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MODELLING OF DECISION SUPPORT USING THE STACKELBERG DUOPOLY MODEL TO BIMATRIX HIERARCHICAL NON-ZERO-SUM GAME¹

Summary. In the work some problem receiving the elements of the bimatrix hierarchical non-zero sum game is presented. In this aim the Stackelberg duopoly model was used. To explain a proposed method the example concerning two mining enterprises output a coal is discussed.

METODA OTRZYMYWANIA ELEMENTÓW BIMACIERZY W HIERARCHICZNEJ GRZE O SUMIE NIEZEROWEJ ZA POMOCĄ MODELU DUOPOLU STACKELBERGA

Streszczenie. W artykule rozważono metodę otrzymywania elementów bimacierzy hierarchicznej gry o sumie niezerowej za pomocą Modelu Duopolu Stackelberga. Zaproponowaną metodę zilustrowano na przykładzie dwóch przedsiębiorstw górniczych, rozwiązujących problem planowania i kontrolowania zapotrzebowania materiałowego.

1. Introduction

Applications of the matrix Stackelberg game in a variety of decision-making situations are numerous and it is impossible to mention them all. Some of them, in our opinion important for this study are presented in [1, 2, 3]. These relate to the game scenario construction methods, methods for obtaining Stackelberg equilibrium in pure or mixed strategies. In all studies we have adopted sample forms of game bimatrix with specified values of these matrix elements (payment in the game for both players). In this case, completely ignored is the

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problem of receiving bimatrix elements in a particular decision making model in the form of the Stackelberg game.

Therefore, the purpose of this paper is to analyze the problem of receiving payoffs in the two-person matrix Stackelberg game.

In this paper a task of the winning strategy development is solved basing on one of game theory methods Stackelberg Duopoly Model (SDM) for receiving the payoffs of the two-person game. Two-person game with one leader and one follower and many followers is considered and discussed. Author proposed to modify a classical algorithm based on original method receiving of the elements of $(n \times m)$ dimensional bimatrix game. This method is built on the SDM ground, and it determines the payoffs players as profit for each player. The next n and m pure strategies for two players are considered. To explain a proposed method the example concerning two mining enterprises output a coal is presented.

Thus, section 2 analyzes the real decision-making situations, their modeling and solving by two-person Stockelberg games. Section 3 identifies the elements of the game scenario, allowing to obtain the possible numerical values of bimatrix in Stockelberg game. For this purpose was used a model of Stackelberg duopoly. In section 4 we present a numerical example to expalain a proposed method of determining payoffs in the Stackelberg game. Final conclusions are presented in section 5.

2. Modeling and solving Stackelberg games

We will write in the general form bimatrix of the Stackelberg game for two players with the m - pure strategies for a leader and n - pure strategies for a follower where: $i = \overline{1, m}$, $j = \overline{1, n}$.

$$[(a_{ij}, b_{ij})]_{m \times n} \quad (1)$$

If more than one follower (leader always remains the same) the number of the game bimatrix increases accordingly. Then the game bimatrix will be as follows:

$$[(a_{ij}, b_{ij})]_{[m \times n]}^{K=1, r} \quad (2)$$

where: $k = \overline{1, r}$, r - number of followers.

Sometimes in the calculations it is easier to use matrices of individual elements for each player. The number of such matrices will be then $(r+1)$. We then say that we are dealing with $(r+1)$ personal Stackelberg game [4]. Numerical example of such matrix with 4 players can be found in [2].

The above shows that the asking numerical value for payoffs to individual players is quite challenging and requires appropriate methodological procedure. So far in the available

literature on Stackelberg games applications in modeling real decision situations, the problem was and is completely ignored. In our opinion, the main reason for this is that most of the studies relate to methods of solving in usually two or multiplayer Stackelberg games in pure or mixed strategies. In this case the numerical values of payoffs for players were selected arbitrarily so as to emphasize the analyzed method of solving.

Regarding the methods for solving Stackelberg games in pure strategies, the solutions are generally based on the procedures of min max or max min of corresponding payoffs for a leader [5]. In case of finding the Stackelberg equilibrium in mixed strategies the problem is greatly complicated. In this situation, we deal with a solution for two-or multi-dimensional discrete problem of nonlinear programming. Therefore we use a discrete two-stage programming, iterated linear programming [6] (also a two-stage), and gradient methods in the case of favoring or not favoring followers. For finding the equilibrium point in mixed strategies in non cooperative bimatrix games we use the Lemke – Howson algorithm [7]. Another class of today's methods for solving multiplayer static Stackelberg games are numerical methods [8, 9]. One of them is so-called extraproximal method [10]. Its version with regularization has been proposed in [2] to solve the four players Stackelberg game (1 leader and 3 followers).

In all discussed cases of Stackelberg games and their methods of solving, the figures for individual bimatrix were adopted arbitrarily without any justification! The problem of determining the payoffs in two- players zero - sum game with symmetric (equal) players is from case to case seen today in a comprehensive literature in search of Nash equilibrium in non cooperative games [5, 11].

Next the paper presents a method of determining payoffs in the two- players matrix Stackelberg game.

3. Determining payoffs in the matrix Stackelberg game

Now we will discuss a method for determining payoffs in the Stackelberg game with m pure strategies for a leader and n pure strategies for a follower. The pure strategies for both players will be to concern two mining enterprise. To specify the way (methodology) to obtain the numerical value of payments in the Stackelberg game, it will be necessary for further analysis to discuss the model of imperfect competition in the Stackelberg model of duopoly.

Historically, the Stackelberg model was based on criticism of the Cournot model. We will not discuss in detail, of course, all the conditions leading from the Cournot model to the Stackelberg model, and consequently to the two-person Stackelberg game. We will note only that in the Cournot model each firm (enterprise) independently determines the volume of production, which guarantees the greatest profit. Unlike the Cournot model, where market

players make decisions simultaneously, since both players have equal rights (are symmetrical), which leads to the game with incomplete information. Heinrich F. von Stackelberg proposed in 1934 another duopoly model [12, 13]. In this model, market players are asymmetrical which results from their mutual dependency. One of them so-called a leader makes a move first, while the other player so-called a follower adjusts its moves (decisions) to the leader. Thus, in effect, the Stackelberg game is hierarchical, which is played in two stages. It is a game with complete information [14, 15].

Stackelberg duopoly model assumes a linear demand and constant marginal costs. A linear model is also assumed for the total production cost of each company. Let the inverse demand function (and thus the function of the price) be as follows:

$$P(Q) = a - b \cdot Q \quad (3)$$

where: $Q = Q_A + Q_B$ is the total production companies A and B, while for total costs we assume that:

$$K_c(Q_i) = f_i + K_{mi} \cdot Q_i, \quad i = A \cup B \quad (4)$$

where: K_{mi} - Incremental costs f_i - certain constant, and $a > K_{mi}$, $b > 0$.

In the Stackelberg model player B - follower, in this case a mining company, will seek a level of production (coal mining) Q_B , which depends on the decision of the leader – player A to maximize its profit Z_B

$$Z_B = [a - b(Q_A + Q_B)] \cdot Q_B - f_B - K_{mB} \cdot Q_B. \quad (5)$$

The leader selects an Q_A , to maximize its profit Z_A .

$$Z_A = [a - b(Q_A + Q_B)]Q_A - f_A - K_{mA} \cdot Q_A. \quad (6)$$

In the Stackelberg duopoly, there is only one equilibrium for the levels of production and profits of both players, ie (Q_A^*, Q_B^*) and (Z_A^*, Z_B^*) . These pairs form the Nash equilibrium in the Stackelberg game. Differentiating (5) after Q_B we obtain:

$$\frac{\partial Z_B}{\partial Q_B} = a - bQ_A - 2bQ_B - K_{mB}. \quad (7)$$

In this formula $a - bQ_A - 2bQ_B$ is the marginal income of the company B.

For equilibrium conditions should occur:

$$a - bQ_A - 2bQ_B = K_{mB} \quad (8)$$

from here:

$$Q_B = \frac{a - b \cdot Q_A - K_{mB}}{2b}. \quad (9)$$

You can see that the size of the company B production is dependent on Q_A .

Market price of the leader is $P_A = a - bQ_A - bQ_B$. Substituting into this relation the value Q_B from (9) we obtain:

$$P_A = a - bQ_A - b\left(\frac{a - bQ_A - K_{mB}}{2b}\right)$$

from:

$$P_A = \frac{a + K_{mB} - bQ_A}{2}. \quad (10)$$

We calculate the total revenue of the leader

$$D_{CA} = P_A \cdot Q_A = Q_A \left(\frac{a + K_{mB} - bQ_A}{2} \right).$$

Differentiating the Q_A we obtain marginal revenue of the leader in the form:

$$\frac{\partial P_A \cdot Q_A}{\partial Q_A} = \frac{a}{2} + \frac{K_{mB}}{2} - bQ_A. \quad (11)$$

Equating these values to the marginal cost of the leader K_{mA} we obtain:

$$\frac{a}{2} + \frac{K_{mB}}{2} - bQ_A = K_{mA}. \quad (12)$$

Hence, we obtain the value Q_A^* , which is in the Stackelberg equilibrium for the leader (Stackelberg cost for the leader):

$$Q_A^* = \frac{a + K_{mB} - 2K_{mA}}{2b}. \quad (13)$$

And the value Q_B^* , which is in the Stackelberg equilibrium for the follower. From formula (9) we have:

$$Q_B^* = \frac{a - bQ_A^* - K_{mB}}{2b}.$$

After the substitution Q_A^* we receive:

$$Q_B^* = \frac{a + 2K_{mA} - 3K_{mB}}{4b}. \quad (14)$$

We can now calculate the price P^* of the market equilibrium for both players:

$$P^* = a - b \cdot Q_A^* - bQ_B^* \quad (15)$$

and profits Z_A^* and Z_B^* . Given that:

$$Z_A = D_{CA} - K_{CA}(Q_A), \quad (16)$$

and:

$$Z_B = D_{CB} - K_{CB}(Q_B), \quad (17)$$

we obtain the pair (Z_A^*, Z_B^*) in the form:

$$Z_A^* = P^* \cdot Q_A^* - K_{CA}(Q_A^*) = P^* \cdot Q_A^* - f_A - K_{mA} \cdot Q_A^* \quad (18)$$

and:

$$Z_B^* = P^* \cdot Q_B^* - K_{mB}(Q_B^*) = P^* \cdot Q_B^* - f_B - K_{mB} \cdot Q_B^* \quad (19)$$

The total market earnings generated by the two companies will be $Z^* = Z_A^* + Z_B^*$.

Let's discuss now the presented Stackelberg model of duopoly and basing on it the ability to set the value of payments in the matrix Stackelberg game with m pure strategies for the leader and n pure strategies for follower. Note that the equilibrium in the Stackelberg duopoly, represented by the pair of numbers (Q_A^*, Q_B^*) and (Z_A^*, Z_B^*) is the differential equilibrium obtained assuming an infinite number of pure strategies for both players. However, in the matrix Stackelberg game, the number of these strategies is always finite and presents the production levels of both players. Thus, to construct a bimatrix Stackelberg game the number of pure strategy m and n the production levels of both players should be specified in advance. The optimization task modeled by the matrix Stackelberg game the numbers are justified on the basis of the game scenario. This issue was discussed more in the work [16]. Payments in the game for both players in this case are defined profits are defined corresponding to a particular pair of strategy.

Now we are going to back to the bimatrix of payoffs of the studied Stackelberg game, modeling a decision-making problem.. Thus, we set m levels of production for the leader. Let's denote these values by $Q_A^i, i = \overline{1, m}$. Also, we set in advance n values for production for the follower. Let's denote these value by $Q_B^j, j = \overline{1, n}$. For each of these players these are their strategies. Te pure strategies for both players in this case reflect the volume of coal production. The extraction of this type is connected with the specific costs of production., which is reflected by the formula (5) and (6).

For illustrate the methodology to obtain the elements of bimatrix in the Stackelberg game will be presented one example.

4. Numerical example

On the basis of data obtained in a coal mine it was assumed that the inverse demand function has the form $P(Q) = 1250 - 180 \cdot Q$ and function for the total cost can be expressed by $K_c(Q_A) = 200 + 30 \cdot Q_A$ for the leader and $K_c(Q_B) = 150 + 45 \cdot Q_B$ for the follower. Let's

interpret these values. Maximum price of coal per ton is in this case 1250 PLN. This price reflects the situation in a duopoly market where demand is the maximum and coal supply is zero, ie the production of coal in both the leader plant and the follower plant is zero. The constant value 180 [PLN / ton] determines the profitability of production at high production level of both mining companies. Next, we have total costs as a constant; $f_A = 200m \ln.zł$ (PLN) and $f_B = 150m \ln.zł$ (PLN) for annual production of the two mining companies. $K_{mA} = 30m \ln.zł$ (PLN) and $K_{mB} = 45m \ln.zł$ (PLN) are the constant marginal costs in the year of production. We assume then pure strategies for both players as their levels of production. These levels are arbitrary, starting with the game scenario and the number of pure strategies. Taking into account formulas (5) and (6) we obtain bimatrix of profits of the Stackelberg game in the following form: The Table 1 shows the Stackelberg (5x5) dimensional bimatrix for two enterprises as profits (in Polish zloty) per each player. In the Table 1 variables Q_A and Q_B are output in millions tons per each enterprise.

Table1

The Stackelberg (5x5) bimatrix game for two enterprises

		Q_B [mln tons]				
		0	1	2	3	4
Q_A [mln tons]	0	(-200, -150)	(-200, 885)	(-200,1590)	(-200, 1045)	(-200, 1794)
	1	(840, -150)	(560, 600)	(480, 1180)	(300, 1305)	(130, 1070)
	2	(1520,- 150)	(1160, 515)	(800, 970)	(490, 770)	(80, 350)
	3	(2130,-150)	(1330, 335)	(760, 460)	(220, 225)	(-320, -370)
	4	(1800, -150)	(1080, 155)	(360, 100)	(-360, -315)	(-1080, -1190)

On the ground of bimatrix game the Stackelberg equilibrium was calculated from the formula:

$$\min_j a_{i_0j} = \max_i \min_j a_{ij} = S^*(A).$$

where $i, j = 1, 2, \dots, 5$. and $S^*(A)$ is profit of Stackelberg for the leader. According to a formula above we determined the optimal strategies as pair (4,1) and the profit **(1080,155)** - corresponding to this pair of strategies - is Stackelberg result of balance.

Simultaneously the equilibrium of the SDM (when the number of pure strategies for two enterprises is infinite) also was calculated (18),(19). The result was **(851,330)** for output $Q_A = 3,4m \ln.tons$ and $Q_B = 1,6m \ln.tons$.

Finally we can conclude that application of the Stackelberg Duopoly Model allows creating correctly the bimatrix hierarchical non-zero-sum game. Proposed method can be recommended for problem solving in uncertain conditions in different domains.

5. Conclusions

The study, discussion and numerical example presented in this paper lead to a few conclusions:

1. The possibility of receiving payoffs in any matrix game for two players with m pure strategies for the first player and n pure strategies for the second player is one of the most important elements of such games to model real decision problems.
2. We should distinguish between the matrix model of the Stackelberg and the Stackelberg duopoly model. Both of these models used together make it possible to calculate the elements of the game bimatrix modeling real decision problem.

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Omówienie

Problem otrzymywania wypłat w grach macierzowych o sumie zerowej lub niezerowej jest zwykle pomijany. Rozważamy więc gotowe macierze w celu znalezienia rozwiązań gier oraz wyznaczenia strategii optymalnych – czystych lub mieszanych. Należy sobie jednak zdać sprawę z tego, że jeżeli te gotowe macierze nie zawierają przypadkowych liczb, będących wypłatami w grze dla obu graczy, to otrzymanie ich dla każdej rzeczywistej gry jest zagadnieniem niezwykle skomplikowanym i dla każdej gry stanowi problem sam w sobie. Rzeczywiste wypłaty w grze dla obu graczy stanowią odzwierciedlenie strategii graczy i rodzaju konfliktu między graczami. Dokładny opis konfliktu daje możliwość zorientowania się w sytuacji obu graczy i zaproponowanie odpowiedniego modelu decyzyjnego. Budowa modelu decyzyjnego powinna pozwolić na skonstruowanie odpowiedniej macierzy gry z wyróżnionymi elementami.

Te i inne problemy zostały dokładnie omówione w niniejszym artykule na przykładzie budowy macierzy hierarchicznej dwuosobowej gry o sumie niezerowej. W tym celu zaproponowano wykorzystanie Modelu Duopolu Stackelberga. Przedstawioną metodę zilustrowano na przykładzie budowy modelu decyzyjnego dla rozwiązania problemu planowania i kontrolowania zapotrzebowania materiałowego w przedsiębiorstwach górniczych.