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Stability of positive fractional switched continuous-time linear systems

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Abstract. The asymptotic stability of positive fractional switched continuous-time linear systems for any switching is addressed. Simple sufficient conditions for the asymptotic stability of the positive fractional systems are established. It is shown that the positive fractional switched systems are asymptotically stable for any switchings if the sum of entries of every column of the matrices of all subsystems is negative.

Key words: fractional, positive, switched, linear, asymptotic stability, continuous-time.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

An overview of the fractional calculus and its applications is given in the books [3–5]. A positive fractional switched systems consists of a collection of positive fractional state space models and a switching function (signal) governing the switching among the models [6–9]. The stability and stabilization of positive 1D systems have been investigated in [10–17] and positive 2D linear systems in [7, 8]. The copositive Lyapunov functions approach to switched linear systems has been applied in [9, 13, 15, 18, 19]. Simple stability conditions for the asymptotic stability of positive switched linear systems for switchings have been established in [14]. The stability of fractional order linear systems has been addressed in [20].

In this paper sufficient conditions for the asymptotic stability of positive fractional switched continuous-time linear systems for any switchings are established.

The paper is organized as follows. In Sec. 2 basic definitions and theorems concerning positive fractional continuous-time systems are recalled and the formulation of the problem is given. The main result of the paper is presented in Sec. 3 where sufficient conditions for the asymptotic stability of positive fractional switched continuous-time linear systems for any switchings are established. Concluding remarks are given in Sec. 4.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

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2. Preliminaries

Consider the continuous-time linear system

$$ {}_0D_t^\alpha x(t) = Ax(t), \quad x_0 = x(0), \quad 0 < \alpha < 1, \tag{1}$$

where $x(t) \in \Re^n$ is the state vector and $A \in \Re^{n\times n}$,

$$ {}_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^\alpha} d\tau, \qquad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}, \tag{2}$$

where $n-1 < \alpha < n$, $n \in N = \{1,2,...\}$ is the Caputo definition of $\alpha \in \Re$ order derivative and

$$ \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{3}$$

is the Euler gamma function.

Definition 1. [5] The fractional system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $t \geq 0$ for any initial conditions $x(0) = x_0 \in \Re_+^n$.

Theorem 1. [5] The fractional system (2.1) is positive if and only if

$$ A \in M_n, \tag{4}$$

where M_n is the set of $n\times n$ Metzler matrices.

Definition 2. [1, 2] The fractional positive system (1) is called asymptotically stable (shortly stable) if

$$ \lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x_0 \in \Re_+^n. \tag{5}$$

Theorem 2. [1, 2, 5, 13] The positive fractional system (1) is asymptotically stable if and only if one of the following conditions is satisfied:

1. the coefficients of the polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \quad (6)$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n-1$.

2. there exists a strictly positive vector $\lambda \in \Re_+^n$ (with all positive components) such that $A^T\lambda$ is a strictly negative vector, i.e.

$$A\lambda < 0. \quad (7)$$

Theorem 3. [5] The solution of equation (1) with initial conditions $x_0 \in \Re^n$ is given by

$$x(t) = \Phi_0(t)x_0, \quad (8)$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \quad (9)$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function.

The matrices $\Phi_0(t)$ can be computed by the use of the Sylvester formula [21, 22] which in the case of distinct eigenvalues of A takes the form

$$\Phi_0(t) = \sum_{k=1}^{n} Z_k E_\alpha(s_k t^\alpha), \quad (10a)$$

where s_k for $k = 1, 2, \ldots, n$ are the eigenvalues of the matrix A and

$$Z_k = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{A - s_j I_n}{s_k - s_j}. \quad (10b)$$

Consider the fractional switched continuous-time linear systems

$$\begin{gathered} {}_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = A_{\delta(t)}x(t), \\ x_0 = x(0), \quad 0 < \alpha < 1, \end{gathered} \quad (11)$$

where $x(t) \in \Re^n$ is the state vector, $A_{\delta(t)} \in \Re^{n\times n}$ and $\delta(t)$ is the switching function which takes its values in the finite set $S = \{1, 2, ..., N\}$, N is the number of subsystems. It is assumed that the state vector $x(t) \in \Re^n$ does not jump at the switching instants $0 \le t_0 < t_1 < ....$ When $t \in [t_k, t_{k+1})$, $k = 0, 1, ...$ then $\delta(t_k)$-th system of (1) is active.

Definition 3. The fractional switched system (11) is called positive if

$$A_{\delta(t)} \in M_n \quad \text{for} \quad \delta(t) \in S. \quad (12)$$

The problem under considerations for the positive fractional switched continuous-time system (11) can be stated as follows:

Find conditions under which the positive fractional switched systems (11) is asymptotically stable for any switchings (finite number for any finite interval).

3. Problem solution

The solution is based on the following lemma.

Lamma 1. The positive fractional continuous-time system (1) with $A = [a_{i,j}] \in M_n$ is asymptotically stable if

$$\begin{gathered} a_{ii} < 0 \quad \text{and} \quad a_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} < 0 \\ \text{for} \quad i = 1, 2, ..., n \end{gathered} \quad (13a)$$

or

$$\begin{gathered} a_{jj} < 0 \quad \text{and} \quad a_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij} < 0 \\ \text{for} \quad j = 1, 2, ..., n. \end{gathered} \quad (13b)$$

Proof. The positive fractional system (1) is asymptotically stable if all eigenvalues of the matrix A are located in the left half of the complex plane [5, 23]. By Gershgorin's Circle Theorem [24] the eigenvalues of A are located in the left half of the complex plane if all discs centered at the point a_{ii} (a_{jj}) with the radii

$$r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}, \quad i = 1, 2, ..., n \left(r_j = \sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij}, \quad j = 1, 2, ..., n \right)$$

are located in the left half of the complex plane.

Remark 1. Asymptotically stable Metzler matrices may satisfy only one of the conditions (13). For example the matrix

$$A = \begin{bmatrix} -0.8 & 1 \\ 0.21 & -1.2 \end{bmatrix} \in M_2 \quad (14)$$

satisfies only the condition (13b) since $a_{11} = -0.8$, $a_{21} = 0.21$ but it does not satisfy the condition (13a) since $a_{12} = 1$. The positive fractional system (1) with (14) is asymptotically stable since the polynomial

$$\begin{aligned} \det[I_2 s - A] &= \begin{vmatrix} s + 0.8 & -1 \\ -0.21 & s + 1.2 \end{vmatrix} \\ &= s^2 + 2s + 0.75 \end{aligned} \quad (15)$$

has all positive coefficients and the condition 1) of Theorem 2 is satisfied.

It is easy to show that the positive fractional systems (11) is asymptotically stable for any switchings only if all systems of (11) are asymptotically stable. Therefore, it is assumed that he systems of (11) are asymptotically stable.

Theorem 4. Let the subsystems of (11) be asymptotically stable, i.e. $A_{\delta(t)} \in M_n$ for $\delta(t) \in S = \{1, 2, ..., N\}$ be asymptotically stable matrices. The positive fractional switched systems (11) is asymptotically stable for any switchings if the sum of entries of every column of the matrices $A_{\delta(t)}$, $\delta(t) \in S$ is negative.

Proof. By Lemma 1 the subsystems of (11) are asymptotically stable since the matrices $A_{\delta(t)}$, $\delta(t) \in S$ satisfy the condition (13b). As a common Lyapunov function for all subsystems we choose

$$V(x(t)) = 1_n^T x(t), \quad (16)$$

where $1_n^T = [\ 1 \quad \dots \quad 1\ ] \in \Re_+^n$. The function (16) is positive definite for all positive subsystems since $1_n^T x(t) > 0$ for any nonzero $x(t) \in \Re_+^n$, $t \geq 0$. From (16) and (11) we have

$$\frac{d^\alpha V(x(t))}{dt^\alpha} = 1_n^T \frac{d^\alpha x(t)}{dt^\alpha} = 1_n^T A_{\delta(t)} x(t) < 0 \tag{17}$$

since by assumption the sum of entries of every column of the matrices $A_{\delta(t)}$, $\delta(t) \in S$ is negative and the row vector $1_n^T A_{\delta(t)}$ has all negative components. Therefore, the positive fractional switched system (11) is asymptotically stable for any switchings.

Example 1. Consider the positive fractional switched system (11) for $N = 2$ with the matrices

$$A_1 = \begin{bmatrix} -0.8 & 0.5 \\ 0.4 & -0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ 0.2 & -1.1 \end{bmatrix}. \tag{18}$$

The switching function $\delta(t)$ is presented on Fig. 1.

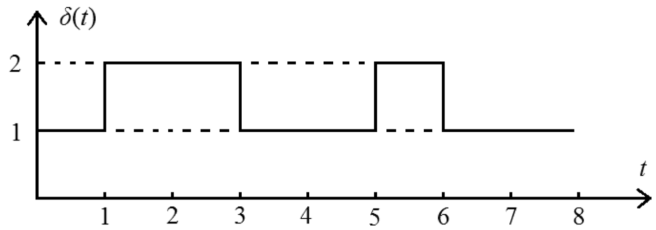

Fig. 1. Switching function $\delta(t)$ for the system (11) with matrices (18)

The matrices (18) are asymptotically stable Metzler matrices with the eigenvalues $s_{11} = -0.3$, $s_{12} = -1.2$ and $s_{21} = -0.6$, $s_{22} = -1.5$, respectively. Note that the matrices (18) satisfy the assumptions of Theorem 4. The solution of the equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = A_1 x(t) = \begin{bmatrix} -0.8 & 0.5 \\ 0.4 & -0.7 \end{bmatrix} x(t), \quad 0 < \alpha < 1 \tag{19}$$

has the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 5 \\ 4 & 5 \end{bmatrix} E_\alpha(-0.3t^\alpha) + \frac{1}{9} \begin{bmatrix} 5 & -5 \\ -4 & 4 \end{bmatrix} E_\alpha(-1.2t^\alpha). \tag{20}$$

The solution of the equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = A_2 x(t) = \begin{bmatrix} -1 & 1 \\ 0.2 & -1.1 \end{bmatrix} x(t) \tag{21}$$

has the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & 10 \\ 2 & 4 \end{bmatrix} E_\alpha(-0.6t^\alpha) + \frac{1}{9} \begin{bmatrix} 4 & -10 \\ -2 & 5 \end{bmatrix} E_\alpha(-1.5t^\alpha). \tag{22}$$

Taking into account the switching function $\delta(t)$ (presented on Fig. 1), (20) and (22) we obtain

$$x(t) = \frac{1}{9} \begin{bmatrix} 4E_\alpha(-0.3t^\alpha) + 5E_\alpha(-1.2t^\alpha) & 5(E_\alpha(-0.3t^\alpha) - E_\alpha(-1.2t^\alpha)) \\ 4(E_\alpha(-0.3t^\alpha) - E_\alpha(-1.2t^\alpha)) & 5E_\alpha(-0.3t^\alpha) + 4E_\alpha(-1.2t^\alpha) \end{bmatrix} x_0 \quad \text{for} \quad 0 \leq t < 1, \tag{23a}$$

$$x(t) = \frac{1}{9} \begin{bmatrix} 5E_\alpha(-0.6(t-1)^\alpha) + 4E_\alpha(-1.5(t-1)^\alpha) & 10(E_\alpha(-0.6(t-1)^\alpha) - E_\alpha(-1.5(t-1)^\alpha)) \\ 2(E_\alpha(-0.6(t-1)^\alpha) - E_\alpha(-1.5(t-1)^\alpha)) & 4E_\alpha(-0.6(t-1)^\alpha) + 5E_\alpha(-1.5(t-1)^\alpha) \end{bmatrix} x(t_1) \quad \text{for} \quad 1 \leq t < 3, \tag{23b}$$

where $x(t_1)$ is given by (23a) for $t = t_1 = 1$ and

$$x(t) = \frac{1}{9} \begin{bmatrix} 4E_\alpha(-0.3(t-3)^\alpha) + 5E_\alpha(-1.2(t-3)^\alpha) & 5(E_\alpha(-0.3(t-3)^\alpha) - E_\alpha(-1.2(t-3)^\alpha)) \\ 4(E_\alpha(-0.3(t-3)^\alpha) - E_\alpha(-1.2(t-3)^\alpha)) & 5E_\alpha(-0.3(t-3)^\alpha) + 4E_\alpha(-1.2(t-3)^\alpha) \end{bmatrix} x(t_2) \quad \text{for} \quad 3 \leq t < 5 \tag{23c}$$

where $x(t_2)$ is given by (23b) for $t = t_2 = 3$.

From (23) it follows that the positive fractional switched system with (18) is asymptotically stable for any switchings.

4. Concluding remarks

Sufficient conditions for the asymptotic stability of positive fractional switched continuous-time linear systems for any switchings have been established. It has been shown that the positive fractional switched continuous-time system is asymptotically stable for any switchings if the sum of entries of every column of the matrices of subsystems is negative (Theorem 4). Note that the well-known [8] condition that the matrices of subsystems commute is not necessary for the asymptotic stability of the positive fractional switched systems for any switchings. The effectiveness of the presented sufficient conditions is demonstrated on a numerical example of positive fractional switched continuous-time linear system.

Following [6, 7] the presented sufficient conditions can be extended to the positive switched 2D linear systems.

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