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Modelling process of municipal wastewater quality – backgrounds and preliminary application

Keywords

sewage, wastewater treatment plant, nutrients, chemical oxygen demand, semi-Markov process

Abstract

The probabilistic model of municipal wastewater quality process is proposed in the chapter. The methods of its characteristics and parameters statistical identification and prediction are presented. Next the proposed model is practically applied to examine and characterized the quality of the municipal wastewater collected in the exemplary sewage treatment plant.

1. Introduction

The impact of wastewater discharges to marine water bodies is of increasing interest. The flow from some wastewater discharges can be an important contribution to the water condition, if they are not of the right environmental quality or managed well. These issues need for a more detailed assessment of the influence of raw sewage composition on proper treatment to support the decision-making process within the treatment plant management.

The chapter is organized into 5 parts, this Introduction as Section 1, Sections 2–4 and Conclusion as Section 5. Section 2 is devoted to the problems and methods of wastewater quality assessment based on its factors such as daily amount delivered to the treatment plant and parameters: chemical oxygen demand and nutrients concentration. In Section 3, the semi-Markov model of municipal wastewater quality process is introduced and presented. In Section 4, the proposed model is applied to the exemplary urbanised area. The model is examined and its parameters and characteristics such as the states of particular wastewater factors, the limit values of transient probabilities and mean total sojourn times staying at the wastewater factor states, for the fixed time interval are determined. Finally, the evaluation of results is discussed. The possibility of the presented model's wider applications in the field considered in this chapter is suggested in Conclusion.

2. Wastewater quality assessment

The municipal wastewater is produced mainly in households and public utility buildings. It contains dispersed in water various organic and inorganic substances coming from: human excrements, foodstuff residues, fats, surfactants, fabric fragments, sand and ash. The wastewater also contains a variety of microorganisms, including bacteria, viruses and parasites.

The basic parameter describing the degree of pollution of wastewater by organic substances is the biological oxygen demand (BOD) or the chemical oxygen demand (COD). In addition, it is necessary to determine the biogenic compounds: nitrogen (ammonium nitrogen – NH_4-N) and phosphorus (phosphate – PO_4-P) (Tchobanoglous et al., 2014).

The priority task of wastewater treatment plants

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(WWTPs) is to reduce pollutants in order to protect the aquatic life in the receiving water body. One of the most serious environmental problems in the Baltic Sea region is eutrophication caused by excessive loads of nitrogen and phosphorus compounds. The Baltic is a shallow, semienclosed brackish sea which receives nutrients and pollutants from agricultural, industrial and wastewater run-off, which has led to the serious deterioration of the water quality. Consequently, the Baltic Sea is considered to be one of the most contaminated sea areas in the world (Baltic University, 2003).

The wastewater management is an essential element of the critical infrastructure. A disruption of the WWTP operation can result in the discharge of wastewater into the environment without adequately treated. Consequently, it can cause health problems, the contamination of soil, groundwater and surface water as well. The effectiveness of wastewater treatment is affected by the composition of the raw sewage entering the treatment plant. The type and concentration of pollutants also has a significant impact on the risk of environmental contamination in the event of pipeline damage. Typically, substances such as heavy metals and organic toxins are the focus of ecological risk assessments associated with the untreated wastewater entering the environment, but sometimes risk assessments focus on nutrients (Efroymson et al., 2007).

2.1. Nutrients

The reduction of biogenic compounds in the wastewater is very important as these compounds, after entering surface waters, cause their eutrophication, and thus bring many negative and very dangerous effects, such as drastic reduction of oxygen, extinction of organisms, acidification and degradation of bottom areas of the ecosystem (HELCOM, 2009, 2021, 2022). On average, the municipal wastewater treatment plant receives in sewage ammonium of concentration range 20–40 g/m^3 and orthophosphate of concentration 2–14 g/m^3 (Tchobanoglous et al., 2014). The concentration of nutrients in the outflow from the WWTP according to the Polish standards is 10 g/m^3 for nitrogen and 1 g/m^3 for phosphorus (Ministry of Maritime Economy and Inland Navigation, 2019).

2.2. Chemical oxygen demand

The chemical oxygen demand (COD) is a measure of the oxygen equivalent of organic matter in a sample that is susceptible to oxidation by a strong chemical oxidant. This parameter is commonly used as a measure of the susceptibility to oxidation of organic and inorganic materials present in the municipal and industrial wastewater. In contrast, the biological oxygen demand (BOD) test is performed to see how much of the organic matter in the wastewater is mineralized by microorganisms. Consequently, COD values are higher compared to BOD. Nevertheless, COD is a useful variable that can be measured quickly; a COD test can be performed in a few hours compared to the 5 days required for a BOD test. It is recognized that COD provides a result that can be used as a basis for calculating a reasonably accurate and reproducible assessment of the oxygen demand characteristics of wastewater.

The range of COD in municipal wastewater, according to various authors, is from 250 g O_2/m^3 up to the value 1600 g O_2/m^3, with an average value of 700 g O_2/m^3 (Tchobanoglous et al., 2014).

2.3. Wastewater process modelling

The improvement of wastewater treatment processes remains one of the most important goals in sanitary engineering. The realization of this goal requires, among other things, the use of modern analytical tools. Using appropriate mathematical models and simulation programs, it is possible to obtain an answer in a relatively short time whether and to what extent a change in wastewater composition, technological parameters or environmental conditions will affect the efficiency of the process. Among the most widely used mathematical models that affect the efficiency of pollutant elimination in biological wastewater treatment plants are Activated Sludge Models (ASMs). They make it possible to predict the biological processes taking place in WWTPs depending on, among other, the parameters of the inflowing wastewater (content of organic and inorganic substrates) the nitrification and denitrification processes, the parameters of the activated sludge (Serdarevic & Dzubur, 2016; Tchobanoglous et al., 2014). These models largely describe the kinetics of biochemical processes, while the variability of the inflowing sewage composition is only one of their components.

The model proposed in this work is not as comprehensive but is focused on the composition of raw wastewater. It defines the quality of wastewater depending on the variability of concentrations of selected forms of biogenic elements, namely, organic carbon, phosphorus, and nitrogen.

3. Process of municipal wastewater quality

3.1. Modelling municipal wastewater quality process

To construct the municipal wastewater quality process, the set of ω, $\omega \in N$, kinds of municipal wastewater quality factors is distinguished and these factors are denoted by $d_1, d_2, \dots, d_\omega$.
This way the set

$$\mathcal{D} = \{d_1, d_2, \dots, d_\omega\}$$

is the set of municipal wastewater quality factors. These factors may attain different levels. Namely, the municipal wastewater quality factor d_i, $i = 1,2, \dots, \omega$, may reach r_i levels

$$d_{i1}, d_{i2}, \dots, d_{ir_i}, i = 1,2, \dots, \omega,$$

that are called the states of this municipal wastewater quality factor.
The set

$$d_i = \{d_{i1}, d_{i2}, \dots, d_{ir_i}\}, i = 1,2, \dots, \omega,$$

is called the set of states of municipal wastewater quality factor d_i.
Under these assumptions, the municipal wastewater quality process is introduced as a vector

$$W(t) = [d_1(t), d_2(t), \dots, d_\omega(t)], t \in \langle 0, +\infty),$$

where

$$d_i(t), t \in \langle 0, +\infty), i = 1,2, \dots, \omega,$$

are the processes of municipal wastewater quality factors defined on the time interval $t \in \langle 0, +\infty)$ and having their values in the municipal wastewater quality factor's state sets d_i, $i = 1,2, \dots, \omega$.

The vector

$$w_i = [c_1, c_2, \dots, c_\omega], \tag{1}$$

where

$$c_i = \begin{cases} 0, & \text{if a municipal wastewater quality factor } d_i \text{ is in the typical range,} \\ d_{ij}, & \text{if a municipal wastewater quality factor } d_i \text{ is in the untypical range } d_{ij}, j = 1,2, \dots r_i \end{cases} \tag{2}$$

for $i = 1,2, \dots, \omega$, is called municipal wastewater quality state. Future, the vectors that cannot occur may be eliminated and the remaining municipal wastewater quality states, defined by (1)–(2), are marked by w_k for $k = 1,2, \dots, v$, and the set is formed

$$W = \{w_k, k = 1,2, \dots, v\}, \tag{3}$$

where

$$w_k \neq w_l, k \neq l, k, l \in \{1,2, \dots, v\}.$$

The set W is called the set of municipal wastewater quality states, while v is called the number of municipal wastewater quality states.
A function

$$W(t), t \in \langle 0, +\infty), \tag{4}$$

having values in the municipal wastewater quality states set W is called the process of municipal wastewater quality.
Next, a semi-Markov model of municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ is assumed. Its random conditional sojourn time at the municipal wastewater quality state w_k while the next transition will be done to the state w_l, $k, l = 1,2, \dots, v$, $k \neq l$ is denoted by θ_{kl}. Then, the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ is described by the following parameters that can be evaluated by experts or identified statistically using the methods given in (Bogalecka, 2020, 2021; Grabski, 2015; Iosifescu, 1980; Kołowrocki, 2014; Limnios & Oprisian, 2005; Smith, 1955):

- the matrix

$$[p_{kl}]_{v\mathrm{x}v} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1v} \\ p_{21} & p_{22} & \cdots & p_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ p_{v1} & p_{v2} & \cdots & p_{vv} \end{bmatrix} \quad (5)$$

of probabilities of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ transitions between the municipal wastewater quality states w_k and w_l, where $k, l = 1,2, \dots, v$, $k \neq l$, and $\forall k = 1,2, \dots, v$, $p_{kk} = 0$,

- the matrix

$$[H_{kl}(t)]_{v\mathrm{x}v} = \begin{bmatrix} H_{11}(t) & H_{12}(t) & \dots & H_{1v}(t) \\ H_{21}(t) & H_{22}(t) & \dots & H_{2v}(t) \\ \vdots & \vdots & \ddots & \vdots \\ H_{v1}(t) & H_{v2}(t) & \dots & H_{vv}(t) \end{bmatrix} \quad (6)$$

of conditional distribution functions of sojourn times θ_{kl} of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ in the state w_k while the next transition will be done to the state w_l, where

$$H_{kl}(t) = P(\theta_{kl} < t), t \in \langle 0, +\infty), \quad (7)$$

for

$k, l = 1,2, \dots, v$, $k \neq l$.

The matrix (6) is complied with the matrix

$$[h_{kl}(t)]_{v\mathrm{x}v} = \begin{bmatrix} h_{11}(t) & h_{12}(t) & \dots & h_{1v}(t) \\ h_{21}(t) & h_{22}(t) & \dots & h_{2v}(t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{v1}(t) & h_{v2}(t) & \dots & h_{vv}(t) \end{bmatrix} \quad (8)$$

of conditional densities of sojourn times θ_{kl} of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ in the state w_k while the next transition will be done to the state w_l, $k, l = 1,2, \dots, v$, $k \neq l$,

$$h_{kl}(t) = \frac{dH_{kl}(t)}{dt}, t \in \langle 0, +\infty), \quad (9)$$

for

$k, l = 1,2, \dots, v$, $k \neq l$,

where by formal agreement

$\forall k = 1,2, \dots, v$, $h_{kk} = 0$.

3.2. Identification of municipal wastewater quality process

In order to estimate parameters of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$, firstly the number of states v of the process $W(t)$, $t \in \langle 0, +\infty)$ are fixed and the states $w_1, w_2, \dots, w_v$ of the set (3) are defined. Further, the matrix of realisations n_{kl}, $k, l = 1,2, \dots, v$, $k \neq l$, of the numbers of the process $W(t)$, $t \in \langle 0, +\infty)$ transitions from the state w_k into the state w_l during the experimental time is fixed

$$[n_{kl}]_{v\mathrm{x}v} = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vv} \end{bmatrix}. \quad (10)$$

Having these numbers, the matrix of realisations of the transition probabilities of the process $W(t)$, $t \in \langle 0, +\infty)$ between states w_k and w_l during the experimental time, given by (5), is fixed according to the formula

$$p_{kl} = \frac{n_{kl}}{n_k}, k, l = 1,2, \dots, v, k \neq l, \quad (11)$$

and

$p_{kk} = 0$, $k = 1,2, \dots, v$,

where

$$\sum_{l \neq k}^{v} n_{kl}, k = 1,2, \dots, v, \quad (12)$$

is the realisation of the total number of the process $W(t)$, $t \in \langle 0, +\infty)$ transitions from the state w_k during the experimental time.

Next, the hypotheses about the conditional distribution functions of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ sojourn times θ_{kl}, $k, l = 1,2, \dots, v$, $k \neq l$ in the state w_k while the next transition is to the state w_l on the base of their realisations θ_{kl}^{γ}, $\gamma = 1,2, \dots, n_{kl}$ are formulated and verified.

Prior to estimating the parameters of the distributions of the conditional sojourn times of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ at its particular states, the following empirical characteristics of the realizations of the conditional sojourn time of the process $W(t)$,

$t \in \langle 0, +\infty)$ at the particular states have to be determined:

- the realizations of the empirical mean values $\bar{\theta}_{kl}$ of the conditional sojourn times θ_{kl}, $k, l = 1,2, \dots, v$, $k \neq l$, in the state w_k while the next transition is to the state w_l on the base of their realizations θ_{kl}^{γ}, $\gamma = 1,2, \dots, n_{kl}$, according to the formula

$$\bar{\theta}_{kl} = \frac{1}{n_{kl}} \sum_{\gamma=1}^{n_{kl}} \theta_{kl}^{\gamma}, \ k, l = 1,2, \dots, v, \ k \neq l, \quad (13)$$

- the number $\bar{r}_{kl}$ of disjoint intervals $I_j = \langle a_{kl}^j, b_{kl}^j)$, $j = 1,2, \dots, \bar{r}_{kl}$, including the realizations θ_{kl}^{γ}, $\gamma = 1,2, \dots, n_{kl}$ of conditional sojourn times θ_{kl}, $k, l = 1,2, \dots, v$, $k \neq l$ at the state w_k while the next transition is to the state w_l, according to the formula

$$\bar{r}_{kl} \cong \sqrt{n_{kl}}, \quad (14)$$

- the length d_{kl} of intervals $I_j = \langle a_{kl}^j, b_{kl}^j)$, $j = 1,2, \dots, \bar{r}_{kl}$, $k, l = 1,2, \dots, v$, $k \neq l$, according to the formula

$$d_{kl} = \frac{\bar{R}_{kl}}{\bar{r}_{kl} - 1}, \quad (15)$$

where

$$\bar{R}_{kl} = \max_{1 \le \gamma \le n_{kl}} \theta_{kl}^{\gamma} - \min_{1 \le \gamma \le n_{kl}} \theta_{kl}^{\gamma}, \quad (16)$$

- the ends a_{kl}^j, b_{kl}^j, $j = 1,2, \dots, \bar{r}_{kl}$, of intervals $I_j = \langle a_{kl}^j, b_{kl}^j)$, $j = 1,2, \dots, \bar{r}^{kl}$, $k, l = 1,2, \dots, v$, $k \neq l$, according to the formulae

$$a_{kl}^1 = \max\{ \min_{1 \le \gamma \le n_{kl}} \theta_{kl}^{\gamma} - \frac{d_{kl}}{2}, 0\}, \quad (17)$$

$$b_{kl}^j = a_{kl}^1 + j d_{kl}, \ j = 1,2, \dots, \bar{r}_{kl}, \quad (18)$$

$$a_{kl}^j = b_{kl}^{j-1}, \ j = 2,3, \dots, \bar{r}_{kl}, \quad (19)$$

in such a way that

$$I_1 \cup I_2 \cup \dots \cup I_{n_{kl}} = \langle a_{kl}^1, b_{kl}^{n_{kl}}),$$

and

$$I_i \cap I_j = \emptyset \text{ for all } i \neq j, \ i, j = 1,2, \dots, \bar{r}_{kl},$$

- the numbers n_{kl}^j of the realizations θ_{kl}^{γ}, $\gamma = 1,2, \dots, n_{kl}$, in the intervals $I_j = \langle a_{kl}^j, b_{kl}^j)$, $j = 1,2, \dots, \bar{r}_{kl}$, $k, l = 1,2, \dots, v$, $k \neq l$, according to the formula

$$n_{kl}^j = \# \left\{ \gamma : \theta_{kl}^{\gamma} \in I_j, \gamma \in \{1,2, \dots, n_{kl}\} \right\}, \quad (20)$$

where

$$\sum_{j=1}^{\bar{r}_{kl}} n_{kl}^j = n_{kl}, \quad (21)$$

whereas the # symbol means the number of elements of the set.

In order to formulate and further to verify the non-parametric hypothesis relating to the distribution form of the municipal wastewater quality process's conditional sojourn time θ_{kl}, $k, l = 1,2, \dots, v$, $k \neq l$ at the state w_k while the next transition is to the state w_l, on the base of its realizations θ_{kl}^{γ}, $\gamma = 1,2, \dots, n_{kl}$, the procedure adopted from (Kołowrocki & Soszyńska-Budny, 2011) is applied as follows:

- to construct and to plot the realization of the histogram of the municipal wastewater quality process's conditional sojourn time θ_{kl} at the state w_k, defined by the following formula

$$\bar{h}_{n_{kl}}(t) = \frac{n_{kl}^j}{n_{kl}} \text{ for } t \in I_j, \quad (22)$$

- to analyse the realization of the histogram $\bar{h}_{n_{kl}}(t)$, comparing it with the graphs of the density functions of the distinguished in Chapter 2 in (Kołowrocki & Soszyńska-Budny, 2011) distributions, to select one of them and to formulate the null hypothesis $\mathcal{H}$, concerning the unknown form of the distribution of the conditional sojourn time θ_{kl} in the following form: the municipal wastewater quality process's conditional sojourn time θ_{kl} at the state w_k while the next transition is to the state w_l has the distribution with the density function $h_{kl}(t)$,
- to join each of the intervals I_j that has the number n_{kl}^j of realizations less than 4 with the neighbour intervals I_{j+1} or I_{j-1} this way that

the numbers of realizations in all intervals are not less than 4,

- to fix a new number of intervals $\bar{\bar{r}}_{kl}$,
- to determine new intervals

$$\bar{I}_j = \langle \bar{a}^j_{kl}, \bar{b}^j_{kl} \rangle,\ j = 1,2,\dots,\bar{\bar{r}}_{kl},$$

- to fix the numbers $\bar{n}^j_{kl}$ of realizations in new intervals $\bar{I}_j$, $j = 1,2,\dots,\bar{\bar{r}}_{kl}$,
- under the assumption that the hypothesis $\mathcal{H}$ is true, to calculate the hypothetical probabilities that the conditional sojourn time θ_{kl} takes values from the new interval $\bar{I}_j$ according to the formula

$$p_j = P(\theta_{kl} \in \bar{I}_j) = P(\bar{a}^j_{kl} \le \theta_{kl} < \bar{b}^j_{kl})$$

$$= H_{kl}(\bar{b}^j_{kl}) - H_{kl}(\bar{a}^j_{kl}), \quad (23)$$

for

$$j = 1,2,\dots,\bar{\bar{r}}_{kl},$$

where $H_{kl}(\bar{b}^j_{kl})$ and $H_{kl}(\bar{a}^j_{kl})$ are the values of the distribution function $H_{kl}(t)$ of the random variable θ_{kl} corresponding to the density function $h_{kl}(t)$ assumed in the null hypothesis $\mathcal{H}$,

- to calculate the realization of the χ^2 −Pearson's statistics $U_{n_{kl}}$, according to the formula

$$u_{n_{kl}} = \sum_{j=1}^{\bar{\bar{r}}_{kl}} \frac{(\bar{n}^j_{kl} - n^j_{kl}\, p_j)^2}{n_{kl}\, p_j}, \quad (24)$$

- to assume the significance level α of the test (for instance $\alpha = 0.05$),
- to fix the number $\bar{\bar{r}}_{kl} - z - 1$ of degrees of freedom, substituting for z the number of unknown parameters of the distribution function $H_{kl}(t)$ estimated on the basis of the sojourn time θ_{kl} realizations,
- to read from the Tables of the χ^2 −Pearson's distribution the value u_α for the fixed values of the significance level α and the number of degrees of freedom $\bar{\bar{r}}_{kl} - z - 1$ such that the following equality holds

$$P(U_{n_{kl}} > u_\alpha) = \alpha, \quad (25)$$

and next to determine the acceptance domain in the form of the interval $\langle 0, u_\alpha \rangle$ and the critical domain in the form of the interval $(u_\alpha, +\infty)$,

- to compare the obtained value $u_{n_{kl}}$ of the realization of the statistics $U_{n_{kl}}$ with the read from the Tables critical value u_α of the χ^2 random variable and to decide on the formulated null hypothesis $\mathcal{H}$ in the following way: if the value $u_{n_{kl}}$ does not belong to the critical domain, i.e. when $u_{n_{kl}} \le u_\alpha$ then the hypothesis $\mathcal{H}$ is not rejected, otherwise if the value $u_{n_{kl}}$ belongs to the critical domain, i.e. when $u_{n_{kl}} > u_\alpha$ then the hypothesis $\mathcal{H}$ is rejected,
- to determine the matrix

$$[M_{kl}]_{v\mathrm{x}v} = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1v} \\ M_{21} & M_{22} & \dots & M_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ M_{v1} & M_{v2} & \dots & M_{vv} \end{bmatrix} \quad (26)$$

of the mean values M_{kl} of the conditional sojourn times θ_{kl}

$$M_{kl} = E[\theta_{kl}] = \int_0^\infty t dH_{kl}(t)$$

$$= \int_0^\infty t h_{kl}(t) dt, \quad (27)$$

for

$$k, l = 1,2,\dots,v,\ k \ne l.$$

3.3. Prediction of municipal wastewater quality process

The unknown parameters of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ are identified using the procedure given in Section 3.2. Then, the main characteristics of the process $W(t)$, $t \in \langle 0, +\infty)$ can be predicted using the procedure adopted from (Kołowrocki & Soszyńska-Budny, 2011). Namely, taking into account the formula for the total probability, the vector

$$[H_k(t)]_{1\mathrm{x}v} = [H_1(t), H_2(t), \dots, H_v(t)] \quad (28)$$

of unconditional distribution functions of sojourn times θ_k, $k = 1,2,\dots,v$ at particular states w_k of the process $W(t)$, $t \in \langle 0, +\infty)$

$$H_k(t) = \sum_{l=1}^{v} p_{kl} H_{kl}(t),\ k = 1,2,\dots,v, \quad (29)$$

and the vector

$$[h_k(t)]_{1xv} = [h_1(t), h_2(t), \ldots, h_v(t)] \quad (30)$$

of their corresponding density functions

$$h_k(t) = \sum_{l=1}^{v} p_{kl} h_{kl}(t),\ k = 1,2,\ldots,v, \quad (31)$$

where p_{kl}, $H_{kl}(t)$ and $h_{kl}(t)$ are defined by (11), (7) and (9) respectively.
Hence, the vector

$$[M_k]_{1xv} = [M_1, M_2, \ldots, M_v] \quad (32)$$

of mean values of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ unconditional sojourn times θ_k, $k = 1,2,\ldots,v$, at the municipal wastewater quality states are given by

$$M_k = E[\theta_k] = \sum_{l=1}^{v} p_{kl} M_{kl},\ k = 1,2,\ldots,v, \quad (33)$$

where p_{kl} and M_{kl} are defined by (11) and (27) respectively.
Next, the vector

$$[p_k]_{1xv} = [p_1, p_2, \ldots, p_v] \quad (34)$$

of limit values of transient probabilities

$$p_k(t) = P(W(t) = w_k), \quad (35)$$

for

$$t \in \langle 0, +\infty),\ k = 1,2,\ldots,v,$$

of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ at the particular states w_k, $k = 1,2,\ldots,v$, are calculated from the formula

$$p_k = \lim_{t \to \infty} p_k(t) = \frac{\pi_k M_k}{\sum_{l=1}^{v} \pi_l M_l},\ k = 1,2,\ldots,v, \quad (36)$$

where M_k, $k = 1,2,\ldots,v$, are given by (33), and the probabilities π_k, $k = 1,2,\ldots,v$, satisfy the system of equations

$$\begin{cases} [\pi_k] = [\pi_k][p_{kl}] \\ \sum_{l=1}^{v} \pi_l = 1 \end{cases} \quad (37)$$

where

$$[\pi_k] = [\pi_1, \pi_2, \ldots, \pi_v],$$

and $[p_{kl}]$ is given by (5).
Finally, other interesting characteristics of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ possible to obtain is the vector

$$[\hat{M}_k]_{1xv} = [\hat{M}_1, \hat{M}_2, \ldots, \hat{M}_v] \quad (38)$$

of the total sojourn times $\hat{\theta}_k$, $k = 1,2,\ldots,v$, at the particular process's states for sufficiently large time θ that have approximately normal distributions with the expected value given by

$$\hat{M}_k = E[\hat{\theta}_k] \cong p_k \theta, \quad (39)$$

where p_k are given by (36).

4. Application of municipal wastewater quality process

Using backgrounds given in Section 3, the proposed municipal wastewater quality process is applied to identify and predict the municipal wastewater quality collecting in the exemplary wastewater treatment plant.
The experiment is performed in the part of Pomerania region (Poland). This zone is situated in the north and seaside part of Poland (Fig. 1) where a wastewater treatment plant serves around 70 thousand people.

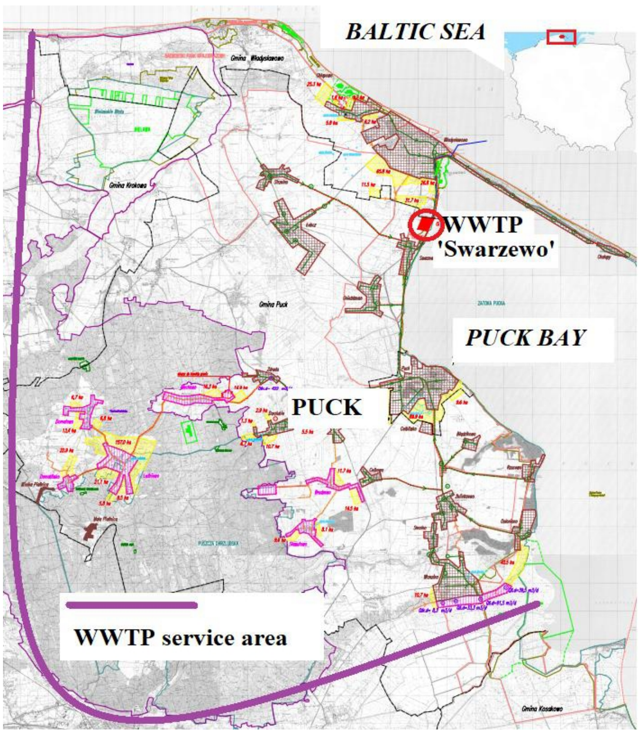

Figure 1. Location of Wastewater Treatment Plant *Swarzewo* and service area.

The WWTP *Swarzewo* collects sewage from the Puck District. This area has an agricultural and tourist character. For this reason, the WWTP receives mainly municipal wastewater. The length of the sewage network carrying the wastewater is 292 km. During the summer months, the WWTP treats up to 15,000 m^3 of sewage per day, whereas during other periods it treats around 5,000 m^3 per day. Annually, WWTP treats 2,500,000 m^3 of raw sewage. The treated wastewater is discharged into the open waters of the Baltic Sea.

The COD values in raw wastewater flowing to this plant are high and usually exceed 1000 g O_2/m^3. This is related to the inflow of high energy substances from the fish industry (proteins, fats), which require significant amounts of oxygen for mineralization. The concentrations of ammonium nitrogen and phosphate are within the range of values recorded for typical municipal wastewater. The quality of wastewater flowing into the mentioned above treatment plan is daily assessed based on $\omega = 3$ kinds of municipal wastewater quality factors:

d_1 – chemical oxygen demand (COD) given in g O_2/m^3,

d_2 – ammonium nitrogen (NH_4-N) concentration given in g/m^3,

d_3 – phosphate (PO_4-P) concentration given in g/m^3.

In addition, and taking into account experts' opinion, each factor may reach $r_i = 3$, $i = 1,2,3$ levels as it is shown in Table 1.

Table 1. Levels of municipal wastewater quality factors

Factor d_i, $i = 1,2,3$	Level of municipal wastewater quality factor		
	$r_1 = 0$ (typical)	$r_2 = 1$ (increased)	$r_3 = 2$ (high)
d_1 (COD)	$\langle 0, 800)$	$\langle 800, 1800)$	$\langle 1800, +\infty)$
d_2 (NH_4-N)	$\langle 0, 50)$	$\langle 50, 80)$	$\langle 80, +\infty)$
d_3 (PO_4-P)	$\langle 0, 15)$	$\langle 15, 40)$	$\langle 40, +\infty)$

Taking this into account and considering (1)–(4), the following states w_k, $k = 1,2,\dots,24$, of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ are distinguished:

$$w_1 = [0,0,0],\ w_2 = [0,0,1],\ w_3 = [0,0,2],\ w_4 = [0,1,0],\ w_5 = [0,1,1],\ w_6 = [0,1,2],\ w_7 = [0,2,1],\ w_8 = [0,2,2],\ w_9 = [1,0,0],\ w_{10} = [1,0,1],\ w_{11} = [1,0,2],\ w_{12} = [1,1,0],\ w_{13} = [1,1,1],\ w_{14} = [1,1,2],\ w_{15} = [1,2,1],\ w_{16} = [1,2,2],\ w_{17} = [2,0,0],\ w_{18} = [2,0,1],\ w_{19} = [2,0,2],\ w_{20} = [2,1,0],\ w_{21} = [2,1,1],\ w_{22} = [2,1,2],\ w_{23} = [2,2,1],\ w_{24} = [2,2,2]. \quad (40)$$

Then, according to (5)–(6) and (8), the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ is described by the probabilities $[p_{kl}]_{24\text{x}24}$ of the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ transitions between the municipal wastewater quality states w_k and w_l, $k, l = 1,2,\dots,24$, $k \neq l$, and the matrix of conditional distribution functions $[H_{kl}]_{24\text{x}24}$ of sojourn times of the process $W(t)$, $t \in \langle 0, +\infty)$ at the particular states or equivalently by corresponding to this matrix the matrix of conditional density functions $[h_{kl}]_{24\text{x}24}$.

On the basis of statistical data coming from the mentioned above WWTP and collected in three-year period, the matrix of realization of numbers of the process $W(t)$, $t \in \langle 0, +\infty)$ transitions from the state w_k into the state w_l during the experimental time $k, l = 1,2,\dots,24$, $k \neq l$ is fixed, according to (10). The realizations n_{kl}, $k, l = 1,2,\dots,24$, of transitions that are not equal to 0 are as follows

$$n_{12} = 6,\ n_{19} = 2,\ n_{1\,10} = 7,\ n_{1\,11} = 1,\ n_{1\,12} = 1,\ n_{1\,13} = 2,\ n_{21} = 3,\ n_{25} = 2,\ n_{29} = 1,\ n_{2\,10} = 14,\ n_{2\,13} = 3,\ n_{2\,14} = 1,\ n_{2\,22} = 1,\ n_{3\,10} = 1,\ n_{4\,12} = 1,\ n_{51} = 1,\ n_{52} = 1,\ n_{57} = 2,\ n_{58} = 1,\ n_{5\,10} = 1,\ n_{5\,12} = 1,\ n_{5\,13} = 3,\ n_{5\,14} = 1,\ n_{5\,15} = 1,\ n_{61} = 1,\ n_{62} = 1,\ n_{75} = 1,\ n_{78} = 1,\ n_{7\,16} = 1,\ n_{7\,21} = 1,\ n_{85} = 1,\ n_{87} = 1,\ n_{92} = 2,\ n_{9\,10} = 8,\ n_{9\,13} = 1,\ n_{9\,18} = 1,\ n_{9\,21} = 1,\ n_{10\,1} = 6,\ n_{10\,2} = 11,\ n_{10\,3} = 1,\ n_{10\,5} = 2,\ n_{10\,9} = 5,\ n_{10\,11} = 9,\ n_{10\,12} = 3,\ n_{10\,13} = 38,\ n_{10\,14} = 6,\ n_{10\,18} = 7,\ n_{10\,19} = 1,\ n_{10\,21} = 3,\ n_{10\,22} = 1,\ n_{11\,2} = 1,\ n_{11\,9} = 1,\ n_{11\,10} = 6,\ n_{11\,13} = 4,\ n_{11\,18} = 1,\ n_{11\,20} = 1,\ n_{12\,10} = 4,\ n_{12\,13} = 6,\ n_{12\,15} = 1,\ n_{12\,17} = 1,\ n_{12\,21} = 1,\ n_{13\,1} = 6,\ n_{13\,2} = 2,\ n_{13\,5} = 3,\ n_{13\,6} = 1,\ n_{13\,9} = 1,\ n_{13\,10} = 32,\ n_{13\,11} = 2,\ n_{13\,12} = 6,\ n_{13\,14} = 10,\ n_{13\,15} = 1,\ n_{13\,16} = 2,\ n_{13\,17} = 1,\ n_{13\,18} = 1,\ n_{13\,19} = 1,\ n_{13\,21} = 9,\ n_{13\,22} = 2,\ n_{13\,23} = 3,\ n_{13\,24} = 2,\ n_{14\,9} = 2,\ n_{14\,10} = 3,\ n_{14\,11} = 1,\ n_{14\,13} = 14,\ n_{14\,21} = 1,\ n_{15\,4} = 1,\ n_{15\,5} = 1,\ n_{15\,13} = 2,\ n_{15\,23} = 1,\ n_{16\,5} = 1,\ n_{16\,7} = 1,\ n_{16\,13} = 1,\ n_{17\,1} = 1,$$

$n_{17\,10} = 1, n_{18\,9} = 1, n_{18\,10} = 9, n_{18\,13} = 1,$
$n_{18\,14} = 1, n_{19\,10} = 1, n_{19\,11} = 1, n_{20\,14} = 1,$
$n_{21\,5} = 1, n_{21\,6} = 1, n_{21\,10} = 4, n_{21\,12} = 1,$
$n_{21\,13} = 7, n_{21\,14} = 1, n_{21\,15} = 1, n_{21\,18} = 2,$
$n_{22\,10} = 2, n_{22\,13} = 1, n_{22\,15} = 1, n_{23\,21} = 1,$
$n_{24\,10} = 1, n_{24\,13} = 1.$ (41)

Hence, according to (12), the vector of realisation of the total numbers of the process $W(t)$, $t \in \langle 0, +\infty)$ transitions from the state w_k, $k = 1,2,\dots,24$, during the experimental time is

$$[n_k]_{1\text{x}24} = [19,25,1,1,12,2,4,2,13,93,14,13,85,21,5,32,12,2,1,18,4,2,2]. \quad (42)$$

Considering (41)–(42) and applying the formula given by (11), the matrix of probabilities of the process $W(t)$, $t \in \langle 0, +\infty)$ transitions between it states w_k and w_l, $k, l = 1,2,\dots,24$, $k \neq l$, is fixed. The probabilities of transitions that are not equal to 0 are as follows

$p_{12} = 0.32, p_{19} = 0.10, p_{1\,10} = 0.37,$
$p_{1\,11} = 0.05, p_{1\,12} = 0.05, p_{1\,13} = 0.11,$
$p_{21} = 0.12, p_{25} = 0.08, p_{29} = 0.04,$
$p_{2\,10} = 0.56, p_{2\,13} = 0.12, p_{2\,14} = 0.04,$
$p_{2\,22} = 0.04, p_{3\,10} = 1, p_{4\,12} = 1, p_{51} = 0.09,$
$p_{52} = 0.08, p_{57} = 0.17, p_{58} = 0.08,$
$p_{5\,10} = 0.08, p_{5\,12} = 0.09, p_{5\,13} = 0.25,$
$p_{5\,14} = 0.08, p_{5\,15} = 0.08, p_{61} = 0.50,$
$p_{62} = 0.50, p_{75} = 0.25, p_{78} = 0.25,$
$p_{7\,16} = 0.25, p_{7\,21} = 0.25, p_{85} = 0.50,$
$p_{87} = 0.50, p_{92} = 0.15, p_{9\,10} = 0.62,$
$p_{9\,13} = 0.08, p_{9\,18} = 0.07, p_{9\,21} = 0.08,$
$p_{10\,1} = 0.06, p_{10\,2} = 0.12, p_{10\,3} = 0.01,$
$p_{10\,5} = 0.02, p_{10\,9} = 0.05, p_{10\,11} = 0.10,$
$p_{10\,12} = 0.03, p_{10\,13} = 0.41, p_{10\,14} = 0.07,$
$p_{10\,18} = 0.08, p_{10\,19} = 0.01, p_{10\,21} = 0.03,$
$p_{10\,22} = 0.01, p_{11\,2} = 0.07, p_{11\,9} = 0.07,$
$p_{11\,10} = 0.43, p_{11\,13} = 0.29, p_{11\,18} = 0.07,$
$p_{11\,20} = 0.07, p_{12\,10} = 0.31, p_{12\,13} = 0.46,$
$p_{12\,15} = 0.08, p_{12\,17} = 0.08, p_{12\,21} = 0.07,$
$p_{13\,1} = 0.07, p_{13\,2} = 0.02, p_{13\,5} = 0.04,$
$p_{13\,6} = 0.01, p_{13\,9} = 0.01, p_{13\,10} = 0.38,$
$p_{13\,11} = 0.03, p_{13\,12} = 0.07, p_{13\,14} = 0.12,$
$p_{13\,15} = 0.01, p_{13\,16} = 0.02, p_{13\,17} = 0.01,$
$p_{13\,18} = 0.01, p_{13\,19} = 0.01, p_{13\,21} = 0.11,$
$p_{13\,22} = 0.02, p_{13\,23} = 0.04, p_{13\,24} = 0.02,$
$p_{14\,9} = 0.09, p_{14\,10} = 0.14, p_{14\,11} = 0.05,$
$p_{14\,13} = 0.67, p_{14\,21} = 0.05, p_{15\,4} = 0.20,$
$p_{15\,5} = 0.20, p_{15\,13} = 0.40, p_{15\,23} = 0.20,$
$p_{16\,5} = 0.33, p_{16\,7} = 0.33, p_{16\,13} = 0.34,$
$p_{17\,1} = 0.50, p_{17\,10} = 0.50, p_{18\,9} = 0.08,$
$p_{18\,10} = 0.75, p_{18\,13} = 0.09, p_{18\,14} = 0.08,$
$p_{19\,10} = 0.50, p_{19\,11} = 0.50, p_{20\,14} = 1,$
$p_{21\,5} = 0.05, p_{21\,6} = 0.05, p_{21\,10} = 0.22,$
$p_{21\,12} = 0.06, p_{21\,13} = 0.39, p_{21\,14} = 0\ 06,$
$p_{21\,15} = 0.06, p_{21\,18} = 0.11, p_{22\,10} = 0.50,$
$p_{22\,13} = 0.25, p_{22\,15} = 0.25, p_{23\,21} = 1,$
$p_{24\,10} = 0.50, p_{24\,13} = 0.50.$ (43)

Next, on the basis of statistical data coming from the experiment, using the procedure and the formulae given in Section 3.2, it is possible to determine the empirical parameters of the conditional sojourn times θ_{kl}, $k, l = 1,2,\dots,24$, $k \neq l$ of the process $W(t)$, $t \in \langle 0, +\infty)$. The application of this procedure is performed below, for $\theta_{10\,13}$ that is one of the conditional sojourn times having most populous set of realizations.
The results for the conditional sojourn time $\theta_{10\,13}$ are:

- the realization $\bar{\theta}_{10\,13}$ of the defined by (13) mean value of the conditional sojourn time $\theta_{10\,13}$ of the municipal wastewater quality process's state w_{10} when the next transition is to the state w_{13}

$$\bar{\theta}_{10\,13} = \frac{1}{38}\sum_{\gamma=1}^{38} \theta_{10\,13}^{\gamma} \cong 1.89, \quad (44)$$

- the number $\bar{r}_{10\,13}$ of disjoint intervals $I_j = \langle a_{10\,13}^j, b_{10\,13}^j)$, $j = 1,2,\dots,\bar{r}_{10\,13}$, including the realizations $\theta_{10\,13}^{\gamma}$, $\gamma = 1,2,\dots,38$ of the conditional sojourn time $\theta_{10\,13}$ at the municipal wastewater quality process's state w_{10} when the next transition is to the state w_{13}, defined by (14)

$$\bar{r}_{10\,13} \cong \sqrt{38} \cong 6,$$

- the length $d_{10\,13}$ of intervals $I_j = \langle a_{10\,13}^j, b_{10\,13}^j)$, $j = 1,2,\dots,6$, defined by (15), after considering (16)

$$\bar{R}_{10\,13} = \max_{1 \le \gamma \le 38} \theta_{10\,13}^{\gamma} - \min_{1 \le \gamma \le 38} \theta_{10\,13}^{\gamma}$$

$$= 7 - 1 = 6$$

is

$$d_{10\,13} = \frac{\bar{R}_{10\,13}}{\bar{r}_{10\,13} - 1} = \frac{6}{5} = 1.2,$$

- the ends $a^j_{10\,13}$, $b^j_{10\,13}$, of intervals $I_j = \langle a^j_{10\,13}, b^j_{10\,13}\rangle$, $j = 1,2,\ldots,6$, defined by (17)–(19), after considering

$$\min_{1\le\gamma\le38} \theta^{\gamma}_{10\,13} - \frac{d_{10\,13}}{2} = 1 - \frac{1.2}{2} = 0.4,$$

are

$$a^1_{10\,13} = \max\{0.4,0\} = 0.4,$$
$$b^1_{10\,13} = a^1_{10\,13} + 1d_{10\,13} = 0.4 + 1.2 = 1.6,$$

$$a^2_{10\,13} = b^1_{10\,13} = 1.6,$$
$$b^2_{10\,13} = a^1_{10\,13} + 2d_{10\,13} = 0.4 + 2\cdot1.2 = 2.8,$$

$$a^3_{10\,13} = b^2_{10\,13} = 2.8,$$
$$b^3_{10\,13} = a^1_{10\,13} + 3d_{10\,13} = 0.4 + 3\cdot1.2 = 4.0,$$

$$a^4_{10\,13} = b^3_{10\,13} = 4.0,$$
$$b^4_{10\,13} = a^1_{10\,13} + 4d_{10\,13} = 0.4 + 4\cdot1.2 = 5.2,$$

$$a^5_{10\,13} = b^4_{10\,13} = 5.2,$$
$$b^5_{10\,13} = a^1_{10\,13} + 5d_{10\,13} = 0.4 + 5\cdot1.2 = 6.4,$$

$$a^6_{10\,13} = b^5_{10\,13} = 6.4,$$
$$b^6_{10\,13} = a^1_{10\,13} + 6d_{10\,13} = 0.4 + 6\cdot1.2 = 7.6, \quad (45)$$

- the numbers $n^j_{10\,13}$ of realizations $\theta^{\gamma}_{10\,13}$ in particular intervals $I_j = \langle a^j_{10\,13}, b^j_{10\,13}\rangle$, $j = 1,2,\ldots,6$, defined by (20) are as follows

$$n^1_{10\,13} = 25,\ n^2_{10\,13} = 5,\ n^3_{10\,13} = 2,$$
$$n^4_{10\,13} = 3,\ n^5_{10\,13} = 2,\ n^6_{10\,13} = 1. \quad (46)$$

Further, using the procedure given in Section 3.2 as well as the data coming from the experiment and the above results, the hypotheses concerning the distribution forms of the municipal wastewater quality process's conditional sojourn times θ_{kl}, $k,l = 1,2,\ldots,24$, $k \ne l$ at the particular states may be verified. To do this, a sufficiently numerous sets of these variables realizations is needed. It means that the sets of particular realizations coming from the experiment should contain at least 30 ones (see: Appendix). The conditional sojourn time $\theta_{10\,13}$ is the one having the most numerous sets of its realizations and preliminarily analysed above in this Section.

The histogram $\bar{h}_{10\,13}(t)$ of the municipal wastewater quality process's conditional sojourn time $\theta_{10\,13}$ realization defined by (22) is presented and illustrated in Table 2 and Figure 2 respectively.

Table 2. Realization of histogram of municipal wastewater quality process's conditional sojourn time $\theta_{10\,13}$

Histogram of the conditional sojourn time $\theta_{10\,13}$						
$I_j = \langle a^j_{10\,13}, b^j_{10\,13})$	⟨0.4, 1.6)	⟨1.6, 2.8)	⟨2.8, 4.0)	⟨4.0, 5.6)	⟨5.6, 6.4)	⟨6.4, 7.6)
$n^j_{10\,13}$	25	5	2	3	2	1
$\bar{h}_{10\,13}(t) = n^j_{10\,13}/n_{10\,13}$	25/38	5/38	2/38	3/38	2/38	1/38

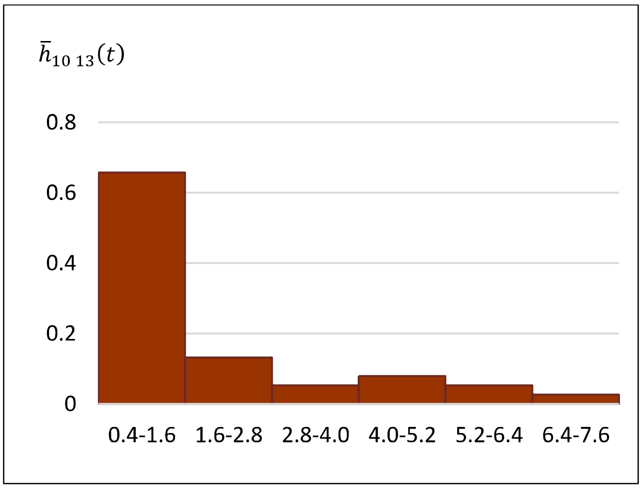

Figure 2. Graph of histogram of municipal wastewater quality process's conditional sojourn time $\theta_{10\,13}$.

After analysing and comparing the realization of histogram $\bar{h}_{10\,13}(t)$ with the graphs of the density function of distributions distinguished in Chapter 2 in (Kołowrocki & Soszyńska-Budny, 2011), the following hypothesis $\mathcal{H}$ is formulated: the municipal wastewater quality process's conditional sojourn time $\theta_{10\,13}$ at the state w_{10} when the next transition is to the state w_{13} has the chimney distribution expressed with the density function of the form

$$h_{10\,13}(t) = \begin{cases} 0 & t < x_{10\,13} \\ \frac{A_{10\,13}}{z^1_{10\,13} - x_{10\,13}} & x_{10\,13} \le t < z^1_{10\,13} \\ \frac{C_{10\,13}}{z^2_{10\,13} - z^1_{10\,13}} & z^1_{10\,13} \le t < z^2_{10\,13} \\ \frac{D_{10\,13}}{y_{10\,13} - z^2_{10\,13}} & z^2_{10\,13} \le t < y_{10\,13} \\ 0 & t \ge y_{10\,13}. \end{cases} \quad (47)$$

Since, according to (4.16)–(4.17) from (Kołowrocki & Soszyńska-Budny, 2011), there is

$$\hat{n}_{10\,13} = \max_{1 \le j \le 6}\{n^i_{10\,13}\} = 25, \quad (48)$$

and

$$n^1_{10\,13} = \hat{n}_{10\,13} = 25, \quad (49)$$

then $i = 1$. Moreover, by (4.22) from (Kołowrocki & Soszyńska-Budny, 2011), there is

$$n^2_{10\,13} = 5. \quad (50)$$

The unknown parameters of the hypothetical density function (47) are estimate using (4.15) and (4.18)–(4.19) in (Kołowrocki & Soszyńska-Budny, 2011), and the results are as follows

$$x_{10\,13} = a^1_{10\,13} = 0.4, \quad (51)$$

$$y_{10\,13} = x_{10\,13} + \bar{r}_{10\,13} d_{10\,13} = 0.4 + 6 \cdot 0.1 = 7.6, \quad (52)$$

$$z^1_{10\,13} = x_{10\,13} + (i-1)d_{10\,13} = 0.4 + (1-1) \cdot 1.2 = 0, \quad (53)$$

$$z^2_{10\,13} = x_{10\,13} + i d_{10\,13} = 0.4 + (1 \cdot 1.2) = 1.6, \quad (54)$$

$$A_{10\,13} = 0, \quad (55)$$

$$C_{10\,13} = \frac{n^i_{10\,13}}{n_{10\,13}} = \frac{25}{38} = 0.658, \quad (56)$$

$$D_{10\,13} = \frac{n^{i+1}_{10\,13} + \cdots + n^{\bar{r}_{10\,13}}_{10\,13}}{n_{10\,13}} = \frac{13}{38} = 0.342. \quad (57)$$

Next, substituting (51)–(57) into (47), the hypothetical density function takes the form

$$h_{10\,13}(t) = \begin{cases} 0 & t < 0.4 \\ \frac{0.658}{1.6-0.4} & 0.4 \le t < 1.6 \\ \frac{0.342}{7.6-1.6} & 1.6 \le t < 7.6 \\ 0 & t \ge 7.6 \end{cases} = \begin{cases} 0 & t < 0.4 \\ 0.548 & 0.4 \le t < 1.6 \\ 0.057 & 1.6 \le t < 7.6 \\ 0 & t \ge 7.6. \end{cases} \quad (58)$$

Taking the integral of the hypothetical density function $h_{10\,13}(t)$ of the conditional sojourn time $\theta_{10\,13}$ expressed by (58), the hypothetical distribution function $H_{10\,13}(t)$ takes the form

$$H_{10\,13}(t) = \begin{cases} 0 & t < 0.4 \\ 0.548t - 0.219 & 0.4 \le t < 1.6 \\ 0.057t + 0.786 & 1.6 \le t < 7.6 \\ 1 & t \ge 7.6. \end{cases} \quad (59)$$

Next, the intervals of the histogram $\bar{h}_{10\,13}(t)$ having the numbers $n^j_{10\,13}$ of realizations less than 4 are jointed into new ones and the following steps are performed

- the new number of intervals are fixed

  $\bar{\bar{r}}_{10\,13} = 3,$

- the new intervals are determined

  $\bar{I}_1 = \langle 0.4, 1.6), \bar{I}_2 = \langle 1.6, 4.0),$
  $\bar{I}_3 = \langle 4.0, 7.6),$

- the numbers of realizations in the new intervals are fixed

  $\bar{n}^1_{10\,13} = 25, \bar{n}^2_{10\,13} = 7, \bar{n}^3_{10\,13} = 6,$

- the hypothetical probabilities that the conditional sojourn time $\theta_{10\,13}$ takes values from the new intervals are calculated using (23)

$$p_1 = P(\theta_{10\,13} \in \bar{I}_1) = P(0.4 \le \theta_{10\,13} < 1.6) = H_{10\,13}(1.6) - H_{10\,13}(0.4) \cong 0.8772 - 0.2192 = 0.6580,$$

$$p_2 = P(\theta_{10\,13} \in \bar{I}_2) = P(1.6 \le \theta_{10\,13} < 4.0) = H_{10\,13}(4.0) - H_{10\,13}(1.6) \cong 1.0140 - 0.8772 = 0.1368,$$

$$p_3 = P(\theta_{10\,13} \in \bar{I}_3) = P(4.0 \le \theta_{10\,13} < 7.6) = H_{10\,13}(7.6) - H_{10\,13}(4.0), \cong 1.2192 - 1.0140 = 0.2052,$$

- the realization of the χ^2 −Pearson's statistics are calculated using (24)

$$u_{10\,13} = \sum_{j=1}^{3} \frac{(\bar{n}_{10\,13}^{j} - n_{10\,13} p_j)^2}{n_{10\,13} p_j} \cong \frac{(25-38 \cdot 0.6580)^2}{38 \cdot 0.6580} + \frac{(7-38 \cdot 0.1368)^2}{38 \cdot 0.1368} + \frac{(6-38 \cdot 0.2052)^2}{38 \cdot 0.2052} \cong 1.04,$$

- the significance level $\alpha = 0.05$ is assumed,
- the number of degrees of freedom for the hypothetical chimney distribution $(z = 0)$ is fixed

$$\bar{\bar{r}}_{10\,13} - z - 1 = 3 - 0 - 1 = 2,$$

- the value u_α for the fixed values of the significance level $\alpha = 0.05$ and the number of degrees of freedom $\bar{\bar{r}}_{10\,13} = 2$ is read from Tables of the χ^2 −Pearson's distribution, that according to (25), amounts $u_\alpha = 5.99$, therefore the acceptance domain in the form of the interval $\langle 0, 5.99 \rangle$ and the critical domain in the form of the interval $(5.99, +\infty)$ are determined as presented in Figure 3,
- the obtained value $u_{10\,13} \cong 1.04$ of the realization of the statistics $U_{10\,13}$ and the critical value $u_\alpha = 5.99$ read from Tables of χ^2 −Pearson's distribution are compared and the hypothesis $\mathcal{H}$ is not rejected since the value $u_{10\,13} \cong 1.04$ belongs to the acceptance domain, i.e.

$$u_{10\,13} \cong 1.04 \le u_\alpha = 5.99.$$

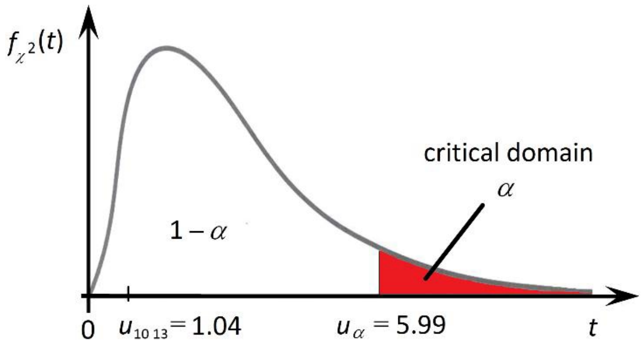

Figure 3. The graphical interpretation of acceptance and critical intervals for the χ^2 goodness-of-fit test.

Proceeding in the analogous way, based on the data given in Appendix, it is identified that the conditional sojourn time $\theta_{13\,10}$ has the exponential distribution expressed by the density function

$$h_{13\,10}(t) = \begin{cases} 0 & t < 0.6 \\ 0.971 \exp[-0.971(t-0.6)] & t \ge 0.6. \end{cases} \quad (60)$$

In the case when there are less than 30 realizations of the municipal wastewater quality process's conditional sojourn times, it is assumed that these ones have the empirical density functions defined by

$$h_{kl}(t) = \frac{1}{n_{kl}} \# \left\{ \gamma : \theta_{kl}^{\gamma} \in I_j, \gamma \in \{1,2,\ldots,n_{kl}\} \right\} \quad (61)$$

where

$$j = 1,2,\ldots,\bar{r}_{kl},$$

and corresponds to the empirical distribution functions defined by

$$H_{kl}(t) = \frac{1}{n_{kl}} \# \left\{ \gamma : \theta_{kl}^{\gamma} < t, \gamma \in \{1,2,\ldots,n_{kl}\} \right\} \quad (62)$$

for

$$t \ge 0,\ k,l = 1,2,\ldots,v,\ k \ne l,$$

whereas # symbol means the number of elements of the set.

For instance, the municipal wastewater quality process's conditional sojourn time $\theta_{13\,14}$ assumed

$n_{13\,14} = 10$ values. The order sample realizations $\theta_{13\,14}$ is 1, 1, 1, 1, 1, 1, 2, 2, 3, 5. Thus it is assumed that the conditional sojourn time $\theta_{13\,14}$ has the empirical density functions given by

$$h_{13\,14}(t) = \begin{cases} 0 & t < 1 \\ 6/10 & 1 \le t < 2 \\ 2/10 & 2 \le t < 3 \\ 1/10 & 3 \le t < 5 \\ 1/10 & t \ge 5 \end{cases} \quad (63)$$

and the distribution function given by

$$H_{13\,14}(t) = \begin{cases} 0 & t < 1 \\ 6/10 & t < 2 \\ 8/10 & t < 3 \\ 9/10 & t < 5 \\ 1 & t < +\infty. \end{cases} \quad (64)$$

For the remaining cases when the number of realizations of the municipal wastewater quality process's conditional sojourn times and their all realizations are equal to an approximate value, it is assumed that these ones have the uniform distribution in the interval from this value minus its half to this value plus its half. For instance, the municipal wastewater quality process's conditional time $\theta_{12\,13}$ assumed $n_{12\,13} = 6$ values equal to 1, thus it is assumed that it has the density function given by

$$h_{12\,13}(t) = \begin{cases} 0 & t < 0.5 \\ 1 & 0.5 \le t < 1.5 \\ 0 & t \ge 1.5 \end{cases} \quad (65)$$

and the distribution function given by

$$h_{12\,13}(t) = \begin{cases} 0 & t < 0.5 \\ t & 0.5 \le t < 1.5 \\ 1 & t \ge 1.5. \end{cases} \quad (66)$$

After accepting the density functions of the particular conditional sojourn times θ_{kl}, $k, l = 1,2,\ldots,24,\quad k \ne l$ of the municipal wastewater quality process $W(t),\ t \in \langle 0, +\infty)$, the general formula (27) is applied to find their mean values $M_{kl} = E[\theta_{kl}]$ of the process $W(t)$, $t \in \langle 0, +\infty)$ conditional sojourn times at its particular states.

Then, the mean values of conditional sojourn time $\theta_{10\,13}$ is

$$M_{10\,13} = \int_0^\infty t h_{10\,13}(t)dt$$
$$= \int_{0.4}^{1.6} 0.548 t dt + \int_{1.6}^{7.6} 0.057 t dt \cong 2.23.$$

Similarly, the mean values of conditional sojourn time $\theta_{12\,13}$ is

$$M_{12\,13} = \int_0^\infty t h_{12\,13}(t)dt = \int_{0.5}^{1.5} 1 t dt = 1.$$

In other cases, when the statistical identification of the municipal wastewater quality process's conditional sojourn times distributions at the particular states is not possible because of the lack of sufficient numbers of their realizations, the approximate empirical values of mean values $M_{kl} = E[\theta_{kl}]$ of the conditional sojourn times at the particular states are calculated using the formula (13). The results are given in the matrix defined by (26) where mean values (given in days) of the conditional sojourn times at the particular states that are not equal to 0 are as follows

$M_{12} = 1.33$, $M_{19} = 1$, $M_{1\,10} = 1.14$, $M_{1\,11} = 1$,
$M_{1\,12} = 1$, $M_{1\,13} = 1$, $M_{21} = 1.67$, $M_{25} = 1$,
$M_{29} = 1$, $M_{2\,10} = 1.43$, $M_{2\,13} = 3.67$,
$M_{2\,14} = 3$, $M_{2\,22} = 1$, $M_{3\,10} = 1$, $M_{4\,12} = 1$,
$M_{51} = 2$, $M_{52} = 1$, $M_{57} = 1.50$, $M_{58} = 1$,
$M_{5\,10} = 1$, $M_{5\,12} = 2$, $M_{5\,13} = 1$, $M_{5\,14} = 1$,
$M_{5\,15} = 1$, $M_{61} = 1$, $M_{62} = 1$, $M_{75} = 2$,
$M_{78} = 1$, $M_{7\,16} = 1$, $M_{7\,21} = 1$, $M_{85} = 1$,
$M_{87} = 1$, $M_{92} = 1.5$, $M_{9\,10} = 1$, $M_{9\,13} = 2$,
$M_{9\,18} = 1$, $M_{9\,21} = 2$, $M_{10\,1} = 1.33$,
$M_{10\,2} = 2.09$, $M_{10\,3} = 1$, $M_{10\,5} = 2$,
$M_{10\,9} = 1.20$, $M_{10\,11} = 1.89$, $M_{10\,12} = 2.33$,
$M_{10\,13} = 2.23$, $M_{10\,14} = 3.50$, $M_{10\,18} = 2.14$,
$M_{10\,19} = 1$, $M_{10\,21} = 1$, $M_{10\,22} = 1$, $M_{11\,2} = 1$,
$M_{11\,9} = 2$, $M_{11\,10} = 1$, $M_{11\,13} = 2$, $M_{11\,18} = 1$,
$M_{11\,20} = 1$, $M_{12\,10} = 1$, $M_{12\,13} = 1$, $M_{12\,15} = 2$,
$M_{12\,17} = 1$, $M_{12\,21} = 1$, $M_{13\,1} = 1.83$,
$M_{13\,2} = 2$, $M_{13\,5} = 3.67$, $M_{13\,6} = 1$, $M_{13\,9} = 1$,

$M_{13\,10} = 1.63$, $M_{13\,11} = 1.50$, $M_{13\,12} = 1.33$, $M_{13\,14} = 1.80$, $M_{13\,15} = 1$, $M_{13\,16} = 2$, $M_{13\,17} = 9$, $M_{13\,18} = 1$, $M_{13\,19} = 1$, $M_{13\,21} = 1.89$, $M_{13\,22} = 2$, $M_{13\,23} = 1$, $M_{13\,24} = 3$, $M_{14\,9} = 1.50$, $M_{14\,10} = 1$, $M_{14\,11} = 1$, $M_{14\,13} = 1.14$, $M_{14\,21} = 1$, $M_{15\,4} = 1$, $M_{15\,5} = 1$, $M_{15\,13} = 1$, $M_{15\,23} = 1$, $M_{16\,5} = 1$, $M_{16\,7} = 1$, $M_{16\,13} = 1$, $M_{17\,1} = 1$, $M_{17\,10} = 1$, $M_{18\,9} = 2$, $M_{18\,10} = 1.22$, $M_{18\,13} = 1$, $M_{18\,14} = 1$, $M_{19\,10} = 1$, $M_{19\,11} = 1$, $M_{20\,14} = 1$, $M_{21\,5} = 1$, $M_{21\,6} = 1$, $M_{21\,10} = 1$, $M_{21\,12} = 1$, $M_{21\,13} = 1.71$, $M_{21\,14} = 1$, $M_{21\,15} = 1$, $M_{21\,18} = 1$, $M_{22\,10} = 1$, $M_{22\,13} = 1$, $M_{22\,15} = 1$, $M_{23\,21} = 1$, $M_{24\,10} = 1$, $M_{24\,13} = 1$. (67)

This way, the municipal wastewater quality process $W(t)$, $t \in \langle 0, +\infty)$ is identified. Now, its main characteristics may be predicted using the procedure presented in Section 3.3.
Applying (33) and considering (43) and (67), the approximate mean values M_k, $k = 1,2,\ldots,24$ of unconditional sojourn times of variables θ_k, $k = 1,2,\ldots,24$ can be evaluated. The values that are not equal to 0 are presented only, and they as follows

$$M_1 = M_{12}p_{12} + M_{19}p_{19} + M_{1\,10}p_{1\,10} + M_{1\,11}p_{1\,11} + M_{1\,12}p_{1\,12} + M_{1\,13}p_{1\,13} = 1.33 \cdot 0.32 + 1 \cdot 0.10 + 1.44 \cdot 0.37 + 1 \cdot 0.05 + 1 \cdot 0.05 + 1 \cdot 0.11 \cong 1.27,$$

$$M_2 = M_{21}p_{21} + M_{25}p_{25} + M_{29}p_{29} + M_{2\,10}p_{2\,10} + M_{2\,13}p_{2\,13} + M_{2\,14}p_{2\,14} + M_{2\,22}p_{2\,22} = 1.67 \cdot 0.12 + 1 \cdot 0.08 + 1 \cdot 0.04 + 1.43 \cdot 0.56 + 3.67 \cdot 0.12 + 3 \cdot 0.04 + 1 \cdot 0.04 \cong 1.72,$$

$$M_3 = M_{3\,10}p_{3\,10} = 1 \cdot 1 = 1.00,$$

$$M_4 = M_{4\,12}p_{4\,12} = 1 \cdot 1 = 1.00,$$

$$M_5 = M_{51}p_{51} + M_{52}p_{52} + M_{57}p_{57} + M_{58}p_{58} + M_{5\,10}p_{5\,10} + M_{5\,12}p_{5\,12} + M_{5\,13}p_{5\,13} + M_{5\,14}p_{5\,14} + M_{5\,15}p_{5\,15} = 2 \cdot 0.09 + 1 \cdot 0.08 + 1.5 \cdot 0.17 + 1 \cdot 0.08 + 1 \cdot 0.08 + 2 \cdot 0.09 + 1 \cdot 0.25 + 1 \cdot 0.08 + 1 \cdot 0.08 \cong 1.27,$$

$$M_6 = M_{61}p_{61} + M_{62}p_{62} = 1 \cdot 0.50 + 1 \cdot 0.50 = 1.00,$$

$$M_7 = M_{75}p_{75} + M_{78}p_{78} + M_{7\,16}p_{7\,16} + M_{7\,21}p_{7\,21} = 2 \cdot 0.25 + 1 \cdot 0.25 + 1 \cdot 0.25 + 1 \cdot 0.25 = 1.25,$$

$$M_8 = M_{85}p_{85} + M_{87}p_{87} = 1 \cdot 0.50 + 1 \cdot 0.50 = 1.00,$$

$$M_9 = M_{92}p_{92} + M_{9\,10}p_{9\,10} + M_{9\,13}p_{9\,13} + M_{9\,18}p_{9\,18} + M_{9\,21}p_{9\,21} = 1.5 \cdot 0.15 + 1 \cdot 0.62 + 2 \cdot 0.08 + 1 \cdot 0.07 + 2 \cdot 0.08 \cong 1.24,$$

$$M_{10} = M_{10\,1}p_{10\,1} + M_{10\,2}p_{10\,2} + M_{10\,3}p_{10\,3} + M_{10\,5}p_{10\,5} + M_{10\,9}p_{10\,9} + M_{10\,11}p_{10\,11} + M_{10\,12}p_{10\,12} + M_{10\,13}p_{10\,13} + M_{10\,14}p_{10\,14} + M_{10\,18}p_{10\,18} + M_{10\,19}p_{10\,19} + M_{10\,21}p_{10\,21} = 1.33 \cdot 0.06 + 2.09 \cdot 0.12 + 1 \cdot 0.01 + 2 \cdot 0.02 + 1.20 \cdot 0.05 + 1.89 \cdot 0.10 + 2.33 \cdot 0.03 + 2.23 \cdot 0.41 + 3.50 \cdot 0.07 + 2.14 \cdot 0.08 + 1 \cdot 0.01 + 1 \cdot 0.03 + 1 \cdot 0.01 \cong 2.08,$$

$$M_{11} = M_{11\,2}p_{11\,2} + M_{11\,9}p_{11\,9} + M_{11\,10}p_{11\,10} + M_{11\,13}p_{11\,13} + M_{11\,18}p_{11\,18} + M_{11\,20}p_{11\,20} = 1 \cdot 0.07 + 2 \cdot 0.07 + 1 \cdot 0.43 + 2 \cdot 0.29 + 1 \cdot 0.07 + 1 \cdot 0.07 \cong 1.36,$$

$$M_{12} = M_{12\,10}p_{12\,10} + M_{12\,13}p_{12\,13} + M_{12\,15}p_{12\,15} + M_{12\,17}p_{12\,17} + M_{12\,21}p_{12\,21} = 1 \cdot 0.31 + 1 \cdot 0.46 + 2 \cdot 0.08 + 1 \cdot 0.08 + 1 \cdot 0.07 \cong 1.08,$$

$$M_{13} = M_{13\,1}p_{13\,1} + M_{13\,2}p_{13\,2} + M_{13\,5}p_{13\,5} + M_{13\,6}p_{13\,6} + M_{13\,9}p_{13\,9} + M_{13\,10}p_{13\,10} + M_{13\,11}p_{13\,11} + M_{13\,12}p_{13\,12} + M_{13\,14}p_{13\,14} + M_{13\,15}p_{13\,15} + M_{13\,16}p_{13\,16} + M_{13\,17}p_{13\,17} + M_{13\,18}p_{13\,18} + M_{13\,19}p_{13\,19} + M_{13\,21}p_{13\,21} + M_{13\,22}p_{13\,22} + M_{13\,23}p_{13\,23} + M_{13\,24}p_{13\,24} = 1.83 \cdot 0.07 + 2 \cdot 0.02 + 3.67 \cdot 0.04 + 1 \cdot 0.01 + 1 \cdot 0.01 + 1.63 \cdot 0.38 + 1.50 \cdot 0.03 + 1.33 \cdot 0.07 + 1.80 \cdot 0.12 + 1 \cdot 0.01 + 2 \cdot 0.02 + 9 \cdot 0.01 + 1 \cdot 0.01 + 1 \cdot 0.01 + 1.89 \cdot 0.11 + 2 \cdot 0.02 + 1 \cdot 0.04 + 3 \cdot 0.02 \cong 1.82,$$

$$M_{14} = M_{14\,9}p_{14\,9} + M_{14\,10}p_{14\,10} + M_{14\,11}p_{14\,11} + M_{14\,13}p_{14\,13} + M_{14\,21}p_{14\,21}$$

$$= 1.50 \cdot 0.09 + 1 \cdot 0.15 + 1 \cdot 0.05 + 1.14 \cdot 0.67 + 1 \cdot 0.05 \cong 1.14,$$

$$M_{15} = M_{15\,4}p_{15\,4} + M_{15\,5}p_{15\,5} + M_{15\,13}p_{15\,13} + M_{15\,23}p_{15\,23} = 1 \cdot 0.20 + 1 \cdot 0.20 + 1 \cdot 0.20 + 1 \cdot 0.20 = 1.00,$$

$$M_{16} = M_{16\,5}p_{16\,5} + M_{16\,7}p_{16\,7} + M_{16\,13}p_{16\,13} = 1 \cdot 0.33 + 1 \cdot 0.33 + 1 \cdot 0.34 = 1.00,$$

$$M_{17} = M_{17\,1}p_{17\,1} + M_{17\,10}p_{17\,10} = 1 \cdot 0.50 + 1 \cdot 0.50 = 1.00,$$

$$M_{18} = M_{18\,9}p_{18\,9} + M_{18\,10}p_{18\,10} + M_{18\,13}p_{18\,13} + M_{18\,14}p_{18\,14} = 2 \cdot 0.08 + 1.22 \cdot 0.75 + 1 \cdot 0.09 + 1 \cdot 0.08 \cong 1.25,$$

$$M_{19} = M_{19\,10}p_{19\,10} + M_{19\,11}p_{19\,11} = 1 \cdot 0.50 + 1 \cdot 0.50 = 1.00,$$

$$M_{20} = M_{20\,14}p_{20\,14} = 1 \cdot 1 = 1.00,$$

$$M_{21} = M_{21\,5}p_{21\,5} + M_{21\,6}p_{21\,6} + M_{21\,10}p_{21\,10} + M_{21\,12}p_{21\,12} + M_{21\,13}p_{21\,13} + M_{21\,14}p_{21\,14} + M_{21\,15}p_{21\,15} + M_{21\,18}p_{21\,18} = 1 \cdot 0.05 + 1 \cdot 0.05 + 1 \cdot 0.22 + 1 \cdot 0.06 + 1.71 \cdot 0.39 + 1 \cdot 0.06 + 1 \cdot 0.06 + 1 \cdot 0.11 \cong 1.28,$$

$$M_{22} = M_{22\,10}p_{22\,10} + M_{22\,13}p_{22\,13} + M_{22\,15}p_{22\,15} = 1 \cdot 0.50 + 1 \cdot 0.25 + 1 \cdot 0.25 = 1.00,$$

$$M_{23} = M_{23\,21}p_{23\,21} = 1 \cdot 1 = 1.00,$$

$$M_{24} = M_{24\,10}p_{24\,10} + M_{24\,13}p_{24\,13} = 1 \cdot 0.50 + 1 \cdot 0.50 = 1.00, \quad (68)$$

Further, to find the limit values of the transient probabilities p_k, $k = 1,2,\dots,24$ at particular states of the process $W(t)$, $t \in \langle 0, +\infty)$, the system of equations (37) has to be solved. It consists of the following equations

$$\pi_1 = 0.12\pi_2 + 0.09\pi_5 + 0.50\pi_6 + 0.06\pi_{10} + 0.07\pi_{13} + 0.50\pi_{17},$$

$$\pi_2 = 0.32\pi_1 + 0.08\pi_5 + 0.50\pi_6 + 0.15\pi_9 + 0.12\pi_{10} + 0.07\pi_{11} + 0.02\pi_{13},$$

$$\pi_3 = 0.01\pi_{10},$$

$$\pi_4 = 0.20\pi_{15},$$

$$\pi_5 = 0.08\pi_2 + 0.25\pi_7 + 0.50\pi_8 + 0.02\pi_{10} + 0.04\pi_{13} + 0.20\pi_{15} + 0.33\pi_{16} + 0.05\pi_{21},$$

$$\pi_6 = 0.01\pi_{13} + 0.05\pi_{21},$$

$$\pi_7 = 0.17\pi_5 + 0.50\pi_8 + 0.33\pi_{16},$$

$$\pi_8 = 0.08\pi_5 + 0.25\pi_7,$$

$$\pi_9 = 0.10\pi_1 + 0.04\pi_2 + 0.05\pi_{10} + 0.07\pi_{11} + 0.01\pi_{13} + 0.09\pi_{14} + 0.08\pi_{18},$$

$$\pi_{10} = 0.37\pi_1 + 0.56\pi_2 + \pi_3 + 0.08\pi_5 + 0.62\pi_9 + 0.43\pi_{11} + 0.31\pi_{12} + 0.38\pi_{13} + 0.14\pi_{14} + 0.50\pi_{17} + 0.75\pi_{18} + 0.50\pi_{19} + 0.22\pi_{21} + 0.50\pi_{22} + 0.50\pi_{24},$$

$$\pi_{11} = 0.05\pi_1 + 0.10\pi_{10} + 0.03\pi_{13} + 0.05\pi_{14} + 0.50\pi_{19},$$

$$\pi_{12} = 0.05\pi_1 + \pi_4 + 0.09\pi_5 + 0.03\pi_{10} + 0.07\pi_{13} + 0.06\pi_{21},$$

$$\pi_{13} = 0.11\pi_1 + 0.12\pi_2 + 0.25\pi_5 + 0.08\pi_9 + 0.41\pi_{10} + 0.29\pi_{11} + 0.46\pi_{12} + 0.67\pi_{14} + 0.40\pi_{15} + 0.34\pi_{16} + 0.09\pi_{18} + 0.39\pi_{21} + 0.25\pi_{22} + 0.50\pi_{24},$$

$$\pi_{14} = 0.04\pi_2 + 0.08\pi_5 + 0.07\pi_{10} + 0.12\pi_{13} + 0.08\pi_{18} + \pi_{20} + 0.06\pi_{21},$$

$$\pi_{15} = 0.08\pi_5 + 0.08\pi_{12} + 0.01\pi_{13} + 0.06\pi_{21} + 0.25\pi_{22},$$

$$\pi_{16} = 0.25\pi_7 + 0.02\pi_{13},$$

$$\pi_{17} = 0.08\pi_{12} + 0.01\pi_{13},$$

$$\pi_{18} = 0.07\pi_9 + 0.08\pi_{10} + 0.07\pi_{11} + 0.01\pi_{13} + 0.11\pi_{21},$$

$$\pi_{19} = 0.01\pi_{10} + 0.01\pi_{13},$$

$$\pi_{20} = 0.07\pi_{11},$$

$$\pi_{21} = 0.25\pi_7 + 0.08\pi_9 + 0.03\pi_{10} + 0.07\pi_{12} + 0.11\pi_{13} + 0.05\pi_{14} + \pi_{23},$$

$$\pi_{22} = 0.04\pi_2 + 0.01\pi_{10} + 0.02\pi_{13},$$

$$\pi_{23} = 0.04\pi_{13} + 0.20\pi_{15},$$

$$\pi_{24} = 0.02\pi_{13},$$

$\pi_1 + \pi_2 + \cdots + \pi_{24} = 1.$ (69)

The solution of system of equations (69) is

$\pi_1 = 0.048, \pi_2 = 0.065, \pi_3 = 0.003,$
$\pi_4 = 0.003, \pi_5 = 0.034, \pi_6 = 0.005,$
$\pi_7 = 0.011, \pi_8 = 0.005, \pi_9 = 0.034,$
$\pi_{10} = 0.260, \pi_{11} = 0.041, \pi_{12} = 0.036,$
$\pi_{13} = 0.239, \pi_{14} = 0.061, \pi_{15} = 0.014,$
$\pi_{16} = 0.008, \pi_{17} = 0.005, \pi_{18} = 0.035,$
$\pi_{19} = 0.005, \pi_{20} = 0.003, \pi_{21} = 0.058,$
$\pi_{22} = 0.010, \pi_{23} = 0.012, \pi_{24} = 0.005.$

Hence, according to (36) and considering (68), the approximate limit values of the transient probabilities p_k, $k = 1,2,\ldots,24$ at the particular states w_k of the process $W(t)$, $t \in \langle 0, +\infty)$ are

$p_1 = 0.038, p_2 = 0.070, p_3 = 0.002,$
$p_4 = 0.002, p_5 = 0.027, p_6 = 0.003,$
$p_7 = 0.009, p_8 = 0.003, p_9 = 0.026,$
$p_{10} = 0.336, p_{11} = 0.035, p_{12} = 0.024,$
$p_{13} = 0.271, p_{14} = 0.043, p_{15} = 0.009,$
$p_{16} = 0.005, p_{17} = 0.003, p_{18} = 0.027,$
$p_{19} = 0.003, p_{20} = 0.002, p_{21} = 0.046,$
$p_{22} = 0.006, p_{23} = 0.007, p_{24} = 0.003.$ (70)

Further, by (39) and considering (70), the approximate mean values of the sojourn total time $\hat{\theta}_k$, $k = 1,2,\ldots,24$, of the process (t), $t \in \langle 0, +\infty)$ in the fixed time interval $\theta = 1$ year $= 365$ days at the particular states w_k, $k = 1,2,\ldots,24$ expressed in days are

$\widehat{M}_1 = E[\hat{\theta}_1] = 13.87, \widehat{M}_2 = E[\hat{\theta}_2] = 25.55,$
$\widehat{M}_3 = E[\hat{\theta}_3] = 0.73, \widehat{M}_4 = E[\hat{\theta}_4] = 0.73,$
$\widehat{M}_5 = E[\hat{\theta}_5] = 9.86, \widehat{M}_6 = E[\hat{\theta}_6] = 1.10,$
$\widehat{M}_7 = E[\hat{\theta}_7] = 3.29, \widehat{M}_8 = E[\hat{\theta}_8] = 1.09,$
$\widehat{M}_9 = E[\hat{\theta}_9] = 9.49, \widehat{M}_{10} = E[\hat{\theta}_{10}] = 122.64,$
$\widehat{M}_{11} = E[\hat{\theta}_{11}] = 12.78, \widehat{M}_{12} = E[\hat{\theta}_{12}] = 8.76,$
$\widehat{M}_{13} = E[\hat{\theta}_{13}] = 98.92, \widehat{M}_{14} = E[\hat{\theta}_{14}] = 15.70,$
$\widehat{M}_{15} = E[\hat{\theta}_{15}] = 3.29, \widehat{M}_{16} = E[\hat{\theta}_{16}] = 1.83,$
$\widehat{M}_{17} = E[\hat{\theta}_{17}] = 1.09, \widehat{M}_{18} = E[\hat{\theta}_{18}] = 9.86,$
$\widehat{M}_{19} = E[\hat{\theta}_{19}] = 1.09, \widehat{M}_{20} = E[\hat{\theta}_{20}] = 0.73,$
$\widehat{M}_{21} = E[\hat{\theta}_{21}] = 16.79, \widehat{M}_{22} = E[\hat{\theta}_{22}] = 2.19,$
$\widehat{M}_{23} = E[\hat{\theta}_{23}] = 2.56, \widehat{M}_{24} = E[\hat{\theta}_{24}] = 1.10.$
(71)

According to (70), states w_{10} and w_{13} reach the highest value of the transient probabilities equal to $p_{10} = 0.336$ and $p_{13} = 0.271$ respectively. Consequently, according to (71), states w_{10} and w_{13} reach the highest value of the sojourn total times equal to $\widehat{M}_{10} = 122.64$ hours and $\widehat{M}_{13} = 98.82$ days per a year respectively. The states w_{10} and w_{13} represent municipal wastewater quality factors such as COD and PO_4-P are in the increased range and the NH_4-N factor is in the typical or increased range. Nevertheless the values (70)–(71) are evaluated based on the experiment and the real statistical data therefore the values (70)–(71) may change and being more precise if the experiment duration is longer.
Moreover the last results (70)–(71) can play a practically role in the optimization of municipal wastewater quality what is the subject of future researches.

5. Conclusion

The model the municipal wastewater quality process is introduced in the Chapter. The procedure of its practical application is illustrated in the modelling, identification and prediction of municipal wastewater quality process concerned with the chemical oxygen demand, NH_4-N and PO_4-P concentration changes in time.
Presented model and tools are supposed to be useful in the sewage treatment plant (as the element of critical infrastructure) and modelling accident consequences of wastewater pipeline failures as well as their losses optimization and mitigation. The model can also be useful for simulating the variability of nutrient inflow from the catchment to the wastewater treatment plant.

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Appendix

Table 3. Realization of conditional sojourn times at states of municipal wastewater quality process

Transitions	Transition time θ_{kl} [day]	Number of transitions n_{kl}
$w_1 \rightarrow w_2$	1, 1, 1, 3, 1, 1	6
$w_1 \rightarrow w_9$	1, 1	2
$w_1 \rightarrow w_{10}$	1, 1, 1, 2, 1, 1, 1	7
$w_2 \rightarrow w_1$	2, 2, 1	3
$w_2 \rightarrow w_5$	1, 1	2
$w_2 \rightarrow w_9$	1	1
$w_2 \rightarrow w_{10}$	1, 1, 1, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1	14
$w_2 \rightarrow w_{13}$	9, 1, 1	3
$w_2 \rightarrow w_{14}$	3	1
$w_2 \rightarrow w_{22}$	1	1
$w_1 \rightarrow w_{11}$	1	1
$w_1 \rightarrow w_{12}$	1	1
$w_1 \rightarrow w_{13}$	1, 1	2
$w_3 \rightarrow w_{10}$	1	1
$w_4 \rightarrow w_{12}$	1	1
$w_5 \rightarrow w_1$	2	1
$w_5 \rightarrow w_2$	1	1
$w_5 \rightarrow w_7$	1, 2	2
$w_5 \rightarrow w_8$	1	1
$w_5 \rightarrow w_{10}$	1	1
$w_5 \rightarrow w_{12}$	2	1
$w_5 \rightarrow w_{13}$	1, 1, 1	3
$w_5 \rightarrow w_{14}$	1	1
$w_5 \rightarrow w_{15}$	1	1
$w_6 \rightarrow w_1$	1	1
$w_6 \rightarrow w_2$	1	1
$w_7 \rightarrow w_5$	2	1
$w_7 \rightarrow w_8$	1	1
$w_7 \rightarrow w_{16}$	1	1
$w_7 \rightarrow w_{21}$	1	1
$w_8 \rightarrow w_5$	1	1
$w_8 \rightarrow w_7$	1	1
$w_9 \rightarrow w_2$	2, 1	2
$w_9 \rightarrow w_{10}$	1, 1, 1, 1, 1, 1, 1, 1	8
$w_9 \rightarrow w_{13}$	2	1
$w_9 \rightarrow w_{18}$	1	1
$w_9 \rightarrow w_{21}$	2	1
$w_{10} \rightarrow w_1$	1, 1, 1, 1, 1, 3	6
$w_{10} \rightarrow w_2$	2, 2, 2, 1, 4, 1, 2, 1, 2, 2, 4	11
$w_{10} \rightarrow w_3$	1	1
$w_{10} \rightarrow w_5$	2, 2	2
$w_{10} \rightarrow w_9$	1, 1, 2, 1, 1	5
$w_{10} \rightarrow w_{11}$	2, 1, 2, 1, 2, 6, 1, 1, 1	9
$w_{10} \rightarrow w_{12}$	2, 2, 3	3
$w_{10} \rightarrow w_{13}$	2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 4, 2, 1, 2, 6, 1, 4, 3, 1, 4, 1, 1, 1, 6, 1, 1, 7, 1, 1, 3	38
$w_{10} \rightarrow w_{14}$	1, 1, 1, 16	4

continuation

Transitions	Transition time θ_{kl} [day]	Number of transitions n_{kl}
$w_{10} \rightarrow w_{18}$	2, 2, 1, 4, 3, 1, 2	7
$w_{10} \rightarrow w_{19}$	1	1
$w_{10} \rightarrow w_{21}$	1, 1	2
$w_{10} \rightarrow w_{22}$	1	1
$w_{11} \rightarrow w_2$	1	1
$w_{11} \rightarrow w_9$	2	1
$w_{11} \rightarrow w_{10}$	1, 1, 1, 1, 1, 1	6
$w_{11} \rightarrow w_{13}$	1, 1, 1, 5	4
$w_{11} \rightarrow w_{18}$	1	1
$w_{11} \rightarrow w_{20}$	1	1
$w_{12} \rightarrow w_{10}$	1, 1, 1, 1	4
$w_{12} \rightarrow w_{13}$	1, 1, 1, 1, 1, 1	6
$w_{12} \rightarrow w_{15}$	2	1
$w_{12} \rightarrow w_{17}$	1	1
$w_{12} \rightarrow w_{21}$	1	1
$w_{13} \rightarrow w_1$	1, 5, 1, 1, 2, 1	6
$w_{13} \rightarrow w_2$	2, 2	2
$w_{13} \rightarrow w_5$	1, 2, 8	3
$w_{13} \rightarrow w_6$	1	1
$w_{13} \rightarrow w_9$	1	1
$w_{13} \rightarrow w_{10}$	2, 3, 1, 1, 2, 5, 2, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 3, 3, 2, 1, 2, 2, 2, 1, 1, 2, 1, 1, 1, 1	32
$w_{13} \rightarrow w_{11}$	1, 2	2
$w_{13} \rightarrow w_{12}$	1, 1, 3, 1, 1, 1	6
$w_{13} \rightarrow w_{14}$	1, 1, 1, 5, 3, 2, 1, 1, 2, 1	10
$w_{13} \rightarrow w_{15}$	6	1
$w_{13} \rightarrow w_{16}$	1, 3	2
$w_{13} \rightarrow w_{17}$	9	1
$w_{13} \rightarrow w_{18}$	1	1
$w_{13} \rightarrow w_{19}$	1	1
$w_{13} \rightarrow w_{21}$	1, 2, 1, 4, 1, 1, 1, 5, 1	9
$w_{13} \rightarrow w_{22}$	1, 3	2
$w_{13} \rightarrow w_{23}$	1, 1, 1	3
$w_{13} \rightarrow w_{24}$	1, 5	2
$w_{14} \rightarrow w_9$	2, 1	2
$w_{14} \rightarrow w_{10}$	1, 1, 1	3
$w_{14} \rightarrow w_{11}$	1	1
$w_{14} \rightarrow w_{13}$	1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1	14
$w_{14} \rightarrow w_{21}$	1	1
$w_{15} \rightarrow w_4$	1	1
$w_{15} \rightarrow w_5$	1	1
$w_{15} \rightarrow w_{13}$	1, 1	2
$w_{15} \rightarrow w_{23}$	1	1
$w_{16} \rightarrow w_5$	1	1
$w_{16} \rightarrow w_7$	1	1
$w_{16} \rightarrow w_{13}$	1	1
$w_{17} \rightarrow w_1$	1	1
$w_{17} \rightarrow w_{10}$	1	1
$w_{18} \rightarrow w_9$	2	1
$w_{18} \rightarrow w_{10}$	1, 1, 1, 2, 1, 1, 1, 2, 1	9
$w_{18} \rightarrow w_{10}$	1, 1, 1, 2, 1, 1, 1, 2, 1	9

continuation

Transitions	Transition time θ_{kl} [day]	Number of transitions n_{kl}
$w_{18} \rightarrow w_{13}$	1	1
$w_{18} \rightarrow w_{14}$	1	1
$w_{19} \rightarrow w_{10}$	1	1
$w_{19} \rightarrow w_{11}$	1	1
$w_{20} \rightarrow w_{14}$	1	1
$w_{21} \rightarrow w_{5}$	1	1
$w_{21} \rightarrow w_{6}$	1	1
$w_{21} \rightarrow w_{10}$	1, 1, 1, 1	4
$w_{21} \rightarrow w_{12}$	1	1
$w_{21} \rightarrow w_{13}$	1, 1, 6, 1, 1, 1, 1	7
$w_{21} \rightarrow w_{14}$	1	1
$w_{21} \rightarrow w_{15}$	1	1
$w_{21} \rightarrow w_{18}$	1, 1	2
$w_{22} \rightarrow w_{10}$	1, 1	2
$w_{22} \rightarrow w_{13}$	1	1
$w_{22} \rightarrow w_{15}$	1	1
$w_{23} \rightarrow w_{21}$	1, 1	2
$w_{24} \rightarrow w_{10}$	1	1
$w_{24} \rightarrow w_{13}$	1	1

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