(x_0, \overline{y_i}) + \overline{z_i''} \subseteq D_{\uparrow}(-\mathcal{H})(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x_0}(0+)) + \rho_i' \|x_0 - x'\|^2 + \mathbb{R}_+.$$

Again, as $z_i'' \in \mathcal{H}(x_0, \overline{y_i})$, we have

$$-z_i'' + \overline{z_i''} \in D_{\uparrow}(-\mathcal{H})(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x_0}(0+)) + \rho_i' \|x_0 - x'\|^2 + \mathbb{R}_+. \tag{23}$$

Since $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, we have

$$x_0^T D_2 v - x'^T D_1 v \geq D_2 v(\mathcal{H}'_{x',x_0}(0+)) + \overline{\rho_i'} \|x_0 - x'\|^2 + \mathbb{R}_+. \tag{24}$$

As $\mathcal{G}_j$ is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $w'_j \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, we have

$$\mathcal{G}_j(x_0) - w'_j \subseteq D_\uparrow \mathcal{G}_j(x', w'_j)(\mathcal{H}'_{x',x_0}(0+)) + \nu_j \|x_0 - x'\|^2 + \mathbb{R}_+.$$

Since $w_j \in \mathcal{G}_j(x_0) \cap (-\mathbb{R}_+)$, we have

$$w_j - w'_j \in D_\uparrow \mathcal{G}_j(x', w'_j)(\mathcal{H}'_{x',x_0}(0+)) + \nu_j \|x_0 - x'\|^2 + \mathbb{R}_+. \tag{25}$$

From (17), (21) - (25), and the constraints of (MWD), we have

$$\begin{aligned}&\sum_{i=1}^{k} z_i^* \Big(z'_i + (x_0^T D_1 w) - z'(z''_i - (x_0^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j \\ &\geq \sum_{i=1}^{k} z_i^* \Big(\overline{z'_i} + (x'^T D_1 w) - z'(\overline{z''_i} - (x'^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w'_j,\end{aligned}$$

which contradicts (20). Therefore,

$$\max \bigcup_{y \in B} \Phi(x_0, y) \not\leq z',$$

which completes the proof of the theorem. □

Theorem 4 *(Strong duality) Let (x', z') be a minimizer of the problem* (MFP) *and $w'_j \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Assume that for some positive integer k, $(1 \leq k \leq n+1)$, $z_i^* \geq 0$, $\overline{y_i} \in B(x')$, $(1 \leq i \leq k)$ with $\sum_{i=1}^{k} z_i^* = 1$ and $w_j^* \geq 0$, $(1 \leq j \leq p)$, Eqs. (2) - (7) are satisfied at (x', z', w', z^*, w^*). Then, (x', z', w', z^*, w^*) is a feasible solution of* (MWD)*. If the weak duality Theorem 3 between* (MFP) *and* (MWD) *holds, then (x', z', w', z^*, w^*) is a maximizer of* (MWD).

Proof As Eqs. (2) - (7) are satisfied at (x', z', w', z^*, w^*),

$$\begin{aligned}\sum_{i=1}^{k} z_i^* \Big(D_\uparrow \mathcal{F}(., \overline{y_i})(x', \overline{y_i}) + D_1 w - z'(D_\uparrow(-\mathcal{H})(., \overline{y_i})(x', \overline{y_i}) - D_2 v) \Big)(\mathcal{H}'_{x',x}(0+)) \\ + \sum_{j=1}^{p} w_j^* D_\uparrow \mathcal{G}_j(x', w'_j)(\mathcal{H}'_{x',x}(0+)) \geq 0, \forall x \in A,\end{aligned}$$

$$\sum_{j=1}^{p} w_j^* w'_j = 0,$$

$$w^T D_1 w \leq 1, v^T D_2 v \leq 1,$$

$$(x'^T D_1 x')^{\frac{1}{2}} = x'^T D_1 w,$$

and

$$(x'^T D_2 x')^{\frac{1}{2}} = x'^T D_2 v.$$

Hence, (x', z', w', z^*, w^*) is a feasible solution of (MWD). Suppose that the weak duality Theorem 3 between (MFP) and (MWD) holds and (x', z', w', z^*, w^*) is not a maximizer of (MWD). Let (x, z, w, z_1^*, w_1^*) be a feasible point for (MWD) such that

$$z' < z.$$

This contradicts the weak duality Theorem 3 between (MFP) and (MWD). Consequently, (x', z', w', z^*, w^*) is a maximizer for (MWD). $\square$

THEOREM 5 *(Converse duality) Let A be an arcwise connected subset of $\mathbb{R}^n$, and (x', z', w', z^*, w^*) be a feasible point of* (MWD). *Assume that $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(., \overline{y_i})$ is ρ'_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, satisfying* (17). *If x' is a feasible point of* (MFP)*, then (x', y') is a minimizer of* (MFP).

PROOF Suppose that (x', z') is not a minimizer of the problem (MFP). Then there exist $x \in S'$ and $z = \max \bigcup_{y \in B} \Phi(x, y)$, with $x \neq x'$, such that

$$z < z'.$$

Since $\overline{y_i} \in B(x')$, $i = 1, ..., k$, we have

$$\max \bigcup_{y \in B} \Phi(x', y) \in \Phi(x', \overline{y_i}).$$

As $z' = \max \bigcup_{y \in B} \Phi(x', y)$, so that we have

$$z' \in \Phi(x', \overline{y_i}), i = 1, ..., k.$$

Let $z_i \in \Phi(x, \overline{y_i})$. Again, as $z = \max \bigcup_{y \in B} \Phi(x, y)$ and $\overline{y_i} \in B(x') \subseteq B$, we have

$$z_i \leq z.$$

Hence, $z_i < z'$. As $z' \in \Phi(x', \overline{y_i})$, there exist $\overline{z'_i} \in \mathcal{F}(x', \overline{y_i})$ and $\overline{z''_i} \in \mathcal{H}(x', \overline{y_i})$ such that

$$z' = \frac{\overline{z'_i} + (x'^T D_1 x')^{\frac{1}{2}}}{\overline{z''_i} - (x'^T D_2 x')^{\frac{1}{2}}}.$$

So,

$$\overline{z_i'} + (x'^T D_1 x')^{\frac{1}{2}} - z'(\overline{z_i''} - (x'^T D_2 x')^{\frac{1}{2}}) = 0, \forall i = 1, ..., k. \tag{26}$$

Since $z_i \in \Phi(x, \overline{y_i})$, there exist $z_i' \in \mathcal{F}(x, \overline{y_i})$ and $z_i'' \in \mathcal{H}(x, \overline{y_i})$ such that

$$z_i = \frac{z_i' + (x^T D_1 x)^{\frac{1}{2}}}{z_i'' - (x^T D_2 x)^{\frac{1}{2}}}.$$

Therefore,

$$\frac{z_i' + (x^T D_1 x)^{\frac{1}{2}}}{z_i'' - (x^T D_2 x)^{\frac{1}{2}}} < z'.$$

So,

$$z_i' + (x^T D_1 x)^{\frac{1}{2}} - z'(z_i'' - (x^T D_2 x)^{\frac{1}{2}}) < 0, \forall i = 1, ..., k. \tag{27}$$

We have

$$\sum_{i=1}^{k} z_i^* \Big(z_i' + (x^T D_1 w) - z'(z_i'' - (x^T D_2 v)) \Big)$$
$$\leq \sum_{i=1}^{k} z_i^* \Big(z_i' + (x^T D_1 x)^{\frac{1}{2}} - z'(z_i'' - (x^T D_2 x)^{\frac{1}{2}}) \Big), \text{(from (2) and the constraints of (MWD))}$$
$$< 0, \text{(from (27))}$$
$$= \sum_{i=1}^{k} z_i^* \Big(\overline{z_i'} + (x'^T D_1 x')^{\frac{1}{2}} - z'(\overline{z_i''} - (x'^T D_2 x')^{\frac{1}{2}}) \Big), \text{(from (26))}$$
$$= \sum_{i=1}^{k} z_i^* \Big(\overline{z_i'} + (x'^T D_1 w) - z'(\overline{z_i''} - (x'^T D_2 v)) \Big), \text{(from the constraints of (MWD))}.$$

As $x \in S'$, there exists $w_j \in \mathcal{G}_j(x) \cap (-\mathbb{R}_+)$. Since $w_j^* \geq 0$ $(1 \leq j \leq p)$, we have

$$w_j^* w_j \leq 0, \forall 1 \leq j \leq p.$$

So,

$$\sum_{j=1}^{p} w_j^* w_j \leq 0.$$

Again, from the constraints of (MWD), we have

$$\sum_{j=1}^{p} w_j^* w_j' \geq 0.$$

Therefore,

$$\sum_{j=1}^{p} w_j^* w_j \leq \sum_{j=1}^{p} w_j^* w_j'.$$

Hence,

$$\begin{aligned} &\sum_{i=1}^{k} z_i^* \Big(z_i' + (x^T D_1 w) - z'(z_i'' - (x^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j \\ &< \sum_{i=1}^{k} z_i^* \Big(\overline{z_i'} + (x'^T D_1 w) - z'(\overline{z_i''} - (x'^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j'. \end{aligned} \tag{28}$$

As $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $\overline{z_i'} \in \mathcal{F}(x', \overline{y_i})$, we have

$$\mathcal{F}(x, \overline{y_i}) - \overline{z_i'} \subseteq D_{\uparrow}\mathcal{F}(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i \|x - x'\|^2 + \mathbb{R}_+.$$

Again, as $z_i' \in \mathcal{F}(x, \overline{y_i})$, we have

$$z_i' - \overline{z_i'} \in D_{\uparrow}\mathcal{F}(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i \|x - x'\|^2 + \mathbb{R}_+. \tag{29}$$

Since $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, we have

$$x^T D_1 w - x'^T D_1 w \geq D_1 w(\mathcal{H}'_{x',x}(0+)) + \overline{\rho_i} \|x - x'\|^2 + \mathbb{R}_+. \tag{30}$$

As $-\mathcal{H}(., \overline{y_i})$ is ρ_i'-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $\overline{z_i''} \in \mathcal{H}(x', \overline{y_i})$, we have

$$-\mathcal{H}(x, \overline{y_i}) + \overline{z_i''} \subseteq D_{\uparrow}(-\mathcal{H})(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i' \|x - x'\|^2 + \mathbb{R}_+.$$

Again, as $z_i'' \in \mathcal{H}(x, \overline{y_i})$, we have

$$-z_i'' + \overline{z_i''} \in D_{\uparrow}(-\mathcal{H})(., \overline{y_i})(x', \overline{y_i})(\mathcal{H}'_{x',x}(0+)) + \rho_i' \|x - x'\|^2 + \mathbb{R}_+. \tag{31}$$

Since $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected with respect to 1, on A, we have

$$x^T D_2 v - x'^T D_1 v \geq D_2 v(\mathcal{H}'_{x',x}(0+)) + \overline{\rho_i'} \|x - x'\|^2 + \mathbb{R}_+. \tag{32}$$

As $\mathcal{G}_j$ is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1, on A and $w_j' \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, we have

$$\mathcal{G}_j(x) - w_j' \subseteq D_{\uparrow}\mathcal{G}_j(x', w_j')(\mathcal{H}'_{x',x}(0+)) + \nu_j \|x - x'\|^2 + \mathbb{R}_+.$$

Since $w_j \in \mathcal{G}_j(x) \cap (-\mathbb{R}_+)$, we have

$$w_j - w'_j \in D_\uparrow \mathcal{G}_j(x', w'_j)((\mathcal{H}'_{x',x}(0+)) + \nu_j \|x - x'\|^2 + \mathbb{R}_+. \tag{33}$$

From (17), (29) - (33), and the constraints of (MWD), we have

$$\begin{aligned}
&\sum_{i=1}^{k} z_i^* \Big(z'_i + (x^T D_1 w) - z'(z''_i - (x^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w_j \\
&\geq \sum_{i=1}^{k} z_i^* \Big(\overline{z'_i} + (x'^T D_1 w) - z'(\overline{z''_i} - (x'^T D_2 v)) \Big) + \sum_{j=1}^{p} w_j^* w'_j,
\end{aligned}$$

which contradicts (28). Consequently, (x', z') is a minimizer of the problem (MFP). □

7. Wolfe type dual

We consider a Wolfe type dual (WD) correspoing to the primal problem (MFP), where the set-valued maps $\mathcal{F}(., \overline{y_i})$, $-\mathcal{H}(., \overline{y_i})$ and $\mathcal{G}_j$ are contingent epiderivable set-valued maps. This amounts to

$$\text{maximize } z' + \sum_{j=1}^{p} w_j^* w'_j \tag{WD}$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{k} z_i^* \Big(D_\uparrow \mathcal{F}(., \overline{y_i})(x', \overline{y_i}) + D_1 w \\
&- z'(D_\uparrow(-\mathcal{H})(., \overline{y_i})(x', \overline{y_i}) - D_2 v) \Big) (\mathcal{H}'_{x',x}(0+)) \\
&+ \sum_{j=1}^{p} w_j^* D_\uparrow \mathcal{G}_j(x', w'_j)(\mathcal{H}'_{x',x}(0+)) \geq 0, \forall x \in A,
\end{aligned}$$

for some $k \in \mathbb{N}, (1 \leq k \leq n+1)$, and $\overline{y_i} \in B(x')$,

$$\begin{aligned}
&w^T D_1 w \leq 1, v^T D_2 v \leq 1, (x'^T D_1 x')^{\frac{1}{2}} = x'^T D_1 w, \\
&(x'^T D_2 x')^{\frac{1}{2}} = x'^T D_2 v, \text{for some } w, v \in \mathbb{R}^n, \\
&x' \in A, z' = \max \bigcup_{y \in B} \Phi(x', y), w' = (w'_1, ..., w'_p), w'_j \in \mathcal{G}_j(x'), \\
&z^* = (z_1^*, ..., z_k^*), w^* = (w_1^*, ..., w_p^*), z_i^* \geq 0, w_j^* \geq 0, \sum_{i=1}^{k} z_i^* = 1,
\end{aligned}$$

where $1 \leq i \leq k$ and $1 \leq j \leq p$.

DEFINITION 15 *A feasible point* (x', z', w', z^*, w^*) *of the problem* (WD) *is called a maximizer of* (WD) *if there exists no feasible point* $(x, z, w, \overline{z}^*, \overline{w}^*)$ *of* (WD) *such that*

$$z' + \sum_{j=1}^{p} w_j^* w_j' < z + \sum_{j=1}^{p} \overline{w}_j^* w_j,$$

where
$w' = (w_1', ..., w_p')$, $w^* = (w_1^*, ..., w_p^*)$, $w = (w_1, ..., w_p)$, *and* $\overline{w}^* = (\overline{w}_1^*, ..., \overline{w}_p^*)$.

We prove the duality results of Wolfe type of the problem (MFP). The proofs are very similar to those of Theorems 3 - 5, and hence are omitted.

THEOREM 6 *(Weak duality) Let A be an arcwise connected subset of $\mathbb{R}^n$, x_0 be a feasible point of* (MFP) *and* (x', z', w', z^*, w^*) *be a feasible point of the problem* (WD). *Assume that $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(., \overline{y_i})$ is ρ_i'-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to* 1, *on A, satisfying* (17). *Then,*

$$\max \bigcup_{y \in B} \Phi(x_0, y) \not< z' + \sum_{j=1}^{p} w_j^* w_j'.$$

THEOREM 7 *(Strong duality) Let* (x', z') *be a minimizer of the problem* (MFP) *and $w_j' \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Assume that for some positive integer k, $(1 \leq k \leq n+1)$, $z_i^* \geq 0$, $\overline{y_i} \in B(x')$, $(1 \leq i \leq k)$ with $\sum_{i=1}^{k} z_i^* = 1$ and $w_j^* \geq 0$, $(1 \leq j \leq p)$, Eqs. (2) - (7) are satisfied at (x', z', w', z^*, w^*). Then (x', z', w', z^*, w^*) is a feasible solution of* (WD). *If the weak duality Theorem 6 between* (MFP) *and* (WD) *holds, then (x', z', w', z^*, w^*) is a maximizer of* (WD).

THEOREM 8 *(Converse duality) Let A be an arcwise connected subset of $\mathbb{R}^n$ and (x', z', w', z^*, w^*) be a feasible point of* (WD), *with $\sum_{j=1}^{p} w_j^* w_j' \geq 0$. Assume that $\mathcal{F}(., \overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(., \overline{y_i})$ is ρ_i'-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to* 1, *on A, satisfying* (17). *If x' is a feasible point of* (MFP), *then (x', y') is a minimizer of* (MFP).

8. Mixed type dual

We consider a mixed type dual (MD) correspoing to the primal problem (MFP), where the set-valued maps $\mathcal{F}(., \overline{y_i})$, $-\mathcal{H}(., \overline{y_i})$ and $\mathcal{G}_j$ are contingent epiderivable.

This amounts to

$$\text{maximize } z' + \sum_{j=1}^{p} w_j^* w_j' \tag{MD}$$

subject to

$$\sum_{i=1}^{k} z_i^* \Big(D_{\uparrow}\mathcal{F}(.,\overline{y_i})(x',\overline{y_i}) + D_1 w$$
$$-z'(D_{\uparrow}(-\mathcal{H})(.,\overline{y_i})(x',\overline{y_i}) - D_2 v)\Big)(\mathcal{H}'_{x',x}(0+))$$
$$+\sum_{j=1}^{p} w_j^* D_{\uparrow}\mathcal{G}_j(x',w_j')(\mathcal{H}'_{x',x}(0+)) \geq 0, \forall x \in A,$$

for some $k \in \mathbb{N}, (1 \leq k \leq n+1)$ and $\overline{y_i} \in B(x')$,

$$\sum_{j=1}^{p} w_j^* w_j' \geq 0,$$
$$w^T D_1 w \leq 1, v^T D_2 v \leq 1, (x'^T D_1 x')^{\frac{1}{2}} = x'^T D_1 w,$$
$$(x'^T D_2 x')^{\frac{1}{2}} = x'^T D_2 v, \text{ for some } w, v \in \mathbb{R}^n,$$
$$x' \in A, z' = \max \bigcup_{y \in B} \Phi(x',y), w' = (w_1',...,w_p'), w_j' \in \mathcal{G}_j(x'),$$
$$z^* = (z_1^*,...,z_k^*), w^* = (w_1^*,...,w_p^*), z_i^* \geq 0, w_j^* \geq 0, \sum_{i=1}^{k} z_i^* = 1,$$

where $1 \leq i \leq k$ and $1 \leq j \leq p$.

Definition 16 *A feasible point (x',z',w',z^*,w^*) of the problem* (MD) *is called a maximizer of* (MD) *if there exists no feasible point $(x,z,w,\overline{z}^*,\overline{w}^*)$ of* (MD) *such that*

$$z' + \sum_{j=1}^{p} w_j^* w_j' < z + \sum_{j=1}^{p} \overline{w}_j^* w_j,$$

where
$w' = (w_1',...,w_p')$, $w^* = (w_1^*,...,w_p^*)$, $w = (w_1,...,w_p)$, and $\overline{w}^* = (\overline{w}_1^*,...,\overline{w}_p^*)$.

We prove the duality results of mixed type of the problem (MFP). The proofs are very similar to those of Theorems 3 - 5, and hence are omitted.

Theorem 9 *(Weak duality) Let A be an arcwise connected subset of $\mathbb{R}^n$, x_0 be a feasible point of* (MFP) *and (x',z',w',z^*,w^*) be a feasible point of the problem* (MD). *Assume that $\mathcal{F}(.,\overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is*

$\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(.,\overline{y_i})$ is ρ'_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to 1*, on A, satisfying* (17)*. Then,*

$$\max \bigcup_{y \in B} \Phi(x_0, y) \not< z'.$$

THEOREM 10 (STRONG DUALITY) *Let (x', z') be a minimizer of the problem* (MFP) *and $w'_j \in \mathcal{G}_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Assume that for some positive integer k, $(1 \leq k \leq n+1)$, $z^*_i \geq 0$, $\overline{y_i} \in B(x')$, $(1 \leq i \leq k)$ with $\sum_{i=1}^{k} z^*_i = 1$ and $w^*_j \geq 0$, $(1 \leq j \leq p)$, Eqs. (2) - (7) are satisfied at (x', z', w', z^*, w^*). Then (x', z', w', z^*, w^*) is a feasible solution of* (MD)*. If the weak duality Theorem 9 between* (MFP) *and* (MD) *holds, then (x', z', w', z^*, w^*) is a maximizer of* (MD).

THEOREM 11 (CONVERSE DUALITY) *Let A be an arcwise connected subset of $\mathbb{R}^n$ and (x', z', w', z^*, w^*) be a feasible point of* (MD)*. Assume that $\mathcal{F}(.,\overline{y_i})$ is ρ_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_1 w$ is $\overline{\rho}_i$-$\mathbb{R}_+$-arcwise connected, $-\mathcal{H}(.,\overline{y_i})$ is ρ'_i-$\mathbb{R}_+$-arcwise connected, $(.)^T D_2 v$ is $\overline{\rho'}_i$-$\mathbb{R}_+$-arcwise connected and $\mathcal{G}_j$, $(1 \leq j \leq p)$, is ν_j-$\mathbb{R}_+$-arcwise connected with respect to* 1*, on A, satisfying* (17)*. If x' is a feasible point of* (MFP)*, then (x', y') is a minimizer of* (MFP).

9. Conclusions

In this paper, we establish the sufficient KKT conditions of a set-valued minimax fractional programming problem (MFP) and study the duality results of Mond-Weir (MWD), Wolfe (WD), and mixed (MD) types under ρ-cone arcwise connectedness assumption.

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