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A novel multiple attribute decision-making method based on Schweizer-Sklar t-norm and t-conorm with q-rung dual hesitant fuzzy information

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The recently proposed q-rung dual hesitant fuzzy sets (q-RDHFSs) not only deal with decision makers' (DMs') hesitancy and uncertainty when evaluating the performance of alternatives, but also give them great liberty to express their assessment information comprehensively. This paper aims to propose a new multiple attribute decision-making (MADM) method where DMs' evaluative values are in form of q-rung dual hesitant fuzzy elements (q-RDHFEs). Firstly, we extend the powerful Schweizer-Sklar t-norm and t-conorm (SSTT) to q-RDHFSs and propose novel operational rules of q-RDHFEs. The prominent advantage of the proposed operations is that they have important parameters q and r, making the information fusion procedure more flexible. Secondly, to effectively cope with the interrelationship among attributes, we extend the Hamy mean (HM) to q-RDHFSs and based on the newly developed operations, we propose the q-rung dual hesitant fuzzy Schweizer-Sklar Hamy mean (q-RDHFSSHM) operator, and the q-rung dual hesitant fuzzy Schweizer-Sklar weighted Hamy mean (q-RDHFSSWHM) operator. The properties of the proposed operators, such as idempotency, boundedness and monotonicity are discussed in detail. Third, we propose a new MADM method based on the q-RDHFSSWHM operator and give the main steps of the algorithm. Finally, the effectiveness, flexibility and advantages of the proposed method are discussed through numerical examples.

Key words: multiple attribute decision-making; Schweizer-Sklar t-norm and t-conorm; q-rung dual hesitant fuzzy Schweizer-Sklar Hamy mean operator; q-rung dual hesitant fuzzy Schweizer-Sklar weighted Hamy mean operator.

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1. Introduction

Multiple attribute decision-making (MADM) theories and methods have been extensively studied by many scholars all around the world [1–7]. MADM refers to a procedure that ranks the feasible alternatives according to some principles and decision makers' (DMs') evaluation information, and selects the optimal one. In most cases, the decision-making information with respect to evaluating alternatives is fuzzy and vague, and DMs can hardly obtain all information of alternatives before providing their evaluations. In light of this, many scholars have put their attention on investigating methods or tools that can portray fuzzy decision-making information appropriately and comprehensively. Atanassov's [8] intuitionistic fuzzy sets (IFSs) and Yager's Pythagorean fuzzy sets (PFSs) [9] are two important extensions of the classical fuzzy sets (FSs) theory. IFSs and PFSs are more powerful and flexible than FSs as they describe fuzzy data or information from not only the membership degrees (MDs) but also the non-membership degrees (NMDs). Due to this characteristic, IFSs and PFSs have attracted widespread attention and been extensively applied in MADM problems. For example, based on consistency and consensus goal programming method, Zhang and Pedrycz [10] introduced a new group decision-making model in which DMs' evaluation information is in term of interval-valued intuitionistic multiplicative preference relations. Garg [11] proposed linguistic intuitionistic fuzzy power aggregation operators based on set pair analysis and investigated their applications in MADM. Zeng et al. [12] proposed a new MADM method based on novel score function of intuitionistic fuzzy numbers and modified VIKOR. Garg and Rani [13] introduced complex intuitionistic fuzzy Archimedean Bonferroni mean operators based MADM method. With respect to intuitionistic fuzzy MADM problems, based on variable weights theory Liu et al. [14] proposed a method to dynamically determine DMs' weights. Liu and Li [15] introduced intuitionistic fuzzy Muirhead mean operators to capture the interrelationship among any numbers of intuitionistic fuzzy values. For MADM problems wherein the decision-making information is given in PFSs, Hussian and Yang [16] introduced Hausdorff metric based distance and similarity measures of PFSs and furthermore a Pythagorean fuzzy TOPSIS method was presented. Zhang et al. [17] introduced a collection of Pythagorean fuzzy generalized Bonferroni mean operators. To make the decision results more reasonable by reducing the negative effects of unreasonable evaluation values, Li et al. [18] proposed the Pythagorean fuzzy power Muirhead mean operators. Xing et al. [19] proposed the Pythagorean fuzzy Choquet integral aggregation operators based on Frank t-norm and t-conorm. For more recent developments of IFSs and PFSs in the field of MADM, we suggest authors to refer [20–28].

In the past years, IFSs and PFSs based MADM theories and methods have received increasing attention. However, the flaw of them is also obvious. That is the rigorous constraints of IFSs and PFSs may cause information loss or distortion to some extent, which narrow their application scope. In light of the shortcomings of IFSs and PFSs, Prof. Yager [29] introduced the generalized orthopair fuzzy sets (GOFSs). The constraint of GOFSs is that the sum of qth power of MD and qth power of NMD does not exceed one, so that GOFSs are also known as q-rung orthopair fuzzy sets (q-ROFSs). Due to the good performance of q-ROFSs in representing fuzzy information, MADM methods based on q-ROFSs have been a new research direction. Liu and Wang [30] proposed the q-rung orthopair fuzzy operations as well as their weighted averaging operators. Afterwards, the q-rung orthopair fuzzy Bonferroni mean (BM) operator [31], the q-rung orthopair fuzzy Archimedean BM operator [32], the q-rung orthopair fuzzy Heronian mean operator [33], the q-rung orthopair fuzzy Hamy mean operator [34], the q-rung orthopair fuzzy Maclaurin symmetric mean (MSM) operator [35], the q-rung orthopair fuzzy power MSM operator [36], the q-rung orthopair fuzzy Muirhead mean operator [37], the q-rung orthopair fuzzy partitioned BM operator [38], the q-rung orthopair fuzzy partitioned Heronian mean [39], and the q-rung orthopair fuzzy partitioned MSM operator [40] have been proposed one after the other. These researches are based on the classical q-ROFSs and have been successfully applied in MADM procedure. Besides, some scholars also studied the extended forms of q-ROFSs and further investigated their applications in MADM. For instance, Wang et al. [41] utilized interval values to represent membership and non-membership degrees in q-ROFSs and proposed q-rung interval-valued orthopair fuzzy sets. P. Liu and W. Liu [42] employed linguistic terms to denote the q-rung orthopair fuzzy MD and NMD and proposed the so-called linguistic q-ROFSs. Additionally, they also proposed operators for linguistic q-ROFSs and applied them in decision-making. Xing et al. [43] extended the classical q-ROFSs to q-rung orthopair fuzzy uncertain linguistic sets, which can more effectively represent DMs' evaluation values.

Although q-ROFSs are efficient to handle MADM problem, their main drawback is that they are powerless to deal with DMs' hesitancy degrees in providing MDs and NMDs of their evaluations, as the MD and NMD of q-ROFSs are denoted by single values. Hence, recently Xu et al. [44] proposed the q-rung dual hesitant fuzzy sets (q-RDHFSs) by allowing the MDs and NMDs in q-ROFSs to be denoted by more than one value. Additionally, Xu et al. [44] proposed a q-rung dual hesitant fuzzy (q-RDHF) Heronian mean aggregation operator (AO) based MADM method. Nevertheless, Xu et al.'s [44] method still have some shortcomings and is still insufficient to deal with complicated realistic MADM problems. First, the operational rules of q-RDHF elements (q-RDHFEs) proposed by Xu et al. [44] are not as flexible. More concretely, the operations of q-RDHFEs given in [44] based on algebraic t-norm and t-conorm, which are stiff in information aggregation process and hence, these operations should be improved. Second, Xu et al.'s [44] MADM method is based on the q-RDHF weighted Heronian mean operators. In other words, although Xu et al.'s [44] method can effectively capture the interrelationship between attributes, it only reflects the interrelationship between any two attributes. If there exists interrelationship among more than two attributes, then Xu et al.'s [44] method is insufficient and inadequate to handle such situations.

Based on the above analysis, the main motivation and aim are to propose a novel MADM method, which overcomes the drawbacks of Xu et al.'s [44] decision-making method. To this end, we first propose some novel operations of q-RDHFEs. The algebraic t-norm and t-conorm are special cases of Archimedean t-norm and t-conorm (ATT). ATT are known as the generalization of many t-norms and t-conorms. Gradually, some new operations based on the special cases of ATT have been introduced, such as Hamacher operations, Frank operations and Schweizer-Sklar operations and so on. The Schweizer-Sklar t-norm and t-conorm (SSTT) is well-known for its ability of producing flexible information aggregation process. Compared with other operational laws, Schweizer-Sklar operations contains a variable parameter, making it more flexible and superior. Owing to this noticeable characteristic, SSTT has been widely investigated in IFSs [45], interval-valued IFSs [46] and single-valued neutrosophic sets [47, 48]. Hence, we propose new operations of q-RDHFEs by extending SSTT into q-RDHFSs. The new proposed operations are more powerful than those presented in [44], as they produce more flexible information process. Second, when considering to propose novel AOs of q-RDHFEs, the property of Hamy mean (HM) in capturing the interrelationship among multiple inputs impresses us deeply and it has been utilized to aggregate IFSs [49], interval-valued IFSs [50], PFSs [51], interval neutrosophic sets [52], etc. Thus, we further generalize HM into q-RDHFSs, and based on the Schweizer-Sklar operations of q-RDHFEs, we propose the q-RDHF Schweizer-Sklar Hamy mean (q-RDHFSSHM) and q-RDHF Schweizer-Sklar weighted Hamy mean (q-RDHFSSWHM) operators. Finally, based on the newly proposed AOs, we introduce a novel MADM method. This method can overcome the above mentioned two shortcomings of Xu et al.'s [44] method.

To clearly present the works of this manuscript, in Section 2 we review basic notions related to q-RDHFSs. Section 3 introduces new operations of q-RDHFEs based on SSTT. Section 4 presents a series of q-RDHF Schweizer-Sklar HM operators and discusses their properties. Section 5 proposes a new MADM method based on the proposed AOs. Section 6 attempts to verify the proposed method and discusses its advantages. The conclusions and problems to be solved in the future are presented in Section 7.

2. Preliminaries

In this section, we introduce some basic concepts, such as q-ROFS, q-RDHFS, HM operator and Schweizer-Sklar operations.

2.1. q-ROFSs and q-RDHFSs

In [29] Prof. Yager proposed a concept of q-ROFSs, which are an extension of traditional IFSs and PFSs. The definition of q-ROFSs is given as follows.

Definition 1 *[29] Let X be an ordinary fixed set, a q-ROFS A defined on X is given by*

$$A = \left\{ \langle x, \mu_A(x), \nu_A(x) \rangle \,\middle|\, x \in X \right\}, \tag{1}$$

where $\mu_A(x)$ *and* $\nu_A(x)$ *represent the MD and NMD respectively, satisfying* $\mu_A(x) \in [0, 1]$, $\nu_A(x) \in [0, 1]$ *and* $0 \leqslant \mu_A(x)^q + \nu_A(x)^q \leqslant 1$, $(q \geqslant 1)$. *The indeterminacy degree is defined as* $\pi_A(x) = (1 - \mu_A(x)^q - \nu_A(x)^q)^{1/q}$. *For convenience,* $(\mu_A(x), \nu_A(x))$ *is called a q-rung orthopair fuzzy number (q-ROFN) by Liu and Wang [29], which can be denoted by* $A = (\mu_A, \nu_A)$.

From Definition 1, we find out that the traditional q-ROFS is characterized by one MD and one NMD. However, in some realistic decision-making scenarios DMs are sometimes hesitant among several values when determining the MD and NMD. Hence, to fully express their evaluation information DMs would like to utilize several values instead of single one to depict the MD and NMD. To comprehensively deal with such kind of situations, Xu et al. [44] introducedthe notion of q-RDHFSs.

Definition 2 *[44] Let X be an ordinary fixed set, a q-RDHFS A defined on X is given by*

$$A = \left\{ \langle x, h_A(x), g_A(x) \rangle \,\middle|\, x \in X \right\}, \tag{2}$$

in which $h_A(x)$ *and* $g_A(x)$ *are two sets of values in [0, 1], denoting the possible membership degrees and non-membership degrees of the element* $x \in X$ *to the set A respectively, with the conditions* $\gamma^q + \eta^q \leqslant 1\,(q \geqslant 1)$, *where* $\gamma \in h_A(x)$, $\eta \in g_A(x)$ *for all* $x \in X$. *For convenience, the pair* $d(x) = (h_A(x), g_A(x))$ *is called a q-RDHFE denoted by* $d = (h, g)$, *with the conditions* $\gamma \in h$, $\eta \in g$, $0 \leqslant \gamma, \eta \leqslant 1$, $0 \leqslant \gamma^q + \eta^q \leqslant 1$. *Evidently, when* $q = 2$, *then q-RDHFS is reduced to Wei and Lu's [53] dual hesitant Pythagorean fuzzy set (DHPFS), and when* $q = 1$, *then q-RDHFS is reduced to Zhu et al.'s [54] dual hesitant fuzzy set (DHFS).*

To compare any two q-RDHFEs, a comparison law for q-RDHFEs was proposed by Xu et al. [44].

Definition 3 *[44] Let $d = (h, g)$ be a q-RDHFE, $S(d) = \left(\frac{1}{\#h}\sum_{\gamma\in h}\gamma\right)^q - \left(\frac{1}{\#g}\sum_{\eta\in g}\eta\right)^q$ be the score function of d, and $H(d) = \left(\frac{1}{\#h}\sum_{\gamma\in h}\gamma\right)^q + \left(\frac{1}{\#g}\sum_{\eta\in g}\eta\right)^q$ be the accuracy function of d, where #h and #g are the numbers of the elements in h and g respectively. For any two q-RDHFEs $d_i = (h_i, g_i)$ $(i = 1, 2)$, we have*

(1) If $S(d_1) > S(d_2)$, then d_1 is superior to d_2, denoted by $d_1 > d_2$;
(2) If $S(d_1) > S(d_2)$, then
If $H(d_1) = H(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1 = d_2$;
If $H(d_1) > H(d_2)$, then d_1 is superior to d_2, denoted by $d_1 > d_2$.

Operational rules of q-RDHFEs are presented as follows.

Definition 4 *[44] Let $d = (h, g)$, $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any three of q-RDHFEs, and λ be a positive real number, then*

(1) $d_1 \oplus d_2 = \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\\ \eta_1\in g_1,\eta_2\in g_2}} \left\{\left\{\left(\gamma_1^q + \gamma_2^q - \gamma_1^q\gamma_2^q\right)^{1/q}\right\}, \{\eta_1\eta_2\}\right\}$;

(2) $d_1 \otimes d_2 = \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\\ \eta_1\in g_1,\eta_2\in g_2}} \left\{\{\gamma_1\gamma_2\}, \left\{\left(\eta_1^q + \eta_2^q - \eta_1^q\eta_2^q\right)^{1/q}\right\}\right\}$;

(3)$\lambda d = \bigcup_{\gamma\in h,\eta\in g} \left\{\left\{\left(1-(1-\gamma^q)^\lambda\right)^{1/q}\right\}, \{\eta^\lambda\}\right\}, \lambda > 0$;

(4) $d^\lambda = \bigcup_{\gamma\in h,\eta\in g} \left\{\{\gamma^\lambda\}, \left\{\left(1-(1-\eta^q)^\lambda\right)^{1/q}\right\}\right\}, \lambda > 0.$

2.2. Hamy mean operator

The HM operator was firstly proposed by Hara et al. [55] for crisp numbers. It can consider the interrelationships among multiple arguments.

Definition 5 *[55] Let α_i $(i = 1, 2, \ldots, n)$ be a collection of crisp numbers, and $k = 1, 2, \ldots, n$, if*

$$HM^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\prod_{j=1}^{k}\alpha_{i_j}\right)^{1/k}}{C_n^k}, \tag{3}$$

then $HM^{(k)}$ is called the Hamy mean, where $(i_1, i_2, \ldots, i_k)$ traversal all the k-tuple combination of $(1, 2, \ldots, n)$, C_n^k is the binomial coefficient.

From Eq. (3), it is clear that the HM satisfies the following properties:
(1) $HM^{(k)}(0, 0, \ldots, 0) = 0$;
(2) $HM^{(k)}(\alpha, \alpha, \ldots, \alpha) = \alpha$;
(3) $HM^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leqslant HM^{(k)}(\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$, if $\alpha_i \leqslant b_i$ for all i;
(4) $\min_i(\alpha_i) \leqslant HM^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leqslant \max_i(\alpha_i)$.

3. Schweizer-Sklar t-norm and t-conorm operational laws of q-RDHFEs

In Ref. [44], Xu et al. proposed some operations of q-RDHFEs, which are shown as Definition 2. It is noted that the operations proposed by Xu et al. [44] are based on algebraic t-norm and t-conorm, which are stiff in fusing information to some extent. Therefore, we try to propose novel and flexible operational rules of q-RDHFEs. As a special case of the ATT, the SSTT is a powerful t-norm and t-conorm, which produces flexible in information aggregation process.

The definition of SSTT is provided as follows.

$$T_{SS,r} = (x^r + y^r - 1)^{1/r}, \tag{4}$$

$$T^*_{SS,r} = 1 - ((1-x)^r + (1-y)^r - 1)^{1/r}, \tag{5}$$

where $r < 0$, $x, y \in [0, 1]$.

In addition, when $r \longrightarrow 0$, we have $T_r(x, y) = xy$ and $T_r * (x, y) = x + y - xy$. They are the algebraic t-norm and algebraic t-conorm.

Based on the SSTT, we propose some new operational laws with respect to q-RDHFEs, which are very useful in the remainder of this paper. Some desirable properties of these operations are also analyzed in the followings.

Definition 6 *Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any two of q-RDHFEs, then the operational laws based on SSTT are defined as follows:*

(1) $$d_1 \oplus_{SS} d_2 = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \eta_1 \in g_1, \eta_2 \in g_2}} \left\{ \left\{ \left(1 - \left(\left(1-\gamma_1^q\right)^r + \left(1-\gamma_2^q\right)^r - 1 \right)^{1/r} \right)^{1/q} \right\}, \left\{ \left(\left(\eta_1^q\right)^r + \left(\eta_2^q\right)^r - 1 \right)^{1/qr} \right\} \right\};$$

(2) $$d_1 \otimes_{SS} d_2 = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \eta_1 \in g_1, \eta_2 \in g_2}} \left\{ \left\{ \left(\left(\gamma_1^q\right)^r + \left(\gamma_2^q\right)^r - 1 \right)^{1/qr} \right\}, \left\{ \left(1 - \left(\left(1-\eta_1^q\right)^r + \left(1-\eta_2^q\right)^r - 1 \right)^{1/r} \right)^{1/q} \right\} \right\}.$$

Theorem 1 *Let $d = (h, g)$ be a q-RDHFEs, then we have multiplication operation $n \cdot_{SS} d$ is a q-RDHFE, and*

$$n \cdot_{SS} d = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1 - \left(n\,(1-\gamma^q)^r - (n-1)\right)^{1/r}\right)^{1/q} \right\},\ \left\{ (n\eta^{qr} - (n-1))^{1/qr} \right\} \right\}, \tag{6}$$

where n is any a positive integer and $n \cdot_{SS} d$ denotes $\overbrace{d \oplus d \oplus \ldots \oplus d}^{n}$.

Proof. At first, we prove the value of $n \cdot_{SS} d$ is a q-RDHFE.

Since $\gamma, \eta \in [0, 1]$ and $0 \leqslant \gamma^q + \eta^q \leqslant 1$, we can obtain

$$0 \leqslant n(1-\gamma^q)^r \leqslant n, \quad \text{and} \quad 0 \leqslant n(1-\gamma^q)^r - (n-1) \leqslant 1.$$

Thus,

$$0 \leqslant \left(1 - \left(n\,(1-\gamma^q)^r - (n-1)\right)^{1/r}\right)^{1/q} \leqslant 1.$$

Similarly, we can get

$$0 \leqslant (n\eta^{qr} - (n-1))^{1/qr} \leqslant 1.$$

Meanwhile,

$$\begin{aligned} 0 &\leqslant \left(\left(1 - \left(n\,(1-\gamma^q)^r - (n-1)\right)^{1/r}\right)^{1/q}\right)^q + \left((n\eta^{q\gamma} - (n-1))^{1/qr}\right)^q \\ &\leqslant 1 - \left(n\,(1-\gamma^q)^r - (n-1)\right)^{1/r} + (n\eta^{qr} - (n-1))^{1/r} \\ &\leqslant 1 - \left(n\,(\eta^q)^r - (n-1)\right)^{1/r} + (n\eta^{qr} - (n-1))^{1/r} = 1. \end{aligned}$$

Therefore, the value of $n \cdot_{SS} d$ satisfies the condition of Definition 2 and is still a q-RDHFE.

In the following, we use mathematical induction on n to prove that Eq. (6) holds for any positive integer n. When $n = 1$, we have

$$1 \cdot_{SS} d = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1 - \left((1-\gamma^q)^r - (1-1)\right)^{1/r}\right)^{1/q} \right\}, \left\{ (\eta^{qr} - (1-1))^{1/qr} \right\} \right\} = (\gamma, \eta) = d,$$

which means that Eq. (6) holds for $n = 1$.

If Eq. (6) holds for $n = k$, that is

$$k \cdot_{SS} d = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1 - \left(k\,(1-\gamma^q)^r - (k-1)\right)^{1r}\right)^{1/q} \right\}, \left\{ (k\eta^{qr} - (k-1))^{1/qr} \right\} \right\}.$$

Then, when $n = k + 1$, based on the Schweizer-Sklar sum operation of two q-RDHFEs, we have

$$
\begin{aligned}
&(k+1)\cdot_{SS} d = (k \cdot_{SS} d) \oplus_{SS} d \\
&= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1-\left(k\,(1-\gamma^q)^r-(k-1)\right)^{1/r}\right)^{1/q} \right\}, \left\{ \left(k\eta^{qr}-(k-1)\right)^{1/qr} \right\} \right\} \oplus_{SS} \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \{\gamma,\eta\} \\
&= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1-\left(\left(\left(k\,(1-\gamma^q)^r-(k-1)\right)^{1/r}\right)^r+(1-\gamma^q)^r-1\right)^{1/r}\right)^{1/q} \right\}, \right. \\
&\qquad\qquad \left. \left\{ \left(k\eta^{qr}-(k-1)+\eta^{qr}-1\right)^{1/qr} \right\} \right\} \\
&= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1-\left(k\,(1-\gamma^q)^r-k+1+(1-\gamma^q)^r-1\right)^{1/r}\right)^{1/q} \right\}, \right. \\
&\qquad\qquad \left. \left\{ \left(k\eta^{qr}-k+1+\eta^{qr}-1\right)^{1/qr} \right\} \right\} \\
&= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1-\left(k\,(1-\gamma^q)^r+(1-\gamma^q)^r-k\right)^{1/r}\right)^{1/q} \right\}, \left\{ \left(k\eta^{qr}+\eta^{qr}-k\right)^{1/qr} \right\} \right\} \\
&= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1-\left((k+1)\,(1-\gamma^q)^r-k\right)^{1/r}\right)^{1/q} \right\}, \left\{ \left((k+1)\eta^{qr}-k\right)^{1/qr} \right\} \right\}.
\end{aligned}
$$

Thus, Eq. (6) holds for $n = k + 1$.

Therefore, Eq. (6) holds for all n, which completes the proof. □

Theorem 2 *Let $d = (h, g)$ be a q-RDHFE, then the power operation $d^{\wedge ssn}$ is a q-RDHFE, and*

$$
d^{\wedge ssn} = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(n\gamma^{qr}-(n-1)\right)^{1/gr} \right\}, \left\{ \left(1-\left(n\,(1-\eta^q)^r-(n-1)\right)^{1/r}\right)^{1/q} \right\} \right\}, \quad (7)
$$

where n is any a positive integer and $d^{\wedge ssn}$ denote $\overbrace{d \otimes d \otimes \ldots \otimes d}^{n}$.

The proof of Theorem 2 is similar to that of Theorem 1, which is omitted here.

Based on Theorems 1 and 2, for any a positive integer$\lambda > 0$, we define the following multiplication and power operations as:

(1) $$\lambda \cdot_{SS} d = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(1-\left(\lambda\left(1-\gamma^q\right)^r-(\lambda-1)\right)^{1/r}\right)^{1/q} \right\}, \left\{ \left(\lambda\eta^{qr}-(\lambda-1)\right)^{1/qr} \right\} \right\}, \quad \lambda > 0;$$

(2) $$d^{\wedge_{SS}\lambda} = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \left\{ \left(\lambda\gamma^{qr}-(\lambda-1)\right)^{1/qr} \right\}, \left\{ \left(1-\left(\lambda\left(1-\eta^q\right)^r-(\lambda-1)\right)^{1/r}\right)^{1/q} \right\} \right\}, \quad \lambda > 0.$$

Moreover, some desirable properties of the operational laws can be easily obtained.

Theorem 3 *Let $d = (h, g)$, $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be any three q-RDHFEs, then*

(1) $$d_1 \oplus_{SS} d_2 = d_2 \oplus_{SS} d_1\,; \tag{8}$$

(2) $$d_1 \otimes_{SS} d_2 = d_2 \otimes_{SS} d_1\,; \tag{9}$$

(3) $$\lambda\left(d_1 \oplus_{SS} d_2\right) = \lambda d_1 \oplus_{SS} \lambda d_2, \quad \lambda \geqslant 0; \tag{10}$$

(4) $$\lambda_1 d \oplus_{SS} \lambda_2 d = \left(\lambda_1+\lambda_2\right) d, \quad \lambda_1, \lambda_2 \geqslant 0; \tag{11}$$

(5) $$d^{\lambda_1} \otimes_{SS} d^{\lambda_2} = (d)^{\lambda_1+\lambda_2}, \quad \lambda_1, \lambda_2 \geqslant 0; \tag{12}$$

(6) $$d_1^{\lambda} \otimes_{SS} d_2^{\lambda} = \left(d_1 \otimes_{SS} d_2\right)^{\lambda}, \quad \lambda \geqslant 0. \tag{13}$$

It is easily to prove Eq. (8) and Eq. (9), so we omit them here. In the following, we prove the remaining formulas.

Proof. According to SSTT operational laws, we can obtain

$$\begin{aligned}
\lambda\left(d_1 \oplus_{SS} d_2\right) &= \bigcup_{\substack{\gamma_1\in h_1,\\ \eta_1\in g_1}} \left\{ \left\{ \left(1-\left(\left(1-\gamma_1^q\right)^r+\left(1-\gamma_2^q\right)^r-1\right)^{1/r}\right)^{1/q} \right\}, \right.\\
&\qquad \left. \left\{ \left(\left(\eta_1^q\right)^r+\left(\eta_2^q\right)^r-1\right)^{1/qr} \right\} \right\} \cdot \lambda \\
&= \bigcup_{\substack{\gamma_1\in h_1,\\ \eta_1\in g_1}} \left\{ \left\{ \left(1-\left(\lambda\left(\left(1-\gamma_1^q\right)^r+\left(1-\gamma_2^q\right)^r-1\right)-(\lambda-1)\right)^{\frac{1}{r}}\right)^{1/q} \right\}, \right.\\
&\qquad \left. \left\{ \left(\lambda\left(\left(\eta_1^q\right)^r+\left(\eta_2^q\right)^r-1\right)-(\lambda-1)\right)^{\frac{1}{qr}} \right\} \right\}
\end{aligned}$$

$$= \bigcup_{\substack{\gamma_1\in h_1,\\ \eta_1\in g_1}} \left\{ \left\{ \left(1-\left(\lambda\left(1-\gamma_1^q\right)^r+\lambda\left(1-\gamma_2^q\right)^r-\lambda-(\lambda-1)\right)^{\frac{1}{r}}\right)^{1/q}\right\}, \left\{\left(\lambda\left(\eta_1^q\right)^r+\lambda\left(\eta_2^q\right)^r-\lambda-(\lambda-1)\right)^{\frac{1}{qr}}\right\}\right\}$$

$$= \bigcup_{\substack{\gamma_1\in h_1,\\ \eta_1\in g_1}} \left\{ \left\{ \left(1-\left(\lambda\left(1-\gamma_1^q\right)^r+\lambda\left(1-\gamma_2^q\right)^r-2\lambda+1\right)^{\frac{1}{r}}\right)^{1/q}\right\}, \left\{\left(\lambda\left(\eta_1^q\right)^r+\lambda\left(\eta_2^q\right)^r-2\lambda+1\right)^{\frac{1}{qr}}\right\}\right\}.$$

Meanwhile, we can obtain that

$$\lambda d_1 \oplus_{SS} \lambda d_2$$
$$= \bigcup_{\substack{\gamma_1\in h_1,\eta_1\in g_1,\\ \gamma_2\in h_2,\eta_2\in g_2}} \left\{ \begin{array}{l} \left\{\left(1-\left(\lambda\left(1-\gamma_1^q\right)^r-(\lambda-1)+\lambda\left(1-\gamma_2^q\right)^r-(\lambda-1)-1\right)^{1/r}\right)^{1/q}\right\}, \\ \left\{\left(\lambda\eta_1^{qr}-(\lambda-1)+\lambda\eta_2^{qr}-(\lambda-1)-1\right)^{1/qr}\right\} \end{array}\right\}$$
$$= \bigcup_{\substack{\gamma_1\in h_1,\eta_1\in g_1,\\ \gamma_2\in h_2,\eta_2\in g_2}} \left\{ \begin{array}{l} \left\{\left(1-\left(\lambda\left(1-\gamma_1^q\right)^r+\lambda\left(1-\gamma_2^q\right)^r-2\left(\lambda-1\right)-1\right)^{1/r}\right)^{1/q}\right\}, \\ \left\{\left(\lambda\eta_1^{qr}+\lambda\eta_2^{qr}-2\left(\lambda-1\right)-1\right)^{1/qr}\right\} \end{array}\right\}$$
$$= \bigcup_{\substack{\gamma_1\in h_1,\eta_1\in g_1,\\ \gamma_2\in h_2,\eta_2\in g_2}} \left\{ \begin{array}{l} \left\{\left(1-\left(\lambda\left(1-\gamma_1^q\right)^r+\lambda\left(1-\gamma_2^q\right)^r-2\lambda+1\right)^{1/r}\right)^{1/q}\right\}, \\ \left\{\left(\lambda\eta_1^{qr}+\lambda\eta_2^{qr}-2\lambda+1\right)^{1/qr}\right\} \end{array}\right\}$$
$$= \lambda(d_1 \oplus_{SS} d_2).$$

Therefore, Eq. (10) holds for $\lambda \geqslant 0$. In addition, we have

$$\lambda_1 d \oplus_{SS} \lambda_2 d$$
$$= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \begin{array}{l} \left\{\left(1-((\lambda_1\left(1-\gamma^q\right)^r-(\lambda_1-1))+(\lambda_2\left(1-\gamma^q\right)^r-(\lambda_2-1))-1)^{1/r}\right)^{1/q}\right\}, \\ \left\{((\lambda_1\eta^{qr}-(\lambda_1-1))+(\lambda_2\eta^{qr}-(\lambda_2-1))-1)^{1/qr}\right\} \end{array}\right\}$$

$$= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \begin{array}{l} \left\{ \left(1 - \left(\lambda_1 \left(1 - \gamma^q \right)^r + \lambda_2 \left(1 - \gamma^q \right)^r - \left(\lambda_1 + \lambda_2 \right) + 1 \right)^{1/r} \right)^{1/q} \right\}, \\ \left\{ \left(\lambda_1 \eta^{qr} + \lambda_2 \eta^{qr} - \left(\lambda_1 + \lambda_2 \right) + 1 \right)^{1/qr} \right\} \end{array} \right\}$$
$$= \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \begin{array}{l} \left\{ \left(1 - \left(\left(\lambda_1 + \lambda_2 \right) \left(1 - \gamma^q \right)^r - \left(\lambda_1 + \lambda_2 \right) + 1 \right)^{1/r} \right)^{1/q} \right\}, \\ \left\{ \left(\left(\lambda_1 + \lambda_2 \right) \eta^{qr} - \left(\lambda_1 + \lambda_2 \right) + 1 \right)^{1/qr} \right\} \end{array} \right\}$$
$$= \left(\lambda_1 + \lambda_2 \right) d.$$

According to the above process, Eq. (11) is kept. Based on the Definition 6 and Theorem 2, we have

$$d^{\lambda_1} \otimes_{SS} d^{\lambda_2} = \bigcup_{\substack{\gamma\in h,\\ \eta\in g}} \left\{ \begin{array}{l} \left\{ \left(\left(\lambda_1 + \lambda_2 \right) \gamma^{qr} - \left(\lambda_1 + \lambda_2 \right) + 1 \right)^{1/qr} \right\}, \\ \left\{ \left(1 - \left(\left(\lambda_1 + \lambda_2 \right) \left(1 - \eta^q \right)^r - \left(\lambda_1 + \lambda_2 \right) + 1 \right)^{1/r} \right)^{1/q} \right\} \end{array} \right\} = (d)^{\lambda_1 + \lambda_2},$$

and

$$d_1^\lambda \otimes_{SS} d_2^\lambda = \bigcup_{\substack{\gamma_1\in h_1, \eta_1\in g_1,\\ \gamma_2\in h_2, \eta_2\in g_2}} \left\{ \begin{array}{l} \left\{ \left(\lambda \left(\gamma_1^{qr} + \gamma_2^{qr} \right) - 2\lambda + 1 \right)^{1/qr} \right\}, \\ \left\{ \left(1 - \left(\lambda \left(\left(1 - \eta_1^q \right)^r + \left(1 - \eta_2^q \right)^r \right) - 2\lambda + 1 \right)^{1/r} \right)^{1/q} \right\} \end{array} \right\}$$
$$= \left(d_1 \otimes_{SS} d_2 \right)^\lambda .$$

So far, Theorem 3 has been proved. □

4. q-Rung dual hesitant fuzzy Schweizer-Sklar Hamy mean operators

In this section, based on the SSTT operational laws of q-RDHFEs, we extend the HM to q-rung dual hesitant fuzzy environment and propose q-rung dual hesitant fuzzy Schweizer-Sklar Hamy mean operator (q-RDHFSSHM) and its weighted form.

4.1. q-RDHFSSHM operator

Definition 7 *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$, then the q-RDHFSSHM operator is defined as*

$$q\text{-}RDHFSSHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{\bigoplus\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(\bigotimes\limits_{j=1}^{k} d_{i_j} \right)^{1/k}}{C_n^k}, \tag{14}$$

where $(i_1, i_2, \ldots, i_k)$ traversal all the k-tuple combination of $(1, 2, \ldots, n)$, and C_n^k is the binomial coefficient.

Theorem 4 *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$, then the aggregated value by the q-RDHFSSHM operator is still a q-RDHFE and*

$$\begin{aligned} & q\text{-}RDHFSSHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) \\ & = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \sum_{j=1}^{k} \left(\gamma_{i_j}^q \right)^r \right)^{1/r} \right) \right)^{1/r} \right)^{1/q} \right\}, \right. \\ & \qquad \left. \left\{ \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \sum_{j=1}^{k} \left(1 - \eta_{i_j}^q \right)^r \right)^{1/r} \right)^r \right)^{1/qr} \right\} \right\}. \end{aligned} \tag{15}$$

Proof. According to Schweizer-Sklar operational laws of q-RDHFEs, based Theorems 1 and 2 it is easy to prove that the aggregated value by the q-RDHFSSHM operator is a q-RDHFE. Besides, we can obtain

$$\begin{aligned} \bigotimes_{j=1}^{k} d_{i_j} = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} & \left\{ \left\{ \left(\sum_{j=1}^{k} \left(\gamma_{i_j}^q \right)^r - (k-1) \right)^{1/qr} \right\}, \right. \\ & \left. \left\{ \left(1 - \left(\sum_{j=1}^{k} \left(1 - \eta_{i_j}^q \right)^r - (k-1) \right)^{1/r} \right)^{1/q} \right\} \right\}, \end{aligned}$$

and

$$\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{1/qr}\right\}, \left\{\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{1/q}\right\}\right\}.$$

Further, we can get,

$$\begin{aligned}&\bigoplus_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k}\\ &= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(1-\left(\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r}-\left(C_n^k-1\right)\right)^{1/r}\right)^{1/q}\right\},\right.\\ &\qquad\left.\left\{\left(\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r}-\left(C_n^k-1\right)\right)^{1/qr}\right\}\right\},\end{aligned}$$

Finally,

$$\begin{aligned}&\frac{\bigoplus_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_n^k}\\ &= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r}\right)^{1/r}\right)^{1/q}\right\},\right.\\ &\qquad\left.\left\{\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r}\right)^{1/qr}\right\}\right\}.\end{aligned}$$

Therefore, the proof of Theorem 4 is completed.

Property 1 *(**Idempotency**) Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs, if all q-RDHFEs are equal, i.e., $d_i = d = (h, g)$ for all i, and d only has one MD and one NMD, then*

$$q\text{-}RDHFSSHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = d. \tag{16}$$

Proof. Since $d_i = d = (h, g)$, we can get

$$
\begin{aligned}
&q\text{-RDHFSSHM}^{(k)}(d_1, d_2, \ldots, d_n) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^q\right)^r\right)^{1/r}\right)^r\right)^{1/r}\right)^{1/q}\right\},\right. \\
&\qquad\left.\left\{\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^q\right)^r\right)^{1/r}\right)^r\right)^{1/qr}\right\}\right\}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^q\right)^r\right)^{1/r}\right)^r\right)^{1/r}\right)^{1/q} \\
&\qquad = \left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\gamma^{qr}\right)^{1/r}\right)^r\right)^{1/r}\right)^{1/q},
\end{aligned}
$$

and

$$
\begin{aligned}
&= \left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\gamma^{qr}\right)^{1/r}\right)^r\right)^{1/r}\right)^{1/q} \\
&\qquad = \left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\gamma^q\right)^r\right)^{1/r}\right)^{1/q},
\end{aligned}
$$

and

$$
= \left(1-\left(\frac{1}{C_n^k}C_n^k\left(1-\gamma^q\right)^r\right)^{1/r}\right)^{1/q} = \left(1-\left(1-\gamma^q\right)\right)^{1/q} = \gamma.
$$

Similarly, we can have

$$
\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^q\right)^r\right)^{1/r}\right)^r\right)^{1/qr} = \eta.
$$

Hence, we can obtain $q\text{-RDHFSSHM}^{(k)}(d_1, d_2, \ldots, d_n) = d$.

Property 2 *(**Monotonicity**) Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ and $d_i = \left(h'_i, g'_i\right)$ $(i = 1, 2, \ldots, n)$ be two collections of q-RDHFEs. If $\gamma_i \geqslant \gamma'_i$ and $\eta_i \leqslant \eta'_i$ hold for all $i = 1, 2, \ldots, n$, where $\gamma_i \in h_i$, $\gamma'_i \in h'_i$, $\eta_i \in g_i$ and $\eta'_i \in g'_i$, then*

$$q\text{-}RDHFSSHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) \geqslant q\text{-}RDHFSSHM^{(k)}\left(d'_1, d'_2, \ldots, d'_n\right). \quad (17)$$

Proof. Let

$$q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \begin{array}{l} \left\{ \left(1 - \left(\frac{1}{C_n^k} \sum\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \sum\limits_{j=1}^{k} \gamma_{i_j}^{qr} \right)^{1/r} \right)^r \right)^{1/r} \right)^{1/q} \right\}, \\ \left\{ \left(\frac{1}{C_n^k} \sum\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \sum\limits_{j=1}^{k} \left(1 - \eta_{i_j}^{q} \right)^r \right)^{1/r} \right)^r \right)^{1/qr} \right\} \end{array} \right\} = (h, g)$$

and

$$q\text{-RDHFSSHM}^{(k)}\left(d'_1, d'_2, \ldots, d'_n\right) = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \begin{array}{l} \left\{ \left(1 - \left(\frac{1}{C_n^k} \sum\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \sum\limits_{j=1}^{k} {\gamma'}_{i_j}^{qr} \right)^{1/r} \right)^r \right)^{1/r} \right)^{1/q} \right\}, \\ \left\{ \left(\frac{1}{C_n^k} \sum\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \sum\limits_{j=1}^{k} \left(1 - {\eta'}_{i_j}^{q} \right)^r \right)^{1/r} \right)^r \right)^{1/qr} \right\} \end{array} \right\} = (h', g')$$

Since $\gamma_i \geqslant \gamma'_i$, $r < 0$, based on the SSTT operational laws of q-RDHFEs, we have

$$\frac{1}{k}\left(\sum_{j=1}^{k} \gamma_{i_j}^{qr}\right) \geqslant \frac{1}{k}\left(\sum_{j=1}^{k} {\gamma'}_{i_j}^{qr}\right)$$

and

$$\left(1 - \left(\frac{1}{k}\left(\sum_{j=1}^{k} \gamma_{i_j}^{qr}\right)\right)^{1/r}\right)^r \leqslant \left(1 - \left(\frac{1}{k}\left(\sum_{j=1}^{k} {\gamma'}_{i_j}^{qr}\right)\right)^{1/r}\right)^r.$$

Then,

$$\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\gamma_{i_j}^{qr}\right)\right)^{1/r}\right)^{r}\right)^{1/r}$$
$$\leqslant\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\gamma'^{qr}_{i_j}\right)\right)^{1/r}\right)^{r}\right)^{1/r}.$$

Furthermore,

$$\left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\gamma_{i_j}^{qr}\right)\right)^{1/r}\right)^{r}\right)^{1/r}\right)^{1/q}$$
$$\geqslant\left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\gamma'^{qr}_{i_j}\right)\right)^{1/r}\right)^{r}\right)^{1/r}\right)^{1/q}.$$

Therefore, we get $\gamma_i \geqslant \gamma'_i$. Similarly, we also yield $\eta_i \leqslant \eta'_i$.

Consequently, we can get

$$q\text{-RDHFSSHM}^{(k)}(d_1, d_2, \ldots, d_n) \geqslant q\text{-RDHFSSHM}^{(k)}(d'_1, d'_2, \ldots, d'_n).$$

Property 3 *(**Boundedness**) Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs, if*

$$d^+ = \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\max_{i=1}^{n}(\gamma_i)\right\},\left\{\min_{i=1}^{n}(\eta_i)\right\}\right\},$$

and

$$d^- = \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\min_{i=1}^{n}(\gamma_i)\right\},\left\{\max_{i=1}^{n}(\eta_i)\right\}\right\},$$

then,

$$d^- \leqslant q\text{-}RDHFSSHM^{(k)}(d_1, d_2, \ldots, d_n) \leqslant d^+. \tag{18}$$

Proof. According to property 2, we can easily obtain that

$$q\text{-RDHFSSHM}^{(k)}(d^-, d^-, \ldots, d^-) \leqslant q\text{-RDHFSSHM}^{(k)}(d_1, d_2, \ldots, d_n),$$

$$q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) \leqslant q - RDHFSSHM^{(k)}\left(d^+, d^+, \ldots, d^+\right).$$

In addition, both d^- and d^+ only have one MD and one NMD. Therefore,

$$q\text{-RDHFSSHM}^{(k)}\left(d^-, d^-, \ldots, d^-\right) = d^-,$$
$$q\text{-RDHFSSHM}^{(k)}\left(d^+, d^+, \ldots, d^+\right) = d^+.$$

Hence, we can get $d^- \leqslant q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) \leqslant d^+$.

Property 4 *(**Commutativity**) Let $d_i' = \left(h_i', g_i'\right)$ $(i = 1, 2, \ldots, n)$ be any permutation of $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$. Then*

$$q\text{-}RDHFSSHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = q\text{-}RDHFSSHM^{(k)}\left(d_1', d_2', \ldots, d_n'\right). \quad (19)$$

Proof. Let

$$\begin{aligned} & q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) \\ &= \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \gamma_{i_j}^{qr} \right) \right)^{1/r} \right)^{r} \right)^{1/r} \right)^{1/q} \right\}, \right. \\ & \qquad \left. \left\{ \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \left(1 - \eta_{i_j}^{q}\right)^{r} \right) \right)^{1/r} \right)^{r} \right)^{1/qr} \right\} \right\}, \end{aligned}$$

and

$$\begin{aligned} & q\text{-RDHFSSHM}^{(k)}\left(d_1', d_2', \ldots, d_n'\right) \\ &= \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} {\gamma'}_{i_j}^{qr} \right) \right)^{1/r} \right)^{r} \right)^{1/r} \right)^{1/q} \right\}, \right. \\ & \qquad \left. \left\{ \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \left(1 - {\eta'}_{i_j}^{q}\right)^{r} \right) \right)^{1/r} \right)^{r} \right)^{1/qr} \right\} \right\}. \end{aligned}$$

Since $\left\{d_1', d_2', \ldots, d_n'\right\}$ is any permutation of $\{d_1, d_2, \ldots, d_n\}$, then we have

$$\bigcup_{\gamma_{i_j} \in h_{i_j}} \left\{ \left(\left(1 - \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \gamma_{i_j}^{qr} \right) \right)^{1/r} \right)^r \right)^{1/r} \right)^{1/q} \right) \right\}$$

$$= \bigcup_{\gamma_{i_j} \in h_{i_j}} \left\{ \left(\left(1 - \left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \gamma'^{qr}_{i_j} \right) \right)^{1/r} \right)^r \right)^{1/r} \right)^{1/q} \right) \right\},$$

and

$$\bigcup_{\eta_{i_j} \in g_{i_j}} \left\{ \left(\left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \left(1 - \eta_{i_j}^q \right)^r \right) \right)^{1/r} \right)^r \right) \right)^{1/qr} \right) \right\}$$

$$= \bigcup_{\eta_{i_j} \in g_{i_j}} \left\{ \left(\left(\frac{1}{C_n^k} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\frac{1}{k} \left(\sum_{j=1}^{k} \left(1 - \eta'^q_{i_j} \right)^r \right) \right)^{1/r} \right)^r \right) \right)^{1/qr} \right) \right\}.$$

Thus, we can get

$$q\text{-RDHFSSHM}^{(k)}(d_1, d_2, \ldots, d_n) = q\text{-RDHFSSHM}^{(k)}\left(d_1', d_2', \ldots, d_n'\right).$$

In the following, we will discuss some special cases of the q-RDHFSSHM operator.

(1) When $r = 0$, the q-RDHFSSHM operator reduces to the q-rung dual hesitant fuzzy Hamy mean (q-RDHFHM).

$$\begin{aligned} &q\text{-RDHFSSHM}^{(k)}_{r=0}(d_1, d_2, \ldots, d_n) \\ &\quad = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \left(1 - \left(\prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\prod_{j=1}^{k} \gamma_{i_j} \right)^{q/k} \right) \right)^{1/C_n^k} \right)^{1/q} \right\}, \right. \\ &\qquad \left. \left\{ \prod_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \eta_{i_j}^q \right) \right)^{1/k} \right)^{1/(qC_n^k)} \right\} \right\}. \end{aligned} \tag{20}$$

(2) When $q = 1$, the q-RDHFSSHM operator reduces to the dual hesitant fuzzy Schweizer-Sklar Hamy mean (DHFSSHM) operator.

$$
\begin{aligned}
&q\text{-RDHFSSHM}^{(k)}_{q=1}\left(d_1, d_2, \ldots, d_n\right) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\gamma_{i_j}^r\right)^{1/r}\right)^r\right)^{1/r}\right\},\right. \\
&\qquad \left.\left\{\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}\right)^r\right)^{1/r}\right)^r\right)^{1/r}\right\}\right\}. \qquad (21)
\end{aligned}
$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the dual hesitant fuzzy Hamy mean (DHFHM) operator.

$$
\begin{aligned}
&q\text{-RDHFSSHM}^{(k)}_{r=0,q=1}\left(d_1, d_2, \ldots, d_n\right) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{1-\left(\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\prod_{j=1}^{k}\gamma_{i_j}\right)^{1/k}\right)\right)^{1/C_n^k}\right\},\right. \\
&\qquad \left.\left\{\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\prod_{j=1}^{k}\left(1-\eta_{i_j}\right)\right)^{1/k}\right)^{1/C_n^k}\right\}\right\}. \qquad (22)
\end{aligned}
$$

(3) When $q = 2$, the q-RDHFSSHM operator reduces to the dual hesitant Pythagorean fuzzy Schweizer-Sklar Hamy mean (DHPFSSHM) operator.

$$
\begin{aligned}
&q\text{-RDHFSSHM}^{(k)}_{q=2}\left(d_1, d_2, \ldots, d_n\right) \\
&= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(1-\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\gamma_{i_j}^{2r}\right)\right)^{1/r}\right)^r\right)^{1/r}\right)^{1/2}\right\},\right. \\
&\qquad \left.\left\{\left(\frac{1}{C_n^k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\left(1-\eta_{i_j}^2\right)^r\right)\right)^{1/r}\right)^r\right)^{1/2r}\right\}\right\}. \qquad (23)
\end{aligned}
$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the dual hesitant Pythagorean fuzzy Hamy mean (DHPFHM) operator.

$$q\text{-RDHFSSHM}^{(k)}_{r=0,q=2}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \left(1 - \left(\prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \left(\prod_{j=1}^{k} \gamma_{i_j} \right)^{2/k} \right) \right)^{1/C_n^k} \right)^{1/2} \right\}, \right.$$
$$\left. \left\{ \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \eta_{i_j}^2\right) \right)^{1/k} \right)^{1/(2C_n^k)} \right\} \right\}. \tag{24}$$

(4) When $k = 1$, the q-RDHFSSHM operator reduces to the q-rung dual hesitant fuzzy Schweizer-Sklar average (q-RDHFSSA) operator.

$$q\text{-RDHFSSHM}^{(k)}_{k=1}(d_1, d_2, \ldots, d_n)$$
$$= \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(\bigotimes\limits_{j=1}^{k} d_{i_j} \right)^{1/k}}{C_n^k} = \frac{1}{n} \bigoplus_{i=1}^{n} d_i$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}} \left\{ \left\{ \left(1 - \left(\frac{1}{n} \left(\sum_{i=1}^{n} \left(1-\gamma_i^q\right)^r \right) \right)^{1/r} \right)^{1/q} \right\}, \left\{ \left(\frac{1}{n} \left(\sum_{i=1}^{n} \left(\eta_i^q\right)^r \right) \right)^{1/qr} \right\} \right\}. \tag{25}$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the q-rung dual hesitant fuzzy average (q-RDHFA) operator.

$$q\text{-RDHFSSHM}^{(k)}_{r=0,k=1}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}} \left\{ \left\{ \left(1 - \left(\prod_{i=1}^{n} \left(1-\gamma_i^q\right) \right)^{1/n} \right)^{1/q} \right\}, \left\{ \left(\prod_{i=1}^{n} \eta_i \right)^{1/n} \right\} \right\}. \tag{26}$$

(5) When $k = n$, the q-RDHFSSHM operator reduces to the q-rung dual hesitant fuzzy Schweizer-Sklar geometric (q-RDHFSSG) operator.

$$q\text{-RDHFSSHM}_{k=n}^{(k)}(d_1, d_2, \ldots, d_n) = \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_n^k} = \left(\bigotimes_{i=1}^{n} d_i\right)^{1/n}$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\frac{1}{n}\left(\sum_{i=1}^{n}\left(\gamma_i^q\right)^r\right)\right)^{1/qr}\right\},\left\{\left(1-\left(\frac{1}{n}\left(\sum_{i=1}^{n}\left(1-\eta_i^q\right)^r\right)\right)^{1/r}\right)^{1/q}\right\}\right\}. \tag{27}$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the q-rung dual hesitant fuzzy geometric average (q-RDHFGA) operator.

$$q\text{-RDHFSSHM}_{r=0,k=n}^{(k)}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_j\in h_j,\\ \eta_j\in g_j}}\left\{\left\{\left(\prod_{j=1}^{n}\gamma_j\right)^{1/n}\right\},\left\{\left(1-\left(\prod_{j=1}^{n}\left(1-\eta_j^q\right)\right)^{1/n}\right)^{1/q}\right\}\right\}. \tag{28}$$

(6) When $k = 1$ and $q = 1$, the q-RDHFSSHM operator reduces to the dual hesitant fuzzy Schweizer-Sklar arithmetic average (DHFSSA) operator.

$$q\text{-RDHFSSHM}_{k=1,q=1}^{(k)}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{1-\left(\frac{1}{n}\left(\sum_{i=1}^{n}\left(1-\gamma_i\right)^r\right)\right)^{1/r}\right\},\left\{\left(\frac{1}{n}\left(\sum_{i=1}^{n}\left(\eta_i\right)^r\right)\right)^{1/r}\right\}\right\}. \tag{29}$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the dual hesitant fuzzy average (DHFA) operator.

$$q\text{-RDHFSSHM}_{r=0,k=1,q=1}^{(k)}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{1-\left(\prod_{i=1}^{n}\left(1-\gamma_i\right)\right)^{1/n}\right\},\left\{\left(\prod_{i=1}^{n}\eta_i\right)^{1/n}\right\}\right\}. \tag{30}$$

(7) When $k = 1$ and $q = 2$, the q-RDHFSSHM operator reduces to the dual hesitant Pythagorean fuzzy Schweizer-Sklar average (DHPFSSA) operator.

$$q\text{-RDHFSSHM}^{(k)}_{k=1,q=2}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(1-\left(\frac{1}{n}\left(\sum_{i=1}^{n}\left(1-\gamma_i^2\right)^r\right)\right)^{1/r}\right)^{1/2}\right\},\left\{\left(\frac{1}{n}\left(\sum_{i=1}^{n}\eta_i^{2r}\right)\right)^{1/2r}\right\}\right\}. \quad (31)$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the dual hesitant Pythagorean fuzzy average (DHPFA) operator.

$$q\text{-RDHFSSHM}^{(k)}_{r=0,k=1,q=2}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(1-\left(\prod_{i=1}^{n}\left(1-\gamma_i^2\right)\right)^{1/n}\right)^{1/2}\right\},\left\{\left(\prod_{i=1}^{n}\eta_i\right)^{1/n}\right\}\right\}. \quad (32)$$

(8) When $k = n$ and $q = 1$, the q-RDHFSSHM operator reduces to the dual hesitant fuzzy Schweizer-Sklar geometric average (DHFSSGA) operator.

$$q\text{-RDHFSSHM}^{(k)}_{k=n,q=1}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\frac{1}{n}\left(\sum_{i=1}^{n}(\gamma_i)^r\right)\right)^{1/r}\right\},\left\{1-\left(\frac{1}{n}\left(\sum_{i=1}^{n}(1-\eta_i)^r\right)\right)^{1/r}\right\}\right\}. \quad (33)$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the dual hesitant fuzzy geometric (DHFG) operator.

$$q\text{-RDHFSSHM}^{(k)}_{r=0,k=n,q=1}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_j\in h_j,\\ \eta_j\in g_j}}\left\{\left\{\left(\prod_{j=1}^{n}\gamma_j\right)^{1/n}\right\},\left\{1-\left(\prod_{j=1}^{n}(1-\eta_j)\right)^{1/n}\right\}\right\}. \quad (34)$$

(9) When $k = n$ and $q = 2$, the q-RDHFSSHM operator reduces to the dual hesitant Pythagorean fuzzy Schweizer-Sklar geometric (DHPFSSG) operator.

$$q-RDHFSSHM^{(k)}_{k=n,q=2}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\frac{1}{n}\left(\sum_{i=1}^{n}\gamma_i^{2r}\right)\right)^{1/2r}\right\},\left\{\left(1-\left(\frac{1}{n}\left(\sum_{i=1}^{n}\left(1-\eta_i^2\right)^r\right)\right)^{1/r}\right)^{1/2}\right\}\right\}. \quad (35)$$

In this case, if $r = 0$, then the q-RDHFSSHM operator reduces to the dual hesitant Pythagorean fuzzy geometric (DHPFG) operator.

$$q\text{-RDHFSSHM}^{(k)}_{r=0,k=n,q=2}(d_1, d_2, \ldots, d_n) = \bigcup_{\substack{\gamma_j \in h_j, \\ \eta_j \in g_j}} \left\{ \left\{ \left(\prod_{j=1}^{n} \gamma_j \right)^{1/n} \right\}, \left\{ \left(1 - \left(\prod_{j=1}^{n} \left(1 - \eta_j^2 \right) \right)^{1/n} \right)^{1/2} \right\} \right\}. \tag{36}$$

4.2. q-RDHFSSWHM operator

The q-RDHFSSHM operator can only consider the interrelationship among attributes, but not the self-importance of the aggregated arguments. To overcome this shortcoming, we propose the q-rung dual hesitant fuzzy Schweizer-Sklar weighted Hamy mean (q-RDHFSSWHM) operator, which can take the corresponding weights of aggregated q-RDHFEs into consideration.

Definition 8 *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of d_i $(i = 1, 2, \ldots, n)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The q-RDHFSSWHM operator is defined as*

$$q\text{-}RDHFSSWHM^{(k)}(d_1, d_2, \ldots, d_n) = \begin{cases} \dfrac{\bigoplus\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \sum\limits_{j=1}^{k} w_{i_j} \right) \left(\bigotimes\limits_{j=1}^{k} d_{i_j} \right)^{1/k}}{C_{n-1}^{k}}, & (1 \leqslant k < n), \\ \bigotimes\limits_{i=1}^{k} d_i^{\frac{1-w_i}{n-1}} & (k = n), \end{cases} \tag{37}$$

where $(i_1, i_2, \ldots, i_k)$ traversal all the k-tuple combination of $(1, 2, \ldots, n)$, and C_n^k is the binomial coefficient.

Similarly, we can obtain the following aggregated value by the q-RDHFSSWHM according to the Schweizer-Sklar operational laws of q-RDHFEs.

Theorem 5 *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$. The aggregated value by the q-RDHFSSWHM operator is still a q-RDHFE and*

$$q\text{-}RDHFSSWHM^{(k)}(d_1, d_2, \ldots, d_n) = \frac{\bigoplus_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\sum_{j=1}^{k} w_{i_j}\right)\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_{n-1}^{k}} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}$$

$$\left\{\begin{array}{l}\left\{\left(1-\left(\frac{1}{C_{n-1}^{k}}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\left(1-\sum_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{\frac{1}{r}}\right)^{r}+\sum_{j=1}^{k} w_{i_j}\right)\right)^{\frac{1}{r}}\right)^{\frac{1}{q}}\right\},\\ \left\{\left(\frac{1}{C_{n-1}^{k}}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\left(1-\sum_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{\frac{1}{r}}\right)^{\frac{r}{q}}+\sum_{j=1}^{k} w_{i_j}\right)\right)^{\frac{1}{qr}}\right\}\end{array}\right\},$$

$$(1 \leqslant k < n), \qquad (38)$$

or

$$q\text{-}RDHFSSWHM^{(k)}(d_1, d_2, \ldots, d_n) = \bigotimes_{i=1}^{k} \alpha_i^{\frac{1-w_i}{n-1}}$$

$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}} \left\{\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\gamma_i^{qr}-\left(\frac{1-w_i}{n-1}-1\right)\right)-(k-1)\right)^{1/qr}\right\},\right.$$

$$\left.\left\{\left(1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(1-\eta^{q}\right)^{r}-\left(\frac{1-w_i}{n-1}-1\right)\right)-(k-1)\right)^{1/r}\right)^{1/q}\right\}\right\} \quad (k=n). \qquad (39)$$

Proof. (1) For the first case, when $1 \leqslant k < n$, according to SSTT operational laws of q-RDHFEs, we can get

$$\bigotimes_{j=1}^{k} d_{i_j} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(\sum_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}-(k-1)\right)^{1/qr}\right\},\right.$$

$$\left.\left\{\left(1-\left(\sum_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}-(k-1)\right)^{1/r}\right)^{1/q}\right\}\right\},$$

and

$$\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{1/qr}\right\},\right.$$

$$\left.\left\{\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{1/q}\right\}\right\}.$$

Further,

$$\left(1-\sum_{j=1}^{k} w_{i_j}\right)\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k}$$

$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\begin{array}{l} \left\{\left(1-\left(\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum\limits_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r}+\sum\limits_{j=1}^{k} w_{i_j}\right)^{1/r}\right)^{1/q}\right\}, \\ \left\{\left(\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum\limits_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r/q}+\sum\limits_{j=1}^{k} w_{i_j}\right)^{1/qr}\right\} \end{array}\right\},$$

and

$$\bigoplus_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\sum_{j=1}^{k} w_{i_j}\right)\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}$$

$$\left\{\begin{array}{l} \left\{\left(1-\left(\sum\limits_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum\limits_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{\frac{1}{r}}\right)^{r}+\sum\limits_{j=1}^{k} w_{i_j}\right)-\left(C_{n-1}^{k}-1\right)\right)^{\frac{1}{r}}\right)^{\frac{1}{q}}\right\}, \\ \left\{\left(\sum\limits_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum\limits_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{\frac{1}{r}}\right)^{\frac{r}{q}}+\sum\limits_{j=1}^{k} w_{i_j}\right)-\left(C_{n-1}^{k}-1\right)\right)^{\frac{1}{qr}}\right\} \end{array}\right\}.$$

Finally,

$$\frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_{n-1}^{k}} = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}$$

$$\left\{\begin{array}{l}\left\{\left(1-\left(\dfrac{1}{C_{n-1}^{k}}\sum\limits_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum\limits_{j=1}^{k}\left(\gamma_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r}+\sum\limits_{j=1}^{k} w_{i_j}\right)\right)^{1/r}\right)^{1/q}\right\},\\ \left\{\left(\dfrac{1}{C_{n-1}^{k}}\sum\limits_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum\limits_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)^{r}\right)^{1/r}\right)^{r/q}+\sum\limits_{j=1}^{k} w_{i_j}\right)\right)^{1/qr}\right\}\end{array}\right\}.$$

(2) For the second case, when $k = n$, according to SSTT operational laws of q-RDHFEs, we have

$$d_i^{\frac{1-w_i}{n-1}} = \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\frac{1-w_i}{n-1}\gamma_i^{qr}-\left(\frac{1-w_i}{n-1}-1\right)\right)^{1/qr}\right\},\right.$$

$$\left.\left\{\left(1-\left(\frac{1-w_i}{n-1}\left(1-\eta^{q}\right)^{r}-\left(\frac{1-w_i}{n-1}-1\right)\right)^{1/r}\right)^{1/q}\right\}\right\},$$

and

$$\bigotimes_{i=1}^{k} d_i^{\frac{1-w_i}{n-1}} = \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\gamma_i^{qr}-\left(\frac{1-w_i}{n-1}-1\right)\right)-(k-1)\right)^{1/qr}\right\},\right.$$

$$\left.\left\{\left(1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(1-\eta^{q}\right)^{r}-\left(\frac{1-w_i}{n-1}-1\right)\right)-(k-1)\right)^{1/r}\right)^{1/q}\right\}\right\}.$$

Besides, it is obviously that the aggregated value q-RDHFSSWHM is also a q-RDHFE. Therefore, Theorem 5 is kept.

Similarly, we introduce some properties of q-RDHFSSWHM.

Property 5 ***(Idempotency)*** *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs and $k = 1, 2, \ldots, n$. If all q-RDHFEs are equal, i.e. $d_i = d = (h, g)$ for all i, and d only has one MD and one NMD, then*

$$q\text{-}RDHFSSWHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = d. \tag{40}$$

The proof of Property 5 is similar to that of Property 1, so we omit it here.

Property 6 ***(Monotonicity)*** *Let $d_i = (h_i, g_i)$ and $d'_i = \left(h'_i, g'_i\right)$ $(i = 1, 2, \ldots, n)$ be two collections of q-RDHFEs. If $\gamma_i \geqslant \gamma'_i$ and $\eta_i \geqslant \eta'_i$ hold for all $i = 1, 2, \ldots, n$, where $\gamma_i \in h_i, \gamma'_i \in h'_i$, $\eta_i \in g_i$ and $\eta'_i \in g'_i$, then*

$$q\text{-}RDHFSSWHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) \geqslant q\text{-}RDHFSSWHM^{(k)}\left(d'_1, d'_2, \ldots, d'_n\right). \tag{41}$$

The proof of Property 6 is similar to that of Property 2, which is omitted here.

Property 7 ***(Boundedness)*** *Let $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-RDHFEs, if*

$$d^+ = \bigcup_{\substack{\gamma_i \in h_i,\\ \eta_i \in g_i}} \left\{ \left\{ \max_{i=1}^{n} (\gamma_i) \right\}, \left\{ \min_{i=1}^{n} (\eta_i) \right\} \right\},$$

and

$$d^- = \bigcup_{\substack{\gamma_i \in h_i,\\ \eta_i \in g_i}} \left\{ \left\{ \min_{i=1}^{n} (\gamma_i) \right\}, \left\{ \max_{i=1}^{n} (\eta_i) \right\} \right\},$$

then

$$d^- \leqslant q\text{-}RDHFSSWHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) \leqslant d^+. \tag{42}$$

The proof of Property 7 is similar to that of Property 3, and we will not describe it in detail here.

Property 8 ***(Commutativity)*** *Let $d'_i = \left(h'_i, g'_i\right)$ $(i = 1, 2, \ldots, n)$ be any permutation of $d_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$, then*

$$q\text{-}RDHFSSWHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = q\text{-}RDHFSSWHM^{(k)}\left(d'_1, d'_2, \ldots, d'_n\right). \tag{43}$$

Proof. Since that $(d_1, d_2, \ldots, d_n)$ is any permutation of $\left(d'_1, d'_2, \ldots, d'_n\right)$, then

$$\frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_{n-1}^{k}} = \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(\bigotimes\limits_{j=1}^{k} d'_{i_j}\right)^{1/k}}{C_{n-1}^{k}} \quad (1 \leqslant k < n),$$

$$\bigotimes_{i=1}^{k} d_i^{\frac{1-w_i}{n-1}} = \bigotimes_{i=1}^{k} {d'}_i^{\frac{1-w_i}{n-1}} \quad (k = n).$$

Therefore,

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = q\text{-RDHFSSWHM}^{(k)}\left(d'_1, d'_2, \ldots, d'_n\right)$$

is proved.

Theorem 6 *When $w_i = 1/n$ $(i = 1, 2\ldots, n)$, the q-RDHFSSWHM operator is simplified to the q-RDHFSSHM operator.*

Proof. According to Theorem 5, we should take two cases into account.

(1) For the first case, when $1 \leqslant k < n$,

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_{n-1}^{k}}$$

$$= \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\frac{k}{n}\right)\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_{n-1}^{k}} = \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\frac{n-k}{n}\right)\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_{n-1}^{k}}$$

$$= \frac{\left(\frac{n-k}{n}\right)\left(\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}\right)}{C_{n-1}^{k}} = \frac{n-k}{nC_{n-1}^{k}}\left(\bigoplus_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes_{j=1}^{k} d_{i_j}\right)^{1/k}\right)$$

$$= \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_n^k} = q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right).$$

(2) For the second case, when $k = n$,

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \bigotimes_{i=1}^{k} d_i^{\frac{1-w_i}{n-1}}$$

$$= \bigotimes_{i=1}^{n} d_i^{\frac{1-w_i}{n-1}} = \bigotimes_{i=1}^{n} d_i^{\frac{1-\frac{1}{n}}{n-1}} = \bigotimes_{i=1}^{n} d_i^{\frac{n-1}{n(n-1)}} = \bigotimes_{i=1}^{n} d_i^{\frac{1}{n}},$$

$$q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(\bigotimes\limits_{j=1}^{k} d_{i_j}\right)^{1/k}}{C_n^k}$$

$$= \frac{\bigoplus\limits_{1\leqslant i_1<\ldots<i_n\leqslant n}\left(\bigotimes\limits_{j=1}^{n} d_{i_j}\right)^{1/n}}{C_n^n} = \bigotimes_{i=1}^{n} d_i^{\frac{1}{n}}.$$

Therefore,

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = q\text{-RDHFSSHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right),$$

and Theorem 6 is proved.

In the following, we will discuss some special cases of the q-RDHFSSWHM operator.

(1) When $r = 0$, the q-RDHFSSWHM operator reduces to the following form.

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}$$

$$\left\{\left\{\left(1 - \left(\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\prod_{j=1}^{k}\gamma_{i_j}\right)^{\frac{q}{k}}\right)^{\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)}\right)^{\frac{1}{C_{n-1}^k}}\right)^{1/q}\right\}\right.,$$

$$\left.\left.\left\{\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\prod_{j=1}^{k}\left(1-\eta_{i_j}^{q}\right)\right)^{\frac{1}{k}}\right)^{\frac{1-\sum_{j=1}^{k}w_{i_j}}{qC_{n-1}^{k}}}\right\}\right)\right\} \quad (1\leqslant k<n) \quad (44)$$

and

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1,d_2,\ldots,d_n\right) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\prod_{i=1}^{k}\gamma_i^{\frac{1-w_i}{n-1}}\right\},\left\{\left(1-\prod_{i=1}^{k}\left(1-\eta_i^{q}\right)^{\frac{1-w_i}{n-1}}\right)^{1/q}\right\}\right\} \quad (k=n) \quad (45)$$

(2) When $q=1$, the q-RDHFSSWHM operator reduces to the dual hesitant fuzzy Schweizer-Sklar weighted Hamy mean (DFHSSWHM) operator.

$$q\text{-RDHFSSWHM}_{q=1}^{(k)}\left(d_1,d_2,\ldots,d_n\right) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}$$

$$\left\{\begin{array}{l}\left\{1-\left(\frac{1}{C_{n-1}^{k}}\sum_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum_{j=1}^{k}w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}\right)^{r}\right)^{1/r}\right)^{r}+\sum_{j=1}^{k}w_{i_j}\right)\right)^{1/r}\right\},\\ \left\{\left(\frac{1}{C_{n-1}^{k}}\sum_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum_{j=1}^{k}w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}\right)^{r}\right)^{1/r}\right)^{r}+\sum_{j=1}^{k}w_{i_j}\right)\right)^{1/r}\right\}\end{array}\right\}$$

$$(1\leqslant k<n) \quad (46)$$

and

$$q\text{-RDHFSSWHM}^{(k)}\left(d_1,d_2,\ldots,d_n\right) = \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}$$

$$\left\{\begin{array}{l}\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\gamma_i^{r}-\left(\frac{1-w_i}{n-1}-1\right)\right)-(k-1)\right)^{1/r}\right\},\\ \left\{1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(1-\eta_i\right)^{r}-\left(\frac{1-w_i}{n-1}-1\right)\right)-(k-1)\right)^{1/r}\right\}\end{array}\right\} \quad (k=n). \quad (47)$$

In this case, if $r = 0$, then the q-RDHFSSWHM operator reduces to the dual hesitant fuzzy weighted Hamy mean (DHFWHM) operator.

$$q\text{-RDHFSSWHM}^{(k)}_{r=0,q=1}(d_1, d_2, \ldots, d_n) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1 - \prod_{j=1}^{k}\gamma_{i_j}^{\frac{1}{k}}\right)^{\frac{1}{C_{n-1}^k}\left(1-\sum\limits_{j=1}^{k} w_{i_j}\right)}\right\}, \left\{\prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\left(1-\left(\prod_{j=1}^{k}\left(1-\eta_{i_j}\right)\right)^{\frac{1}{k}}\right)^{\frac{1-\sum\limits_{j=1}^{k} w_{i_j}}{C_{n-1}^k}}\right\}\right\} \quad (1 \leqslant k < n) \tag{48}$$

and

$$q\text{-RDHFSSWHM}^{(k)}_{r=0,q=1}(d_1, d_2, \ldots, d_n) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\prod_{i=1}^{k}\gamma_i^{\frac{1-w_i}{n-1}}\right\}, \left\{1-\prod_{i=1}^{k}\left(1-\eta_i\right)^{\frac{1-w_i}{n-1}}\right\}\right\}. \quad (k = n) \tag{49}$$

(3) When $q = 2$, the q-RDHFSSWHM operator reduces to the dual hesitant Pythagorean fuzzy Schweizer-Sklar weighted Hamy mean (DHPFSSWHM) operator.

$$q - RDHFSSWHM^{(k)}_{q=2}(d_1, d_2, \ldots, d_n) = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(1-\left(\frac{1}{C_{n-1}^k}\sum_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum_{j=1}^{k}w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(\gamma_{i_j}^2\right)^r\right)^{\frac{1}{r}}\right)^r+\sum_{j=1}^{k}w_{i_j}\right)\right)^{\frac{1}{r}}\right)^{\frac{1}{2}}\right\}, \left\{\left(\frac{1}{C_{n-1}^k}\sum_{\substack{1\leqslant i_1<\ldots\\<i_k\leqslant n}}\left(\left(1-\sum_{j=1}^{k}w_{i_j}\right)\left(1-\left(\frac{1}{k}\sum_{j=1}^{k}\left(1-\eta_{i_j}^2\right)^r\right)^{\frac{1}{r}}\right)^{\frac{r}{2}}+\sum_{j=1}^{k}w_{i_j}\right)\right)^{\frac{1}{2r}}\right\}\right\} \quad (1 \leqslant k < n) \tag{50}$$

and

$$q\text{-RDHFSSWHM}^{(k)}_{q=2}(d_1, d_2, \ldots, d_n) = \bigcup_{\substack{\gamma_i \in h_i,\\ \eta_i \in g_i}} \left\{ \begin{array}{l} \left\{ \left(\sum\limits_{i=1}^{n} \left(\frac{1-w_i}{n-1} \gamma_i^{2r} - \left(\frac{1-w_i}{n-1} - 1 \right) \right) - (k-1) \right)^{1/2r} \right\}, \\ \left\{ \left(1 - \left(\sum\limits_{i=1}^{n} \left(\frac{1-w_i}{n-1} \left(1 - \eta_i^2\right)^r - \left(\frac{1-w_i}{n-1} - 1 \right) \right) - (k-1) \right)^{1/r} \right)^{1/2} \right\} \end{array} \right\} \quad (k = n). \tag{51}$$

In this case, if $r = 0$, then the q-RDHFSSWHM operator reduces to the dual hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator.

$$\begin{aligned} & q\text{-RDHFSSWHM}^{(k)}_{r=0,q=2}(d_1, d_2, \ldots, d_n) \\ & = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j},\\ \eta_{i_j} \in g_{i_j}}} \left\{ \begin{array}{l} \left\{ \left(1 - \prod\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod\limits_{j=1}^{k} \gamma_{i_j}^{\frac{2}{k}} \right)^{\frac{1 - \sum_{j=1}^{k} w_{i_j}}{C_{n-1}^k}} \right)^{1/2} \right\}, \\ \left\{ \prod\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \eta_{i_j}^2 \right)^{\frac{1}{k}} \right)^{\frac{1 - \sum_{j=1}^{k} w_{i_j}}{2C_{n-1}^k}} \right\} \end{array} \right\}, (1 \leqslant k < n) \end{aligned} \tag{52}$$

and

$$\begin{aligned} & q\text{-RDHFSSWHM}^{(k)}_{\substack{r=0,\\ q=2}}(d_1, d_2, \ldots, d_n) \\ & = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j},\\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \prod_{i=1}^{k} \gamma_i^{\frac{1-w_i}{n-1}} \right\}, \left\{ \left(1 - \prod_{i=1}^{k} \left(1 - \eta_i^2 \right)^{\frac{1-w_i}{n-1}} \right)^{\frac{1}{2}} \right\} \right\} \quad (k = n). \end{aligned} \tag{53}$$

(4) When $k = 1$, the q-RDHFSSWHM operator reduces to the q-rung dual hesitant fuzzy Schweizer-Sklar weighted average (q-RDHFSSWA) operator.

$$q\text{-RDHFSSWHM}^{(k)}_{k=1}(d_1, d_2, \ldots, d_n) = \bigoplus_{i=1}^{n}\left(\frac{1-w_i}{n-1}d_i\right)$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}} \left\{\left\{\left(1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\left(1-\gamma_i^q\right)^r-1\right)+1\right)-n+1\right)^{1/r}\right)^{1/q}\right\},\right.$$
$$\left.\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\eta_i^{qr}-1\right)+1\right)-n+1\right)^{1/qr}\right\}\right\}. \quad (54)$$

In this case, if $r = 0$, then the q-RDHFSSWHM operator reduces to the q-rung dual hesitant fuzzy weighted averaging (q-RDHFWA) operator

$$q\text{-RDHFSSWHM}^{(k)}_{r=0,k=1}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\left(1-\left(\prod_{i=1}^{n}\left(1-\gamma_{i_j}^q\right)^{\left(1-w_{i_j}\right)}\right)^{\frac{1}{n-1}}\right)^{1/q}\right\}, \left\{\prod_{i=1}^{n}\eta_{i_j}^{\frac{1-w_{i_j}}{n-1}}\right\}\right\}. \quad (55)$$

(5) When $k = n$, the q-RDHFSSWHM operator reduces to the q-rung dual hesitant fuzzy Schweizer-Sklar weighted geometric (q-DHFSSWG) operator.

$$q\text{-RDHFSSWHM}^{(k)}_{k=n}(d_1, d_2, \ldots, d_n) = \bigotimes_{i=1}^{k} d_i^{\frac{1-w_i}{n-1}}$$
$$= \bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}} \left\{\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\gamma_i^{qr}-1\right)+1\right)-k+1\right)^{\frac{1}{qr}}\right\},\right.$$
$$\left.\left\{\left(1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\left(1-\eta_i^q\right)^r-1\right)+1\right)-k+1\right)^{\frac{1}{r}}\right)^{1/q}\right\}\right\}. \quad (56)$$

In this case, if $r = 0$, the q-RDHFSSWHM operator reduces to the q-rung dual hesitant fuzzy weighted geometric (q-DHFWG) operator.

$$q\text{-RDHFSSWHM}^{(k)}_{r=0,k=n}(d_1, d_2, \ldots, d_n)$$
$$= \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{\left\{\prod_{i=1}^{n}\gamma_i^{\frac{1-w_i}{n-1}}\right\}, \left\{\left(1-\prod_{i=1}^{n}\left(1-\eta_i^q\right)^{\frac{1-w_i}{n-1}}\right)^{1/q}\right\}\right\}. \quad (57)$$

(6) When $k = 1$ and $q = 1$, the q-RDHFSSWHM operator reduces to the dual hesitant fuzzy Schweizer-Sklar weighted average (DHFSSWA) operator.

$$
\begin{aligned}
&q\text{-RDHFSSWHM}^{(k)}_{k=1,q=1}(d_1, d_2, \ldots, d_n) \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ \eta_i \in g_i}} \left\{ \left\{ 1 - \left(\sum_{i=1}^{n} \left(\frac{1-w_i}{n-1} (1-\gamma_i)^r - \left(\frac{1-w_i}{n-1} - 1 \right) \right) - (n-1) \right)^{\frac{1}{r}} \right\}, \right. \\
&\qquad\qquad \left. \left\{ \left(\sum_{i=1}^{n} \left(\frac{1-w_i}{n-1} \eta_i^r - \left(\frac{1-w_i}{n-1} - 1 \right) \right) - (n-1) \right)^{\frac{1}{r}} \right\} \right\}.
\end{aligned} \tag{58}
$$

In this case, if $r = 0$, then the q-RDHFSSWHM operator reduces to the dual hesitant fuzzy weighted average (DHFWA) [56] operator.

$$
\begin{aligned}
&q\text{-RDHFSSWHM}^{(k)}_{r=0,k=1,q=1}(d_1, d_2, \ldots, d_n) \\
&= \bigcup_{\substack{\gamma_{i_j} \in h_{i_j}, \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ 1 - \left(\prod_{i=1}^{n} \left(1 - \gamma_{i_j} \right)^{\left(1-w_{i_j}\right)} \right)^{\frac{1}{n-1}} \right\}, \left\{ \prod_{i=1}^{n} \eta_{i_j}^{\frac{1-w_{i_j}}{n-1}} \right\} \right\}.
\end{aligned} \tag{59}
$$

(7) When $k = 1$ and $q = 2$, the q-RDHFSSWHM operator reduces to the dual hesitant Pythagorean fuzzy Schweizer-Sklar weighted average (DHPFSSWA) operator.

$$
\begin{aligned}
&q\text{-RDHFSSWHM}^{(k)}_{k=1,q=2}(d_1, d_2, \ldots, d_n) \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ \eta_i \in g_i}} \left\{ \left\{ \left(1 - \left(\sum_{i=1}^{n} \left(\frac{1-w_i}{n-1} \left(\left(1 - \gamma_i^2 \right)^r - 1 \right) + 1 \right) - n + 1 \right)^{\frac{1}{r}} \right)^{1/2} \right\}, \right. \\
&\qquad\qquad \left. \left\{ \left(\sum_{i=1}^{n} \left(\frac{1-w_i}{n-1} \left(\eta_i^{2r} - 1 \right) + 1 \right) - n + 1 \right)^{\frac{1}{2r}} \right\} \right\}.
\end{aligned} \tag{60}
$$

In this case, if $r = 0$, then the q-RDHFSSWHM operator reduces to the dual hesitant Pythagorean fuzzy weighted average (DHPFWA) [53] operator.

$$q\text{-RDHFSSWHM}^{(k)}_{r=0,k=1,q=2}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\left(1-\left(\prod_{i=1}^{n}\left(1-\gamma_{i_j}^2\right)\right)^{\frac{1-w_{i_j}}{n-1}}\right)^{1/2}\right\},\left\{\prod_{i=1}^{n}\eta_{i_j}^{\frac{1-w_{i_j}}{n-1}}\right\}\right\}. \tag{61}$$

(8) When $k=n$ and $q=1$, the q-RDHFSSWHM operator reduces to the dual hesitant fuzzy Schweizer-Sklar weighted geometric (DHFSSWG) operator.

$$q\text{-RDHFSSWHM}^{(k)}_{k=n,q=1}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\gamma_i^r-1\right)+1\right)-(k-1)\right)^{1/r}\right\},\right.$$
$$\left.\left\{1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left((1-\eta_i)^r-1\right)+1\right)-(k-1)\right)^{1/r}\right\}\right\}. \tag{62}$$

In this case, if $r=0$, then the q-RDHFSSWHM operator reduces to the dual hesitant fuzzy weighted geometric (DHFWG) [56] operator.

$$q\text{-RDHFSSWHM}^{(k)}_{r=0,k=n,q=1}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}}\left\{\left\{\prod_{i=1}^{n}\gamma_i^{\frac{1-w_i}{n-1}}\right\},\left\{1-\prod_{i=1}^{n}(1-\eta_i)^{\frac{1-w_i}{n-1}}\right\}\right\}. \tag{63}$$

(9) When $k=n$ and $q=2$, the q-RDHFSSWHM operator reduces to the dual hesitant Pythagorean fuzzy Schweizer-Sklar weighted geometric (DHPFSSWG) operator.

$$q\text{-RDHFSSWHM}^{(k)}_{k=n,q=2}(d_1,d_2,\ldots,d_n)$$
$$=\bigcup_{\substack{\gamma_i\in h_i,\\ \eta_i\in g_i}}\left\{\left\{\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\gamma_i^{2r}-1\right)+1\right)-k+1\right)^{\frac{1}{2r}}\right\},\right.$$
$$\left.\left\{\left(1-\left(\sum_{i=1}^{n}\left(\frac{1-w_i}{n-1}\left(\left(1-\eta_i^2\right)^r-1\right)+1\right)-k+1\right)^{\frac{1}{r}}\right)^{\frac{1}{2}}\right\}\right\}. \tag{64}$$

In this case, if $r = 0$, then the q-RDHFSSWHM operator reduces to the dual hesitant Pythagorean fuzzy weighted geometric (DHPFWG) operator.

$$
\begin{aligned}
&q\text{-RDHFSSWHM}^{(k)}_{r=0,k=n,q=2}(d_1, d_2, \ldots, d_n)\\
&\quad = \bigcup_{\substack{\gamma_{i_j}\in h_{i_j},\\ \eta_{i_j}\in g_{i_j}}} \left\{ \left\{ \prod_{i=1}^{n} \gamma_i^{\frac{1-w_i}{n-1}} \right\}, \left\{ \left(1 - \prod_{i=1}^{n} \left(1-\eta_i^2\right)^{\frac{1-w_i}{n-1}} \right)^{1/2} \right\} \right\}.
\end{aligned}
\tag{65}
$$

5. A novel approach to MADM based on the proposed operators

In this section, we apply the proposed AOs to solving MADM problems in q-rung dual hesitant fuzzy environment.

5.1. Description of a typical MADM problem with q-rung dual hesitant fuzzy information

A typical MADM problem where attribute values are in the form of q-RDHFEs is expressed as: Let $A = \{A_1, A_2, \ldots, A_m\}$ be a collection of feasible alternatives, $G = \{G_1, G_2, \ldots, G_n\}$ be n attributes. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of attributes, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Suppose the evaluation value of G_j $(j = 1, 2, \ldots, n)$ of alternative A_i $(i = 1, 2, \ldots, m)$ is denoted as $d_{ij} = (h_{ij}, g_{ij})$ and hence a q-rung dual hesitant fuzzy decision matrix $D = (d_{i_j})_{m\times n} = (h_{i_j}, g_{i_j})_{m\times n}$ is obtained.

5.2. An algorithm to q-rung dual hesitant fuzzy MADM problems

In the followings, we present a novel algorithm to MADM based on the proposed operators.

Step 1 Standardize the original decision matrix. In real decision-making problems, there are two kinds of attributes: benefit attributes and cost attributes. Therefore, the original decision matrix should be normalized by

$$
d_{ij} = \begin{cases} \left(h_{ij}, g_{ij}\right) & G_j \in I_1\,, \\ \left(g_{ij}, h_{ij}\right) & G_j \in I_2\,, \end{cases}
\tag{66}
$$

where I_1 represents benefit attributes and I_2 represents cost attributes.

Step 2 For alternative A_i $(i = 1, 2, \ldots, m)$, utilize the q-RDHFSSWHM operator

$$
d_i = q\text{-RDHFSSWHM}^{(k)}\left(d_{i1}, d_{i2}, \ldots, d_{in}\right),
\tag{67}
$$

to aggregate all the attributes values, and a series of comprehensive preference values can be obtained.

Step 3 Rank the overall d_i $(i = 1, 2, \ldots, m)$ values based on their scores according to Definition 3.

Step 4 Rank the corresponding alternatives according to step 3 and select the best alternative.

6. Numerical examples

Example 1 As life style changes, chronic diseases have seriously affected our health and quality of life. Therefore, the development of prevention and treatment of chronic disease is very important for us, and the quality evaluation of chronic disease health management (CDHM) is an important means to do. The purpose of CDHM is to help patient master the knowledge of self-management of disease and develop healthy habits, so that patients can maintain their health status and health functions in a satisfactory state and better return to society. In order to understand the current situation of CDHM in a Chinese hospital, we utilize the evaluation indexes proposed by Donabedian [57] to evaluate the process quality of CDHM in the hospital. As the real evaluation environment is particularly complex, DMs may be hesitant to express their evaluate information, so we allow them to give their decision information with a fuzzy set with both the membership degree and non-membership degree. Besides, for the experts who are still irresolute about the decision values in membership and non-membership degrees, we endow them with freedom to give multiple degrees of membership and non-membership. Summing up the above, q-rung dual hesitant fuzzy set can be utilized to describe the DMs' decision information.

Table 1: The q-rung dual hesitant fuzzy decision matrix of Example 1

	G_1	G_2	G_3	G_4
A_1	$\{\{0.5, 0.6\}, \{0.2, 0.3\}\}$	$\{\{0.8, 0.9\}, \{0.1, 0.2\}\}$	$\{\{0.6, 0.7\}, \{0.2, 0.3\}\}$	$\{\{0.5, 0.6\}, \{0.1, 0.2\}\}$
A_2	$\{\{0.8, 0.9\}, \{0.1, 0.2\}\}$	$\{\{0.6, 0.7\}, \{0.2\}\}$	$\{\{0.7, 0.8\}, \{0.2\}\}$	$\{\{0.8, 0.9\}, \{0.1, 0.2\}\}$
A_3	$\{\{0.6, 0.8\}, \{0.1, 0.2\}\}$	$\{\{0.5, 0.6\}, \{0.2\}\}$	$\{\{0.7, 0.8\}, \{0.1\}\}$	$\{\{0.7, 0.8\}, \{0.2, 0.3\}\}$
A_4	$\{\{0.6, 0.7\}, \{0.2, 0.3\}\}$	$\{\{0.5, 0.7\}, \{0.4\}\}$	$\{\{0.6, 0.7\}, \{0.3\}\}$	$\{\{0.7, 0.8\}, \{0.1, 0.2\}\}$

Suppose that we investigate four hospitals A_i $(i = 1, 2, 3, 4)$, and four attributes G_j $(j = 1, 2, 3, 4)$ of Donabedian's [57] process quality evaluation index are selected as the final evaluation indicators: detection networks (G_1); disease surveillance (G_2); behavioral interventions (G_3) and health education (G_4), with weight vector $w = (0.17, 0.32, 0.38, 0.13)^T$ to evaluate the process quality of CDHM. The decision matrix is shown in Table 1.

6.1. The decision-making process

Step 1 As all the attributes are benefit type, the original decision matrix does not need to be standardized.

Step 2 Utilize the q-RDHFSSWHM ($k = 2, r = -2, q = 3$) operator to aggregate attribute values of each alternative. The comprehensive evaluation values of alternatives are complicated and we omit them.

Step 3 Calculate the scores of the comprehensive evaluation values, and we can obtain

$$S(d_1) = 0.0020, \quad S(d_2) = 0.0356, \quad S(d_3) = 0.0083, \quad S(d_4) = 0.0045.$$

Step 4 According to the Definition 3, we can get the ranking order $A_2 \succ A_3 \succ A_4 \succ A_1$, which means the best hospital for process quality of CDHM is A_2.

6.2. The validity of our proposed method

In this subsection, to prove the validity and the effectiveness of the proposed method, we utilize our proposed method based on q-RDHFSSWHM operator, that proposed by Wei and Lu [53] based on dual hesitant Pythagorean fuzzy weighted average (DHPFWA) operator, and that proposed by Xu et al. [44] based on q-rung dual hesitant fuzzy weighted Heronian mean (q-RDHFWHM) operator to solve Example 1 mentioned above, and the score values and the ranking results are shown in Table 2.

Table 2: Score functions and ranking orders by different methods

Method	Score value $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking results
Wei and Lu's [53] method based on DHPFWA operator	$S(d_1) = 0.4633$, $S(d_2) = 0.6502$, $S(d_3) = 0.5031$, $S(d_4) = 0.4623$	$A_2 \succ A_3 \succ A_1 \succ A_4$
Xu et al.'s [44] method based on q-RDHFWHM operator (when $s = t = 1/2$, $q = 2$)	$S(d_1) = 0.4036$, $S(d_2) = 0.5387$, $S(d_3) = 0.4203$, $S(d_4) = 0.3261$	$A_2 \succ A_3 \succ A_1 \succ A_4$
The proposed method based on q-RDHFSSWHM operator (when $r = 0$, $k = 1$, $q = 3$)	$S(d_1) = 0.2586$, $S(d_2) = 0.4324$, $S(d_3) = 0.2869$, $S(d_4) = 0.2500$	$A_2 \succ A_3 \succ A_1 \succ A_4$

From Table 2, we can see that although the score values of different methods are different, the ranking result derived by our proposed method is the same as that obtained by Wei and Lu's [53] and Xu et al.'s [44] methods, i.e., $A_2 \succ A_3 \succ A_1 \succ A_4$, which illustrates the validity and effectiveness of the proposed method.

6.3. The influence of the parameters on the ranking results

It is obvious that the parameters r, k, and q have great influence on the score values and decision results. Hence, it is necessary to investigate how these parameters affect the final decision results. We firstly study the influence of the parameter r and to this end we assign different values of r in the q-RDHFSSWHM operator and the score values of alternatives are presented as Fig. 1. It is obvious that the parameters r, k, and q have great influence on the score values and decision results. Hence, it is necessary to investigate how these parameters affect the final decision results. We firstly study the influence of the parameter r and to this end we assign different values of r in the q-RDHFSSWHM operator and the score values of alternatives are presented as Fig. 1.

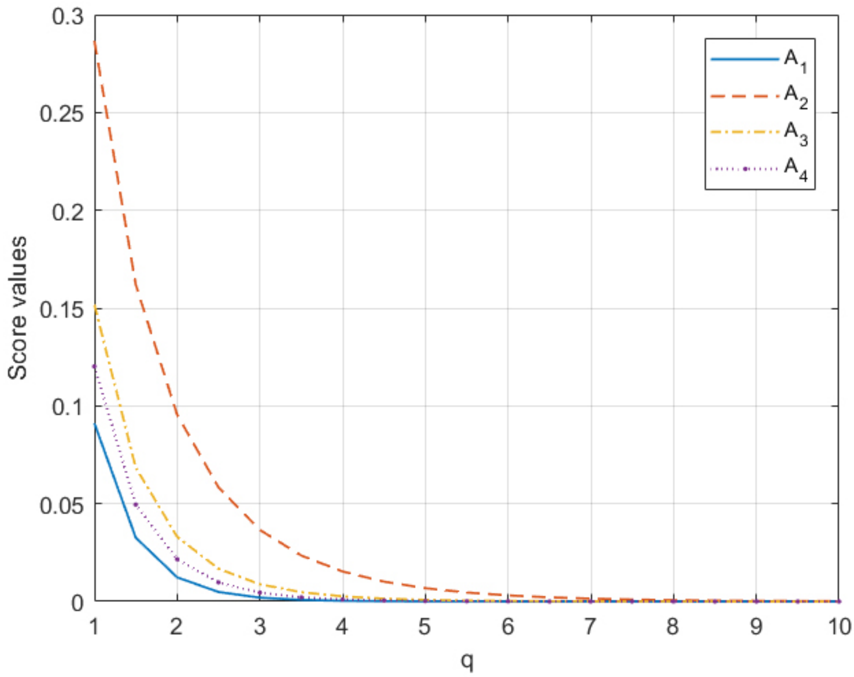

Figure 1: Scores of alternatives A_i ($i = 1, 2, 3, 4$) when $r \in [-4, 0)$ based on the q-RDHFSSWHM operator ($k = 2$, $q = 3$)

The parameter r plays an important role in the operations of q-RDHFEs, and the SSTT operations reduces to the algebraic operation when r approaches to zero. In addition, it has a significant influence on the SSTT operational laws of q-RDHFEs as we can see from Definition 6 and Theorem 2. From Fig. 1, we can observe that the score of alternatives may change with different values of r, the ranking result is always $A_2 \succ A_3 \succ A_4 \succ A_1$ except where $r \in [-0.85, -0.5]$. In other words, the best alternative is always A_2 although the ranking result of the remaining alternatives may change according to different value of parameter r, which illustrate the effectiveness and flexibility of the proposed method. Besides, to better distinguish the alternatives, the value of r should not be very small or close to -0.1, and it is easy to find that the score difference among alternatives come to maximized when $r = -0.9$.

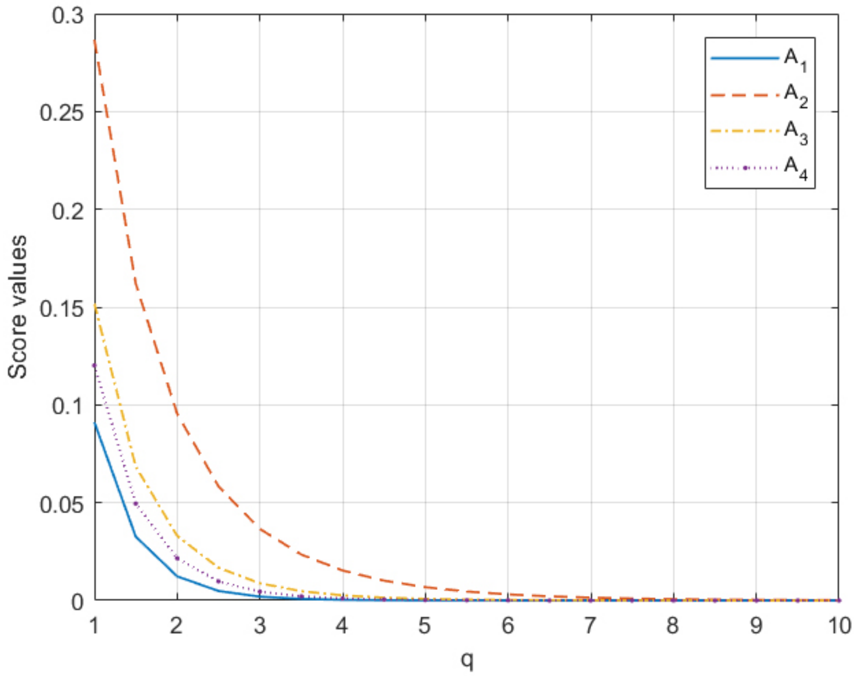

Figure 2: Scores of alternatives A_i $(i = 1, 2, 3, 4)$ when $q \in [1, 10]$ based on the q-RDHFSSWHM operator $(r = -2,\ k = 2)$

Then we investigate the effect of the parameter q on the decision results and similarly we assign different values to q in the q-RDHFSSWHM operator and present the score values of alternatives as Fig. 2. From Fig. 2, we find out that no matter how the parameter q changes, the ranking results of alternatives are always $A_2 \succ A_3 \succ A_4 \succ A_1$, and the score values derived by the q-RDHFSSWHM operator are becoming smaller and smaller and gradually approach to zero with the increase of q. Hence, how to select a proper value of q is an important problem. In [44], Xu et al. introduced the principle of choosing an appropriate value of q, i.e. the value of q should be taken as the smallest integer that makes $\gamma^q + \eta^q \leqslant 1$, where γ and η denote all possible MDs and NMDs. For example, if a DM provides a q-RDHFE $d = \{\{0.1, 0.5, 0.8\},\ \{0.2, 0.4, 0.6, 0.8\}\}$ as his/her evaluation value. Then, as $0.8^3 + 0.8^3 = 1.024 > 1$ and $0.8^4 + 0.8^4 = 0.8192 < 1$, then the value of q should be taken as 4.

In the followings, we investigate the influence of the parameter k on the final decision results. We assign different values to k in the q-RDHFSSWHM operator, and the score values of alternatives and corresponding ranking orders are presented in Table 3. As we can see from Table 3, different score values of alternatives are derived with different values of k in the q-RDHFSSWHM operator. In addition, the ranking orders are slightly different, however, the optimal alternatives are the same, i.e. A_2. Actually, the parameter k manipulates the number of interacted q-RDHFEs. In real MADM problems, the value of k denotes the number of dependent attributes. When $k = 1$, the q-RDHFSSWHM operator does not consider the interrelationship among attributes and when $k = 2, 3, 4$, the interrelationship among attributes is taken into account in the process of computing the comprehensive evaluation values of alternatives. This is why the ranking order obtained by the q-RDHFSSWHM operator when $k = 1$ is different from those when $k = 2, 3, 4$. Additionally, when $k = 2$ the interrelationship between any two attributes is considered. When $k = 3$, the q-RDHFSSWHM operator reflects the interrelationship among triple attributes and when $k = 4$ the interrelationship among the four attributes is reflected. This characteristic illustrates the flexibility and universality of our proposed method. DMs can choose a proper value of k according to actual needs. If there is indeed no interrelationship among attributes, then we can set $k = 1$.

Table 3: The score values and ranking results with different parameter k (suppose $r = -2$, $q = 2$)

k	Score value $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking results
$k = 1$	$S(d_1) = 0.5055$, $S(d_2) = 0.6528$, $S(d_3) = 0.4784$, $S(d_4) = 0.4276$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$k = 2$	$S(d_1) = 0.0126$, $S(d_2) = 0.0936$, $S(d_3) = 0.0315$, $S(d_4) = 0.0214$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$k = 3$	$S(d_1) = 0.0046$, $S(d_2) = 0.0366$, $S(d_3) = 0.0120$, $S(d_4) = 0.0092$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$k = 4$	$S(d_1) = -0.0037$, $S(d_2) = 0.1611$, $S(d_3) = 0.0401$, $S(d_4) = -0.0030$	$A_2 \succ A_3 \succ A_4 \succ A_1$

6.4. Advantages of our proposed method

In this section, to better demonstrate the advantages of our proposed method, we apply it and some existing MADM methods in solving numerical examples and conduct comparative analyzes. These methods involve that proposed by Wang and Liu [45] based on the intuitionistic fuzzy Schweizer-Sklar weighted Maclaurin symmetric mean (IFSSWMSM) operator, that introduced by Xu et al. [44] based on the q-rung dual hesitant fuzzy weighted Heronian mean (q-RDHFWHM) operator, that presented by Tang et al. [58] based on the dual hesitant Pythagorean fuzzy generalized weighted Heronian mean (DHPFGWHM) operator.

6.4.1. The flexibility of aggregating DMs' fuzzy judgements

The proposed MADM method is based on q-RDHFSs and SSTT. Due to the characteristic of SSTT, our method provides a flexibility manner to fuse DMs' evaluation information. To better demonstrate this advantage, we give the following numerical example.

Example 2 (Revised from [44]) There is a supplier selection problem in supply chain management, and five prospective suppliers A_i $(i = 1, 2, 3, 4, 5)$ are required to be evaluated with four attributes G_j $(j = 1, 2, 3, 4)$): (1) relationship closeness G_1; (2) product quality G_2; (3) price competitiveness G_3; (4) delivery performance G_4, whose weight vector is $w = (0.17, 0.32, 0.38, 0.13)^T$. A set of DMs are invited to evaluate each alternative with respect to each attribute using a q-RDHFE, and the decision matrix is shown in Table 4. We utilize the proposed method and Xu et al.'s [44] method to solve this example and present the decision results in Table 5.

Table 4: The dual hesitant q-rung dual hesitant fuzzy decision matrix of Example 2

	G_1	G_2	G_3	G_4
A_1	$\{\{0.3, 0.4\}, \{0.6\}\}$	$\{\{0.7, 0.8\}, \{0.2\}\}$	$\{\{0.4\}, \{0.2, 0.3\}\}$	$\{\{0.5, 0.6\}, \{0.2\}\}$
A_2	$\{\{0.2, 0.3\}, \{0.5\}\}$	$\{\{0.6, 0.7\}, \{0.2\}\}$	$\{\{0.7, 0.8\}, \{0.2\}\}$	$\{\{0.6\}, \{0.1, 0.2, 0.3\}\}$
A_3	$\{\{0.4\}, \{0.2, 0.3\}\}$	$\{\{0.5, 0.6\}, \{0.2\}\}$	$\{\{0.7, 0.8\}, \{0.1\}\}$	$\{\{0.7\}, \{0.2, 0.3\}\}$
A_4	$\{\{0.6, 0.8\}, \{0.3\}\}$	$\{\{0.5\}, \{0.4\}\}$	$\{\{0.3, 0.4\}, \{0.3\}\}$	$\{\{0.4, 0.6\}, \{0.1\}\}$

Table 5: Score functions and ranking orders of Example 2 by different methods

Method	Score values $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking results
Xu et al.'s [44] method based on q-RDHFWHM operator (when $q = 2$, $s = t = 1$)	$S(d_1) = 0.2215$, $S(d_2) = 0.3326$, $S(d_3) = 0.3572$, $S(d_4) = 0.1702$	$A_3 \succ A_2 \succ A_1 \succ A_4$
The proposed method based on q-RDHFSSWHM operator (when $r = -1$, $k = 2$, $q = 2$)	$S(d_1) = -0.0165$, $S(d_2) = 0.0438$, $S(d_3) = 0.1275$, $(d_4) = 0.0102$	$A_3 \succ A_2 \succ A_4 \succ A_1$
The proposed method based on q-RDHFSSWHM operator (when $r \to 0$, $k = 2$, $q = 2$)	$S(d_1) = 0.0091$, $S(d_2) = 0.0128$, $S(d_3) = 0.0165$, $S(d_4) = 0.0101$	$A_3 \succ A_2 \succ A_4 \succ A_1$
The proposed method based on q-RDHFSSWHM operator (when $r = -2$, $k = 2$, $q = 2$)	$S(d_1) = 0.0025$, $S(d_2) = 0.0079$, $S(d_3) = 0.0102$, $S(d_4) = 0.0029$	$A_3 \succ A_2 \succ A_4 \succ A_1$

From Table 5, we can find that although the score values and ranking results by different methods are slightly different, the best supplier company is always A_3, which illustrates the validity of the proposed method. In addition, Xu et al.'s [44] method is based on the algebraic t-norm and t-conorm and our proposed method is based on SSTT. As we know, the algebraic t-norm and t-conorm is a special case of the SSTT. When $r \to 0$, then SSTT reduces to the simple algebraic t-norm and t-conorm. Hence, our method provides more flexible information aggregation process than Xu et al. [44] method.

6.4.2. Its ability of capturing the interrelationship among multiple attributes

In real MADM problems, it often happens that the input arguments have interrelationship with others. Our proposed method is based on HM operator, which is famous for its capability to capture the relationship among multiple attributes. In the following, we compare the proposed method with Xu et al.'s [44] method based on q-RDHFWHM operator, and Tang et al.'s [58] method based on DHPFGWHM operator. We utilize these methods to solve Example 2 and present the decision results in Table 6.

Table 6: Score functions and ranking orders of Example 2 by different methods

Method	Score values $S(d_i)$ $(i = 1, 2, 3, 4))$	Ranking results
Xu et al.'s [44] method based on q-RDHFWHM operator (when $q = 3$, $s = t = 1$)	$S(d_1) = 0.1735$, $S(d_2) = 0.2631$, $S(d_3) = 0.2663$, $S(d_4) = 0.1182$	$A_3 \succ A_2 \succ A_1 \succ A_4$
Tang et al.'s [58] method based on DHPFGWHM operator (when $s = t = 1$)	$S(d_1) = 0.1654$, $S(d_2) = 0.2673$, $S(d_3) = 0.3437$, $S(d_4) = 0.1502$	$A_3 \succ A_2 \succ A_1 \succ A_4$
The proposed method based on q-RDHFSSWHM operator (when $r = -1$, $k = 2$, $q = 3$)	$S(d_1) = 0.0211$, $S(d_2) = 0.0568$, $S(d_3) = 0.0930$, $S(d_4) = 0.0247$	$A_3 \succ A_2 \succ A_4 \succ A_1$
The proposed method based on q-RDHFSSWHM operator (when $r = -1$, $k = 3$, $q = 3$)	$S(d_1) = -0.1183$, $S(d_2) = -0.0763$, $S(d_3) = 0.0044$, $S(d_4) = -0.0676$	$A_3 \succ A_4 \succ A_2 \succ A_1$
The proposed method based on q-RDHFSSWHM operator (when $r = -1$, $k = 4$, $q = 3$)	$S(d_1) = 0.0029$, $S(d_2) = 0.0033$, $S(d_3) = 0.1295$, $S(d_4) = 0.0658$	$A_3 \succ A_4 \succ A_2 \succ A_1$

As we can see from Table 6, different score values are obtained by using different MADM methods. In addition, the ranking orders derived by different methods are different, but the best alternative is always A_3, which also illustrates the validity of our proposed method. Moreover, it is noted that Xu et al.'s [44] and Tang et al.'s [58] methods based on Heronian mean, which only considers the interrelationship between any two attributes. Hence, our method is more flexible than these methods as it can consider the interrelationship among multiple attributes. Specifically, when $k = 2$ our method reflects the interrelationship between any two attributes, which is same as Xu et al.'s [44] and Tang et al.'s [58] methods. When $k = 3, 4$, our method reflects the interrelationship among multiple attributes while Xu et al.'s [44] and Tang et al.'s [58] methods fail to do consider the interrelationship among multiple attributes. Basically, in real MADM problems the interrelationship exists among multiple attributes and hence our proposed method is more powerful and useful than those proposed by Xu et al. [44] and Tang et al. [58].

6.4.3. Its ability to comprehensively express DMs' evaluations

Owing to the high complexity and uncertainty of practical MADM problem, it is not easy to comprehensively express DMs' evaluation information. Our proposed method is based on q-RDHFSs, which give DMs great freedom to express their judgments. In addition, q-RDHFSs can efficiently handle DMs' high hesitancy in providing their evaluation values. Hence, our method is more suitable to deal with complicated and vague MADM problems. To better illustrate this advantage of our method, we provide the following example.

Example 3 (Revised from [59]). A city plans to build a municipal library with an air conditioning system and the contractor offers five feasible alternatives A_i $(i = 1, 2, 3, 4, 5)$, which might be suitable for the physical structure of the library. These alternatives are required to be evaluated based on three indexes: economics G_1; functional G_2 and operational G_3, whose weight vector is $w = (0.3, 0.5, 0.2)^T$. Assume that the characteristics of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ with respect to the indexes G_j $(j = 1, 2, 3)$ are represented by the intuitionistic fuzzy number (IFN) $d_{ij} = (\gamma_{ij}, \eta_{ij})$, where γ_{ij} expresses the degree that A_i satisfies the index G_j and η_{ij} indicates the degree that the alternative A_i does not satisfy the index G_j. The decision matrix $D = (d_{ij})_{5\times3}$ given by DMs is shown in Table 7. We employ our proposed method and Wang and Liu's [45] method to solve this example and present the decision results in Table 8. (It should be noted that the IFS is also a special case of q-RDHFS. When $q = 1$ and there are only one MD and one MDM, then q-RDHFS reduces to IFS. Hence, our proposed method can be applied in MADM problems where DMs' evaluation values are in the form of IFNs.

As seen from Table 8, although the score values of each alternative are different, the ranking orders produced by Wang and Liu's [45] method and our proposed method are the same, i.e.$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$, and the optimal alternative is A_2. Actually, the method introduced by Wang and Liu [45] has similar characteristics as our proposed method. First, Wang and Liu's [45] method is based on SSTT which is the same as our proposed method. Second, Wang and Liu's [45] method is on the basis of Maclaurin symmetric mean, which also has the ability of capture the interrelationship among multiple attributes.

Table 7: The intuitionistic fuzzy decision matrix of Example 3

	G_1	G_2	G_3
A_1	(0.3, 0.4)	(0.7, 0.2)	(0.5, 0.3)
A_2	(0.5, 0.2)	(0.4, 0.1)	(0.7, 0.1)
A_3	(0.4, 0.5)	(0.7, 0.2)	(0.4, 0.4)
A_4	(0.2, 0.6)	(0.8, 0.1)	(0.8, 0.2)
A_5	(0.9, 0.1)	(0.6, 0.3)	(0.2, 0.5)

Table 8: Score functions and ranking orders of Example 3 by different methods

Method	Score value $S(d_i)$ $(i = 1, 2, 3, 4)$	Ranking results
Wang and Liu's [45] method based on IFSSWMSM operator (when $r = -1$, $k = 2$)	$S(d_1) = -0.3494$, $S(d_2) = -0.0325$, $S(d_3) = -0.4346$, $S(d_4) = -0.2097$, $S(d_5) = -0.3647$	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
The proposed method based on q-RDHFSSWHM operator (when $r = -1$, $k = 2$, $q = 2$)	$S(d_1) = -0.1611$, $S(d_2) = 0.0652$, $S(d_3) = -0.2723$, $S(d_4) = -0.0126$, $S(d_5) = -0.1928$	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$

Nevertheless, our proposed method is still more powerful and useful than Wang and Liu's [45] method, due to its flexible information expression form. First, our proposed method allows the MD and NMD of DMs' fuzzy judgements to be represented by two sets of values, instead of single ones. While, Wang and Liu's [45] method only permits one MD and one NMD. Hence, our proposed method can effectively deal with DMs' high hesitancy in MADM procedure. Second, our method is based on the q-RDHFS, whose constraint is the sum of qth power of MD and qth power of NMD is less than or equal to one. Wang and Liu's [45] method is based on IFS, which should satisfy the condition that the sum of MD and NMD should be no greater than one. Hence, when the sum of MD and NMD is greater than one, then Wang and Liu's [45] method does no work, while our method can still deal with such a case. Therefore, our proposed method is more powerful due to its good performance and high efficiency in portraying DMs' complicated preference information. To better demonstrate the characteristics of above-mentioned MADM methods, we provide Table 9.

Table 9: Characteristics of different methods

Methods	Whether permits the sum of MD and NMD to be greater than one	Whether permits the square sum of MD and NMD to be greater than one	Whether allows the MD and NMD to be denoted by more than one values	The flexibility of the operational laws	Whether captures the interrelationship between any two attributes	Whether captures the interrelationship among multiple attributes
Wang and Liu's [45] method based on IFSSWMSM operator	No	No	No	High	Yes	Yes
Xu et al.'s [44] method based on q-RDHFWHM operator	Yes	Yes	Yes	Medium	Yes	No
Tang et al.'s [58] method based on DHPFGWHM operator	Yes	No	Yes	Medium	Yes	No
The proposed method based on q-RDHFSSWHM operator	Yes	Yes	Yes	High	Yes	Yes

7. Conclusions

This paper proposed a novel MADM method based on q-RDHFEs. To this end, we firstly introduced new operations of q-RHFEs based on SSTT. Then, based on the new operational laws we extend the classical HM operator to q-RDHFSs and proposed the q-RDHFSSHM and q-RDHFSSWHM operators. Afterwards, we explained the main steps of the new MADM method. We also investigated the applications of the proposed method in actual MADM methods. To sum up, the advantages of our MADM method are three-fold. First, it employs q-RDHFSs to depict DMs' evaluation information, which not only deals with human beings' inherent hesitancy in making-decisions, but also provides DMs' great freedom to comprehensively express their assessment information. Second, it is based on the powerful SSTT, making the information process more flexible. Finally, it utilizes the HM to fuse DMs' evaluation information, so that the interrelationship among multiple attributes can be effectively captured. Due to these features, our proposed method is efficient in dealing with practical MADM problems. In the future, we plan to investigate more methods to effectively aggregate q-RDHFEs and propose more powerful MADM methods.

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