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GROWTH OF SOLUTIONS OF A CLASS OF LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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Abstract. This paper is devoted to the study of the growth of solutions of certain class of linear fractional differential equations with polynomial coefficients involving the Caputo fractional derivatives by using the generalized Wiman–Valiron theorem in the fractional calculus.

Keywords: linear fractional differential equations, growth of solutions, Caputo fractional derivative operator.

Mathematics Subject Classification: 34M10, 26A33.

1. INTRODUCTION

The order of growth of an entire function $f(z)$ is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ m(r,f)}{\log r},$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi,$$

and we have

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$ (for more details see [5,8,14]). Also, the order of an entire function given by $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ is equal to

$$\sigma(f) = \limsup_{n\to+\infty} \frac{n\log n}{-\log|a_n|}$$

(see [2]).

Consider the linear differential equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \ldots + P_1(z)f' + P_0(z)f = 0, \tag{1.1}$$

where $P_0(z) \not\equiv 0, P_1(z), \ldots, P_{n-1}(z)$ are polynomials. It is well known that every solution f of equation (1.1) is an entire function of finite rational order $\sigma(f)$ satisfying

$$\sigma(f) \leq 1 + \max_{0 \leq k \leq n-1} \frac{\deg P_k}{n-k} \tag{1.2}$$

(see [6, 8, 12, 13]). In [4], Gundersen *et al.* gave the possible orders of solutions of (1.1). As particular cases, Belaidi and Hamani proved the following result.

Theorem 1.1 ([1])**.** *Let $P_0(z), P_1(z), \ldots, P_{n-1}(z)$ be nonconstant polynomials with degrees $d_k = \deg P_k(z)$ $(k = 0, 1, \ldots, n-1)$.*

(i) *If $\frac{d_0}{n} \geq \frac{d_k}{n-k}$ holds for all $k = 1, \ldots, n-1$, then any solution $f \not\equiv 0$ of (1.1) satisfies $\sigma(f) = 1 + \frac{d_0}{n}$.*
(ii) *If $d_k < d_{n-1} - (n-k-1)$ holds for all $k = 0, \ldots, n-2$, then any solution $f \not\equiv 0$ of (1.1) satisfies $\sigma(f) = 1 + d_{n-1}$.*

Fractional order differential equations have become a very important tool for modeling phenomena in many diverse fields of science and engineering which traditional differential modeling cannot accomplish (see, for example, Kilbas *et al.* [7]). In present, three kinds of fractional derivatives are often used, the Grünwald–Letnikov derivative, the Riemann–Liouville derivative and the Caputo derivative. There are many discussions for properties of these derivatives, see [9, 10]. All these studies are limited in real line. In this paper, we will use the Caputo derivative which is defined as follows.

Definition 1.2 ([7, 10, 11])**.** Suppose that $\alpha > 0$ and $r > 0$. The fractional operator

$$\mathcal{D}^\alpha f(r) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^r \frac{f^{(n)}(t)}{(r-t)^{\alpha+1-n}} dt, & n-1 < \alpha < n, \\ \frac{d^n}{dr^n} f(r), & \alpha = n \in \mathbb{N} \setminus \{0\} \end{cases}$$

is called the Caputo derivative. It is understood that f should be n time continuously differentiable.

Consider the function $f(z) = \sum_{j=0}^{+\infty} a_j z^j$, where $z = re^{i\theta}$. By using the properties of the Caputo operator derivative, for $n-1 < \alpha < n$, we have

$$\mathcal{D}^\alpha f(z) = \sum_{j=n}^{+\infty} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} a_j r^{j-\alpha} e^{ji\theta}, \tag{1.3}$$

$$r^\alpha \mathcal{D}^\alpha f(z) = \sum_{j=n}^{+\infty} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} a_j z^j.$$

For $\alpha = n \in \mathbb{N} \setminus \{0\}$,

$$\mathcal{D}^{\alpha} f(z) = \frac{d^n}{dr^n} f\left(re^{i\theta}\right) \neq \frac{d^n}{dz^n} f\left(z\right),$$

while

$$\frac{r^{\alpha}}{z^{[\alpha]}} \mathcal{D}^{\alpha} f(z) = \frac{d^n}{dz^n} f(z).$$

Proposition 1.3. *The two functions $f(z) = \sum_{j=0}^{+\infty} a_j z^j$ and $r^{\alpha} D^{\alpha} f(z)$ have the same radius of convergence. Consequently, if $f(z)$ is an entire function, then $r^{\alpha} D^{\alpha} f(z)$ is equally an entire function.*

Proof. To prove that the two power series

$$f(z) = \sum_{j=0}^{+\infty} a_j z^j, \quad r^{\alpha} D^{\alpha} f(z) = \sum_{j=n}^{+\infty} \frac{\Gamma\left(j+1\right)}{\Gamma\left(j-\alpha+1\right)} a_j z^j,$$

have the same radius of convergence, we have just to show that

$$\lim_{j\to+\infty} \frac{\Gamma\left(j+1\right)}{\Gamma\left(j-\alpha+1\right)} \frac{\Gamma\left(j-\alpha+2\right)}{\Gamma\left(j+2\right)} = 1.$$

By the property asymptotic of Gamma function near the infinity, we have

$$\lim_{j\to+\infty} \frac{\Gamma\left(j+1\right)}{\Gamma\left(j-\alpha+1\right)} \frac{\Gamma\left(j-\alpha+2\right)}{\Gamma\left(j+2\right)} = \lim_{j\to+\infty} j^{\alpha}.j^{-\alpha}\left(1+o\left(1\right)\right) = 1.$$ □

Recently, Chyzhykov and Semochko generalized the Wiman–Valiron method for fractional derivatives and as an application to fractional differential equations, they proved the following result.

Theorem 1.4 ([3]). *Let $a(z)$ be a polynomial of degree $m \geq 0$. Then all nontrivial solutions f of the equation*

$$\widetilde{\mathcal{D}}^{q}\left(r^{q} f(z)\right) + z a(z) f(z) = 0 \tag{1.4}$$

have the order of growth $\rho = \frac{m+1}{q}$, where $\widetilde{\mathcal{D}}^{q} f(z) = \mathcal{D}^{q}_{RL} f(z) - \Gamma\left(q+1\right) f\left(0\right)$ and $\mathcal{D}^{q}_{RL} f(z)$ is the Riemann–Liouville fractional derivative operator.

Remark 1.5. By using the power series method, we can confirm that (1.4) does not admit any entire solutions $f \not\equiv 0$. Corollary 1.9 below might be the alternative result of Theorem 1.4.

In this paper, we will investigate the growth of solutions of certain class of linear fractional differential equations by using the Caputo fractional derivative operator as the following.

Theorem 1.6. *Let $P_0(z) \not\equiv 0, P_1(z), \dots, P_{n-1}(z)$ be polynomials such that $P_0(0) = 0$. Let $0 = q_0 < q_1 < q_2 < \dots < q_n$. Then all solutions of the linear fractional differential equation*

$$\begin{aligned} &\frac{r^{q_n}}{z^{[q_n]}}\mathcal{D}^{q_n} f(z) + P_{n-1}(z)\frac{r^{q_{n-1}}}{z^{[q_{n-1}]}}\mathcal{D}^{q_{n-1}} f(z) + \dots + P_1(z)\frac{r^{q_1}}{z^{[q_1]}}\mathcal{D}^{q_1} f(z) \\ &+ P_0(z)f(z) = 0. \end{aligned} \tag{1.5}$$

are entire functions of order of growth $\sigma(f)$ satisfying

$$\sigma(f) \leq \max_{0 \leq k \leq n-1} \left\{ \frac{d_k + [q_n] - [q_k]}{q_n - q_k} \right\},$$

where $d_k = \deg P_k(z)$ and $[x]$ is the greatest integer less than or equal to the real number x.

Corollary 1.7. *Let $P_0(z) \not\equiv 0, P_1(z), \dots, P_n(z)$ be polynomials such that $P_0(0) = 0$ and $0 < \alpha < 1$. Then, every solution of the linear fractional differential equation*

$$\begin{aligned} &\frac{r^{n+\alpha}}{z^n}\mathcal{D}^{n+\alpha} f(z) + P_n(z)\frac{r^{n-1+\alpha}}{z^{n-1}}\mathcal{D}^{n-1+\alpha} f(z) + \dots + P_2(z)\frac{r^{1+\alpha}}{z}\mathcal{D}^{1+\alpha} f(z) \\ &+ P_1(z) r^\alpha \mathcal{D}^\alpha f(z) + P_0(z)f(z) = 0. \end{aligned} \tag{1.6}$$

is an entire function of order of growth $\sigma(f)$ satisfying

$$\sigma(f) \leq \frac{1}{\alpha} + \max_{0 \leq k \leq n-1} \left\{ \frac{d_k}{\alpha(n-k)} \right\}.$$

In the following theorem, we give the precise value of the order of growth of solutions of (1.5).

Theorem 1.8. *Suppose that we have the same assumptions of Theorem 1.6.*

(i) *If*

$$\frac{d_0 + [q_n]}{q_n} \geq \frac{d_k + [q_n] - [q_k]}{q_n - q_k}, \tag{1.7}$$

holds for all $k = 1, \dots, n-1$, then every solution $f \not\equiv 0$ of (1.5) is an entire function of order of growth

$$\sigma(f) = \frac{d_0 + [q_n]}{q_n}.$$

(ii) *If*

$$d_k - [q_k] < d_{n-1} - [q_{n-1}] \tag{1.8}$$

holds for all $k = 0, 1, \dots, n-2$, then every solution $f \not\equiv 0$ of (1.5) is an entire function of order of growth

$$\sigma(f) = \frac{d_{n-1} + [q_n] - [q_{n-1}]}{q_n - q_{n-1}}.$$

Corollary 1.9. *Let $P_0(z) \not\equiv 0$ be polynomial of degree d_0 such that $P_0(0) = 0$ and $\alpha > 0$. Then every solution $f \not\equiv 0$ of the linear fractional differential equation*

$$\frac{r^{\alpha}}{z^{[\alpha]}}\mathcal{D}^{\alpha} f(z) + P_0(z) f(z) = 0$$

is an entire function of order of growth

$$\sigma(f) = \frac{d_0 + [\alpha]}{\alpha}.$$

Corollary 1.10. *Let $0 < \alpha < \beta$. Let $P_0(z) \not\equiv 0$ and $P_1(z) \not\equiv 0$ be polynomials of degrees d_0 and d_1, respectively, such that $P_0(0) = 0$ and $d_0 < d_1 - [\alpha]$. Then, every solution $f \not\equiv 0$ of the linear fractional differential equation*

$$\frac{r^{\beta}}{z^{[\beta]}}\mathcal{D}^{\beta} f(z) + P_1(z)\frac{r^{\alpha}}{z^{[\alpha]}}\mathcal{D}^{\alpha} f(z) + P_0(z) f(z) = 0$$

is an entire function of order of growth

$$\sigma(f) = \frac{d_1 + [\beta] - [\alpha]}{\beta - \alpha}.$$

Example 1.11. Consider the fractional differential equation

$$r^{\alpha}\mathcal{D}^{\alpha} f(z) + z f(z) = 0, \tag{1.9}$$

where $0 < \alpha < 1$. By Corollary 1.9, every solution $f \not\equiv 0$ of (1.9) is an entire function of order of growth $\sigma(f) = \frac{d_0+[\alpha]}{\alpha} = \frac{1}{\alpha}$. We confirm this by using the power series method. Set $f(z) = \sum_{j=0}^{+\infty} a_j z^j$. By (1.9), we find that

$$a_j = (-1)^j a_0 \prod_{k=1}^{j} \frac{\Gamma(k - \alpha + 1)}{\Gamma(k+1)}.$$

By the property asymptotic of Gamma function near the infinity, we have

$$\frac{\Gamma(j - \alpha + 1)}{\Gamma(j+1)} = j^{-\alpha}(1 + o(1)), \quad j \to +\infty,$$

so there exist $c > 0$, $j_0 > 0$ such that

$$|a_j| = c j^{-(j-j_0)\alpha}(1 + o(1)), \quad j \to +\infty$$

and then $\sigma(f) = \frac{1}{\alpha}$.

Example 1.12. Consider the fractional differential equation

$$\frac{r^{\alpha}}{z}\mathcal{D}^{\alpha} f(z) + zf\left(z\right) = 0, \tag{1.10}$$

where $1 < \alpha < 2$. By Corollary 1.9, every solution $f \not\equiv 0$ of (1.10) is an entire function of order of growth $\sigma\left(f\right) = \frac{d_0+[\alpha]}{\alpha} = \frac{2}{\alpha}$. In fact, the solutions of (1.10) are in the form $f(z) = \sum_{j=0}^{+\infty} a_j z^j$ such that

$$a_{2j} = \left(-1\right)^j a_0 \prod_{k=1}^{j} \frac{\Gamma\left(2k-\alpha+1\right)}{\Gamma\left(2k+2\right)}, \quad j \geq 1,$$

$$a_{2j+1} = \left(-1\right)^j a_1 \prod_{k=1}^{j} \frac{\Gamma\left(2k-\alpha+3\right)}{\Gamma\left(2k+3\right)}, \quad j \geq 0.$$

Set

$$f_1(z) = \sum_{j=0}^{+\infty} b_{2j} z^{2j}, \quad f_2(z) = \sum_{j=0}^{+\infty} b_{2j+1} z^{2j+1},$$

where $b_{2j} = \frac{a_{2j}}{a_0}$ and $b_{2j+1} = \frac{a_{2j+1}}{a_1}$. Hence $\{f_1, f_2\}$ forms the fundamental set of solutions of (1.10). By the same method of Example 1.11, we find that $\sigma\left(f_1\right) = \sigma\left(f_2\right) = \frac{2}{\alpha}$. Since $f_1(z)$ and $f_2(z)$ are linearly independents, we conclude that every solution $f \not\equiv 0$ of (1.10) is an entire function of order of growth $\sigma\left(f\right) = \frac{2}{\alpha}$.

2. PRELIMINARY LEMMAS

For the proof of our results we need the following lemmas.

Lemma 2.1 ([3])**.** *Let $f(z)$ be an entire function, $\alpha > 0$, $0 < \delta < \frac{1}{4}$ and z be such that $|z| = r$ and that*

$$|f(z)| > M\left(r, f\right) \nu\left(r\right)^{-\frac{1}{4}+\delta}$$

holds; where $\nu\left(r\right)$ is the central index of f. Then there exists a set $E \subset (0, +\infty)$ of finite logarithmic measure, that is $\int_E \frac{dt}{t} < +\infty$, such that

$$\frac{r^{\alpha}\mathcal{D}^{\alpha} f(z)}{f\left(z\right)} = \left(\nu\left(r\right)\right)^{\alpha}\left(1 + o\left(1\right)\right) \tag{2.1}$$

holds for $r \to +\infty$ and $r \notin E$.

Remark 2.2. We signal here that the fractional derivative used in the proof of Lemma 2.1 is the Riemann–Liouville operator and for an entire function $f(z) = \sum_{j=0}^{+\infty} a_j z^j$ we have

$$\mathcal{D}_{RL}^{\alpha} f(z) = \sum_{j=0}^{+\infty} \frac{\Gamma\left(j+1\right)}{\Gamma\left(j-\alpha+1\right)} a_j r^{j-\alpha} e^{ji\theta}. \tag{2.2}$$

By (1.3) and (2.2), we conclude that the proof of Lemma 2.1 is valid also for the Caputo fractional derivative operator.

Lemma 2.3 ([8]). *Let $f(z)$ be an entire function of finite order $\sigma(f) < +\infty$. Then*

$$\limsup_{r\to+\infty}\frac{\log^+\nu(r)}{\log r}=\sigma(f),$$

where $\nu(r)$ is the central index of f.

Lemma 2.4 ([8]). *Let $P(z) = a_n z^n + \ldots + a_0$ be a polynomial of degree n. Then, for any given $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r = |z| > r_0$ the inequalities*

$$(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n$$

hold.

Lemma 2.5. *Let $P_0(z) \not\equiv 0, P_1(z), \ldots, P_{n-1}(z)$ be polynomials such that $P_0(0) = 0$ and let $0 < q_1 < q_2 < \ldots < q_n$ be real constants. Then, all solutions of (1.5) are entire functions.*

Proof. We will use the power series method. Set $f(z) = \sum_{j=0}^{+\infty} a_j z^j$. Without lost of generality, we can suppose that $k-1 < q_k < k$ $(k = 1, 2, \ldots, n)$. Then we have

$$\begin{aligned}
P_k(z)\frac{r^{q_k}}{z^{[q_k]}}\mathcal{D}^{q_k}f(z) &= P_k(z)\frac{r^{q_k}}{z^{k-1}}\sum_{j=k}^{+\infty}\frac{\Gamma(j+1)}{\Gamma(j-q_k+1)}a_j r^{j-q_k}e^{ji\theta}\\
&= P_k(z)\sum_{j=k}^{+\infty}\frac{\Gamma(j+1)}{\Gamma(j-q_k+1)}a_j z^{j-k+1}\\
&= P_k(z)\sum_{j=k}^{+\infty}b_{k,j}a_j z^{j-k+1},
\end{aligned}$$

where $b_{k,j} = \frac{\Gamma(j+1)}{\Gamma(j-q_k+1)}$. Since $P_0(0) = 0$, we can write $P_0(z) = z\widetilde{P}_0(z)$ and by dividing the equation (1.5) by z we get

$$\begin{aligned}
&\sum_{j=n}^{+\infty}b_{n,j}a_j z^{j-n} + P_{n-1}(z)\sum_{j=n-1}^{+\infty}b_{n-1,j}a_j z^{j-n+1} + \ldots\\
&+ P_1(z)\sum_{j=1}^{+\infty}b_{1,j}a_j z^{j-1} + \widetilde{P}_0(z)\sum_{j=0}^{+\infty}a_j z^j = 0.
\end{aligned}$$

What remains in this method is well known: as in the classical case of linear differential equations, by identification, we can determine a_j $(j = n, n+1, \ldots)$ by the first n terms a_j $(j = 0, 1, \ldots, n-1)$ and then we conclude that the global solutions of (1.5) are entire functions that contains n parameters. □

3. PROOF OF THEOREMS

Proof of Theorem 1.6. We suppose to the contrary that there exists a solution f of (1.5) of order

$$\sigma(f) > \max_{0\le k\le n-1}\left\{\frac{d_k+[q_n]-[q_k]}{q_n-q_k}\right\}, \tag{3.1}$$

and we prove that this leads to a contradiction. From (1.5) we can write

$$\begin{aligned}\left|\frac{r^{q_n}\mathcal{D}^{q_n}f(z)}{z^{[q_n]}f(z)}\right| \le |P_{n-1}(z)|\left|\frac{r^{q_{n-1}}\mathcal{D}^{q_{n-1}}f(z)}{z^{[q_{n-1}]}f(z)}\right| + \dots \\ + |P_1(z)|\left|\frac{r^{q_1}\mathcal{D}^{q_1}f(z)}{z^{[q_1]}f(z)}\right| + |P_0(z)|.\end{aligned} \tag{3.2}$$

By Lemma 2.4, there exists $c_j > 0$ and $r_a \ge 0$ such that for all $r \ge r_a$ we have

$$|P_j(z)| \le c_j r^{d_j}. \tag{3.3}$$

By Lemma 2.1, there exists a set $E \subset (0,+\infty)$ of finite logarithmic measure, such that for $r \to +\infty$ and $r \notin E$, we have

$$\left|\frac{r^{q_k}\mathcal{D}^{q_k}f(z)}{f(z)}\right| = (\nu(r))^{q_k}(1+o(1)), \quad j=1,\dots,n. \tag{3.4}$$

Using (3.3)–(3.4) in (3.2), we obtain

$$\frac{1}{r^{[q_n]}}(\nu(r))^{q_n}(1+o(1)) \le \sum_{k=0}^{n-1} c_k r^{d_k}\frac{1}{r^{[q_k]}}(\nu(r))^{q_k}(1+o(1)). \tag{3.5}$$

By Lemma 2.3, for every $\varepsilon > 0$ there exist $r_b \ge 0$ such that for all $r \ge r_b$ we have

$$\nu(r) \le r^{\sigma+\varepsilon}, \tag{3.6}$$

where $\sigma(f) = \sigma$. On the other hand for $\varepsilon > 0$, there exists a sequence $r_m \to +\infty$ when $m \to +\infty$ such that

$$\nu(r_m) \ge r_m^{\sigma-\varepsilon}, \tag{3.7}$$

Combining (3.6)–(3.7) with (3.5), we get

$$\frac{1}{2}r_m^{q_n(\sigma-\varepsilon)} \le 2\sum_{k=0}^{n-1} c_k r_m^{[q_n]-[q_k]} r_m^{d_k} r_m^{q_k(\sigma+\varepsilon)},$$

which implies

$$1 \le 4\sum_{k=0}^{n-1} c_k r_m^{[q_n]-[q_k]-q_n(\sigma-\varepsilon)+d_k+q_k(\sigma+\varepsilon)}. \tag{3.8}$$

Now we prove that all the powers of r_m in (3.8) are negative as $m \to +\infty$. From (3.1) there exists $\varepsilon > 0$ small enough such that

$$\sigma - \max_{0 \le k \le n-1} \left\{ \frac{d_k + [q_n] - [q_k]}{q_n - q_k} \right\} > \beta\varepsilon \tag{3.9}$$

where β is any constant such that $\beta > \max_{0 \le k \le n-1} \left\{ \frac{q_n + q_k}{q_n - q_k} \right\}$. We have

$$\begin{aligned} &[q_n] - [q_k] - q_n (\sigma - \varepsilon) + d_k + q_k (\sigma + \varepsilon) \\ &= (q_n - q_k) \left(\frac{d_k + [q_n] - [q_k]}{q_n - q_k} - \sigma \right) + (q_n + q_k)\, \varepsilon \end{aligned} \tag{3.10}$$

From (3.9) we get

$$(q_n - q_k) \left(\frac{d_k + [q_n] - [q_k]}{q_n - q_k} - \sigma \right) + (q_n + q_k)\, \varepsilon < 0,$$

and then a contradiction follows in (3.8) as $m \to +\infty$. □

Proof of Theorem 1.8. (i) From Theorem 1.6 and the assumption (1.7), we have $\sigma(f) \le \frac{d_0 + [q_n]}{q_n}$. It remains to prove the inverse inequality $\sigma(f) \ge \frac{d_0 + [q_n]}{q_n}$. We suppose to the contrary that $\sigma(f) < \frac{d_0 + [q_n]}{q_n}$ and we prove that this leads to a contradiction. Set $\sigma = \sigma(f) = \frac{d_0 + [q_n]}{q_n} - 2\varepsilon$ for $\varepsilon > 0$ From (1.5) we can write

$$\begin{aligned} |P_0(z)| \le & \left| \frac{r^{q_n} \mathcal{D}^{q_n} f(z)}{z^{[q_n]} f(z)} \right| + |P_{n-1}(z)| \left| \frac{r^{q_{n-1}} \mathcal{D}^{q_{n-1}} f(z)}{z^{[q_{n-1}]} f(z)} \right| + \dots \\ & + |P_1(z)| \left| \frac{r^{q_1} \mathcal{D}^{q_1} f(z)}{z^{[q_1]} f(z)} \right|. \end{aligned} \tag{3.11}$$

By Lemma 2.4, there exists $c > 0$ and $r_a > 0$ such that for all $r \ge r_a$ we have

$$|P_0(z)| \ge c r^{d_0}. \tag{3.12}$$

Using (3.3), (3.4), (3.6) and (3.12) in (3.11), for r large enough, we obtain

$$c r^{d_0} \le r^{q_n(\sigma + \varepsilon) - [q_n]} + \sum_{k=1}^{n-1} c_k r^{d_k + q_k(\sigma + \varepsilon) - [q_k]},$$

which implies

$$c \le r^{q_n(\sigma + \varepsilon) - [q_n] - d_0} + \sum_{k=1}^{n-1} c_k r^{d_k + q_k(\sigma + \varepsilon) - [q_k] - d_0}. \tag{3.13}$$

Now, we will prove that all the powers of r in (3.13) are negative. Since $\sigma = \frac{d_0 + [q_n]}{q_n} - 2\varepsilon$, first we have

$$q_n (\sigma + \varepsilon) - [q_n] - d_0 = q_n \left(\frac{d_0 + [q_n]}{q_n} - \varepsilon \right) - [q_n] - d_0 = -q_n \varepsilon. \tag{3.14}$$

Secondly, by taking account the assumption (1.7), we get

$$\begin{aligned} & d_k + q_k(\sigma+\varepsilon) - [q_k] - d_0 \\ & = d_k + q_k\left(\frac{d_0+[q_n]}{q_n} - \varepsilon\right) - [q_k] - d_0 \\ & = \frac{d_k+[q_n]-[q_k]}{q_n-q_k}(q_n-q_k) + q_k\left(\frac{d_0+[q_n]}{q_n} - \varepsilon\right) - [q_n] - d_0 \\ & \le \frac{d_0+[q_n]}{q_n}(q_n-q_k) + q_k\left(\frac{d_0+[q_n]}{q_n} - \varepsilon\right) - [q_n] - d_0 \le -q_k\varepsilon. \end{aligned} \tag{3.15}$$

So, a contradiction follows when $r \to +\infty$ in (3.13), and then the proof of (i) is completed.

(ii) From the assumption (1.8) we have

$$\frac{d_k+[q_n]-[q_k]}{q_n-q_k} < \frac{d_{n-1}+[q_n]-[q_{n-1}]}{q_n-q_{n-1}} \tag{3.16}$$

for all $k = 0, 1, \dots, n-2$, and by Theorem 1.6, we obtain

$$\sigma(f) \le \frac{d_{n-1}+[q_n]-[q_{n-1}]}{q_n-q_{n-1}}.$$

For the inverse inequality we suppose to the contrary that

$$\sigma(f) < \frac{d_{n-1}+[q_n]-[q_{n-1}]}{q_n-q_{n-1}}$$

and we prove that this leads to a contradiction. Set

$$\sigma = \sigma(f) = \frac{d_{n-1}+[q_n]-[q_{n-1}]}{q_n-q_{n-1}} - \lambda\varepsilon, \tag{3.17}$$

where λ is any constant such that $\lambda > \frac{q_n+q_{n-1}}{q_n-q_{n-1}}$ and $\varepsilon > 0$ small enough. From (1.5) we can write

$$\begin{aligned} |P_{n-1}(z)|\left|\frac{r^{q_{n-1}}\mathcal{D}^{q_{n-1}}f(z)}{z^{[q_{n-1}]}f(z)}\right| \le & \left|\frac{r^{q_n}\mathcal{D}^{q_n}f(z)}{z^{[q_n]}f(z)}\right| \\ & + |P_{n-2}(z)|\left|\frac{r^{q_{n-2}}\mathcal{D}^{q_{n-2}}f(z)}{z^{[q_{n-2}]}f(z)}\right| + \dots \\ & + |P_1(z)|\left|\frac{r^{q_1}\mathcal{D}^{q_1}f(z)}{z^{[q_1]}f(z)}\right| + |P_0(z)|. \end{aligned} \tag{3.18}$$

By using (3.4), Lemma 2.3, Lemma 2.4, (3.6) and (3.7) in (3.18), we obtain

$$c_{n-1}r_m^{d_{n-1}+q_{n-1}(\sigma-\varepsilon)-[q_{n-1}]} \le r_m^{q_n(\sigma+\varepsilon)-[q_n]} + \sum_{k=0}^{n-2} c_k r_m^{d_k+q_k(\sigma+\varepsilon)-[q_k]},$$

which implies

$$1 \leq \frac{1}{c_{n-1}} r_m^{q_n(\sigma+\varepsilon)-[q_n]-d_{n-1}-q_{n-1}(\sigma-\varepsilon)+[q_{n-1}]} + \sum_{k=0}^{n-2} \frac{c_k}{c_{n-1}} r_m^{d_k+q_k(\sigma+\varepsilon)-[q_k]-d_{n-1}-q_{n-1}(\sigma-\varepsilon)+[q_{n-1}]}. \tag{3.19}$$

As above we prove that all the powers of r_m of (3.19) are negative. By (3.17), we get

$$\begin{aligned} &q_n(\sigma+\varepsilon)-[q_n]-d_{n-1}-q_{n-1}(\sigma-\varepsilon)+[q_{n-1}] \\ &= (q_n+q_{n-1})\,\varepsilon + (q_n-q_{n-1})\left(\sigma - \frac{d_{n-1}+[q_n]-[q_{n-1}]}{q_n-q_{n-1}}\right) \\ &= -\lambda\varepsilon\,(q_n-q_{n-1}) + (q_n+q_{n-1})\,\varepsilon \\ &= [(q_n+q_{n-1}) - \lambda\,(q_n-q_{n-1})]\,\varepsilon < 0. \end{aligned}$$

Also, from the assumption (1.8), we have

$$\begin{aligned} &d_k + q_k(\sigma+\varepsilon) - [q_k] - d_{n-1} - q_{n-1}(\sigma-\varepsilon) + [q_{n-1}] \\ &< q_k(\sigma+\varepsilon) - q_{n-1}(\sigma-\varepsilon) \\ &< (q_k - q_{n-1})\,\sigma + (q_n+q_{n-1})\,\varepsilon < 0, \end{aligned}$$

for $0 < \varepsilon < \frac{(q_{n-1}-q_k)\sigma}{q_n+q_{n-1}}$. So, (3.19) leads to a contradiction when $r_m \to +\infty$, which completes the proof of (ii). □

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