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Passivity-based optimal control of discrete-time nonlinear systems*

by

Tahereh Binazadeh and Mohammad Hossein Shafiei

School of Electrical and Electronic Engineering, Shiraz University of Technology,
Modares Blvd., Shiraz, P.O. Box 71555/313, Iran
{binazadeh,shafiei}@sutech.ac.ir

Abstract: In this paper, a passivity-based optimal control method for a broad class of nonlinear discrete-time systems is proposed. The resulting control law is a static output feedback law which is practically preferred with respect to the state feedback law and is simple to implement. The control law has a general structure with adjustable parameters which are tuned, using an optimization method (genetic algorithm), to minimize an arbitrary cost function. By choosing this cost function it is possible to shape the transient response of the closed-loop system, as it is desirable. An illustrative example shows the effectiveness of the proposed approach.

Keywords: nonlinear discrete-time systems, optimal passivity-based control, genetic optimization algorithm;

1. Introduction

The concept of dissipativity and its particular case, passivity, were born from the observation of physical systems behaviors. These concepts have provided a useful tool for analysis of nonlinear systems. One of the main motivations, in the study of passivity in the system theory, is its connection with stability (Willems, 1972; Hill and Moylan, 1976; Sengor, 1995; Byrnes, Isidori and Willems, 1991). Passive systems have this valuable property that with a special kind of output feedback, their closed-loop stability is guaranteed.

A very important tool to check the dissipativity or passivity of continuous-time and discrete-time systems is the well-known Kalman-Yakubovich-Popov (KYP) Lemma or its equivalent for linear systems, i.e. Positive Real Lemma (PR) (Willems, 1972). In the paper of Hill and Moylan (1976), the KYP conditions have been developed for a broad class of nonlinear continuous-time systems. This work was continued by Isidori and Willems (1991) for affine nonlinear

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systems. In the case of discrete-time systems, first, KYP Lemma was developed for linear systems (Hitz and Anderson, 1969) and then, KYP conditions were developed for nonlinear systems which are affine in control (Byrnes and Lin, 1993, 1994; Lin and Byrnes, 1995). In the paper of Sengor (1995), new definitions of lossless and dissipative systems in the framework of abstract dynamical energy systems were proposed and then the appropriate KYP conditions were developed. In the case of non-affine discrete-time systems, the KYP lemma has been presented for single-input single-output systems by Navarro-Lopez, Sira-Ramirez and Fossas-Colet (2002). Then, KYP conditions for single-input multiple-output systems which are non-affine-in-input, have been presented by Monaco and Normand-Cyrot (1997, 1999). Navarro-Lopez (2002, 2007) developed KYP conditions for a class of dissipativity called *Quadratic Storage Supply*-dissipativity for MIMO general systems. There are also other studies in this regard (for instance, see Navarro-Lopez, 2005).

The close connection between stability and passivity concepts has caused the development of passivity-based control methods. In this regard, Byrnes and Isidori (1991) have shown that a number of stabilization theorems may be derived from the basic stability property of passive systems. In Byrnes and Lin (1993); Lin and Byrnes (1995); Navarro-Lopez (2002, 2007); Navarro-Lopez, Sira-Ramirez and Fossas-Colet (2002); Navarro-Lopez and Fossas-Colet (2004) nonlinear discrete-time systems which are affine in the control input, have been studied and some theorems on passivity-based control of such systems have been presented.

However, in the case of non-passive system, the above mentioned methods were inapplicable. Therefore, the action of making a system passive using a static state feedback, which is known as feedback passivity (passification), was studied. Consequently, sufficient conditions to convert MIMO non-passive systems to passive ones were proposed in the series of papers (Shuping, Guoshan and Wanquan, 2010; Navarro-Lopez, 2002, 2007; Navarro-Lopez, Sira-Ramirez and Fossas-Colet, 2002; Navarro-Lopez and Fossas-Colet, 2004).

The problem of stabilization of passive systems may be summarized as follows: If a nonlinear system is zero-state detectable and passive (with a positive definite storage function), the origin can be globally stabilized by the output feedback $u = -\varphi(y)$, where φ is any locally Lipschitz function such that $\varphi(0) = 0$ and $y^T\varphi(y) > 0$, for all $y \neq 0$ (Lin and Byrnes, 1995). The output feedback is more preferred with respect to another type of feedback law (state feedback). This is because that the state variables in many of applications are not available or measurable. However, the system output is almost always available. Moreover, the proposed control law is static which is, in practice, much preferred with respect to a dynamic output control law.

There is a great freedom in choosing this function. For any first-third quadrant sector function ($\varphi(y)$), the closed-loop system is asymptotically stable. However, the choice of this function affects the transient response of the system. The above mentioned papers have not considered the desired transient response in their design method.

One of the significant approaches in shaping the transient response of a system (the trajectories of state variables and the control input) is to define a cost function in the design procedure. Placing appropriate terms with proper weighting coefficients in this cost function can force the closed-loop system to have the desirable transient response (Lewis, Vrabie and Syrmos, 2012).

In this paper, presenting the first study connecting passivity-based control and optimality concepts for discrete-time nonlinear systems, a design method based on choosing an appropriate cost function is presented. The purpose of this paper is to use the freedom in choosing the function $\varphi(y)$, in such a way that a given cost function be minimized. Therefore, a general structure for $\varphi(y)$ (which satisfies the above conditions) is considered. In the proposed structure, there are adjustable parameters which may be found by an optimization algorithm like genetic algorithm (GA).

The remainder of this paper is organized as follows: In the next section, the basic definitions and a theorem about passive nonlinear discrete-time systems are presented. Section 3 presents a scheme to design a controller for passive systems in such a way that an appropriate cost function is minimized. A design example is given in Section 4. Finally, conclusions are presented in Section 5.

2. Preliminary definitions

This section introduces some basic definitions concerning the concept of passivity in the nonlinear discrete-time systems, based on the definitions from the papers of Willems (1972) and Lin and Byrnes (1995).

A general class of nonlinear discrete-time systems can be described by the following state-space equations:

$$\begin{aligned} x(k+1) &= F(x(k), u(k)), \\ y(k) &= H(x(k), u(k)), \end{aligned} \tag{1}$$

where $x \in D \subseteq R^n$ is the state vector, $u \in \mathrm{U} \subseteq R^m$ is the control input, and $y \in R^m$ is the system output. Suppose that F and H are both smooth mappings with the appropriate dimensions. Moreover, assume that $F(0,0) = 0$ and $H(0,0) = 0$. In this situations, a positive definite scalar function $V(x(k)) : D \to R$ (where $V(0) = 0$) is addressed as storage function and system (1) is said to be locally passive, if there exists a storage function $V(x(k))$ such that:

$$V(F(x,u)) - V(x) \leq y^T u \qquad \forall (x,u) \in \mathrm{D} \times \mathrm{U} \tag{2}$$

where $\mathrm{D} \times \mathrm{U}$ is a neighborhood of x=0, u=0.

Definition 2.1 *The zero dynamics of system (1) is defined by* $F^* = F(x, u^*)$, *where* $(x, u^*) = \{(x,u) : \text{ s.t. } H(x,u) = 0\}$. *A system of the form (1) has a locally passive zero dynamics, if there exists a positive definite function* $V(x(k)) : D \to R$, *such that:*

$$V(F(x,u^*)) \leq V(x) \qquad \forall x \in D. \tag{3}$$

Definition 2.2 *A system (1) has local relative degree zero at $x = 0$, if*

$$\left.\frac{\partial H(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}} \tag{4}$$

is nonsingular.

Now, assume that the nonlinear discrete-time system (1) is affine in the control input (*i.e.,* in the following form):

$$\begin{aligned} x(k+1) &= f(x(k)) + g(x(k))u(k), \\ y(k) &= h(x(k)) + J(x(k))u(k). \end{aligned} \tag{5}$$

The system (5) has local relative degree zero, if $J(0)$ is nonsingular. This system has uniform relative degree zero, if $J(x)$ is nonsingular for all $x \in$ D. Additionally, the system (5) is locally zero-state observable, if for all $x \in$ D,

$$y(k)|_{u(k)=0} = h(\phi(k,x,0)) = 0 \quad \forall k \in Z^+ \Rightarrow x = 0 \tag{6}$$

where $\phi(k,x,0) = f^k(x) = f(f^{k-1}(x)), \forall k > 1$, and $f^0(x) = x$. Also, $f^k(x)$ is the trajectory of the unforced dynamics, $x(k+1) = f(x(k))$, from $x(0)$=x. If D $= R^n$, the system is globally zero-state observable. Moreover, system (5) is locally zero-state detectable, if for all $x \in$ D and $y(k)|_{u(k)=0} = h(\phi(k,x,0)) = 0$ and for all $k \in Z^+$, $\lim_{k\to\infty} \phi(k,x,0) = 0$. Also, if D $= R^n$, the system is globally zero-state detectable.

Another important benefit from passive systems is their highly desirable stability property which may simplify system analysis and controller design procedure. Therefore, transformation of a non-passive system into a passive one is desirable. The use of feedback to transform a non-passive system into a passive one is known as feedback passivation (Lin and Byrnes, 1995).

Definition 2.3 *Let $\alpha(x)$ and $\beta(x)$ be smooth functions. Consider a static state feedback control law of the following form:*

$$u(x) = \alpha(x) + \beta(x)w(k) \tag{7}$$

A feedback control law of the form (7) is regular, if $\beta(x)$ is invertible for all $x \in$ D.

In order to analyze feedback passivation, the following theorem is taken from Navarro-Lopez (2007).

Theorem 2.1 *Considering a system in the form (5), suppose that $h(0) = 0$ and there exists a positive definite C^2 storage function V (i.e., the storage function and its first and second drivatives are continuous), where $V(0) = 0$ and $V(f(x)+g(x)u)$ is quadratic in u. Then, the system (5) is locally feedback equivalent to a passive system with V as the storage function by means of a regular feedback control law of the form (7), if and only if the system (5) has local relative degree zero at $x = 0$ and its zero dynamic is locally passive in a neighborhood of $x = 0$.*

It has been shown by Navarro-Lopez (2007) that the control law in the form (7) with

$$\alpha(x) = -J^{-1}(x)h(x) + J^{-1}(x)\bar{h}(x) \tag{8}$$

$$\beta(x) = J^{-1}(x)\bar{J}(x) \tag{9}$$

converts the non-passive nonlinear discrete-system (5) to a new passive dynamic described by:

$$\begin{aligned} x(k+1) &= f^*(x(k)) + g^*(x(k))\bar{h}(x(k)) + g^*(x(k))\bar{J}(x)w(k) \\ y(k) &= \bar{h}(x(k)) + \bar{J}(x)w(k), \end{aligned} \tag{10}$$

where

$$f^*(x) = f(x) - g(x)J^{-1}(x)h(x) \tag{11}$$

$$g^*(x) = g(x)J^{-1}(x), \tag{12}$$

$$\bar{J}(x) = (\frac{1}{2}g^{*T}\left.\frac{\partial^2 V}{\partial z^2}\right|_{z=f^*(x)} g^*(x))^{-1} \tag{13}$$

$$\bar{h}(x) = -\bar{J}(x)(\left.\frac{\partial V}{\partial z}\right|_{z=f^*(x)} g^*(x))^{-1}. \tag{14}$$

3. Passivity-based optimal control

Suppose that a system of the form (5) is passive with a positive definite storage function V. Let φ be any smooth mapping such that $\varphi(0) = 0$ and $y^T\varphi(y) > 0$, for all $y \neq 0$. The basic idea of the passivity-based control method is illustrated in the next theorem (Lin and Byrnes, 1995).

Theorem 3.1 *If system (5) is zero-state detectable and passive with storage function V which is proper on R^n, then the following smooth output feedback globally asymptotically stabilizes the equilibrium $x=0$:*

$$u = -\varphi(y) \qquad u, y \in R^m. \tag{15}$$

There is a freedom in selection of vector function $\varphi(y)$. In this paper, it is desirable to design $\varphi(y)$ such that, in addition to globally asymptotically stabilizing the nonlinear system (5), a given cost function is also minimized.

For this purpose, the following general structure for the vector function $\varphi(y) = [\varphi_1(y_1), \varphi_2(y_2), ..., \varphi_m(y_m)]^T$ is proposed:

$$\varphi_i(y_i) = a_{i1}y_i + a_{i2}y_i^3 + ... + a_{il}y_i^{2l-1} \quad \text{for } i = 1, ..., m \tag{16}$$

where $a_{i1}, a_{i2}, ..., a_{il}$ belong to $R^+$and the suitable $l \in Z^+ \geq 1$ may be chosen by the designer. In the suggested structure, each φ_i belongs to the first-third quadrant sector, and $y^T\varphi(y) > 0$, for all $y \neq 0$.

The task is to find the unknown coefficients a_i, such that the proposed static output feedback, minimizes an appropriate cost function in the general form $I(k) = \sum_{\bar{k}=0}^{k} L(x(\bar{k}); u(\bar{k}))$.

In order to obtain the minimum value of the considered cost function, the optimization procedure based on the theory of genetic algorithms (GA) is used. The genetic algorithms constitute a class of search and optimization methods, which imitate the principles of natural evolution (Goldberg, 1989). The flow diagram of the GA method used in this paper is depicted in Fig. 1.

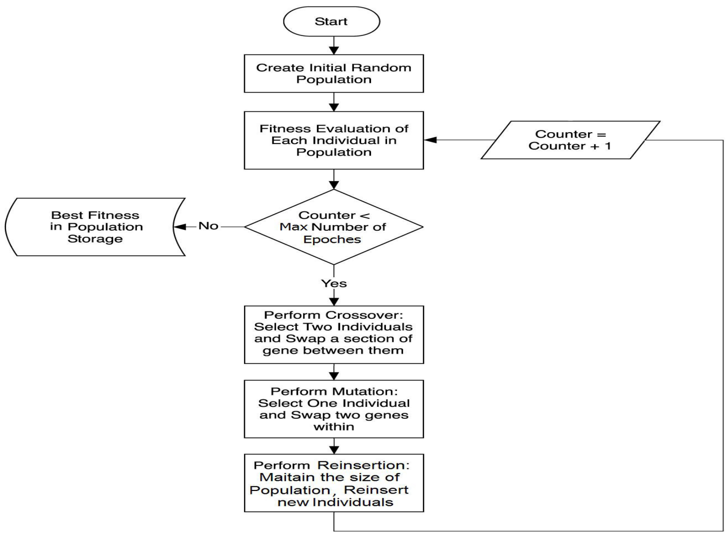

Figure 1. Flow diagram of GA method used

The genetic algorithm differs substantially from more traditional search and optimization methods, like gradient-based optimization (for more information see Belegundu and Chandrupatla, 1999; Nocedal and Wright, 1999). The most significant differences are as follow:

(a) GAs search through a population of points in parallel rather than point-after-point.
(b) GAs do not require derivative information on an objective function or other auxiliary knowledge. Only the objective function and the corresponding fitness levels influence the directions of search.
(c) GAs use probabilistic transition rules, not deterministic ones.

The basic structure of genetic algorithms consists of the following steps:

(a) Initialize a population of chromosomes.
(b) Evaluate each chromosome in the population.
(c) Create new chromosomes by mating current chromosomes.
(d) Remove weaker members of the population, according to the fitness functions for each chromosome, to make room for the new chromosomes.

(e) Insert the new chromosomes into the population.

(f) Stop and return the best chromosome if time is up, otherwise, go to (c).

Following the above structure, a pseudo-code outline of genetic algorithms is shown below. The population of chromosomes at time t is represented by the time-dependent variable $P(t)$, with the initial population of random estimates $P(0)$ (Goldberg, 1989).

```
procedure GA
begin
    t=0;
    initialize P(t) = P(0);
    evaluate P(t);
    while not finished do
    begin
        t=t+1;
        select P(t)from P(t -1);
        reproduce pairs in P(t) by
        begin
            crossover;
            mutation;
            reinsertion;
        end
        evaluate P(t);
    end
end
```

Therefore, by using the GA-based optimization, the best coefficients of the proposed structure (Equation (16)) may be found in such a way that the given cost function is minimized. In the optimization process, the corresponding cost function is considered as the fitness function of the genetic algorithm.

4. Design example

Consider the following nonlinear discrete-time system, taken from Navarro-Lopez (2007):

$$\begin{aligned} &x_1(k+1) = (x_1^2(k) + x_2^2(k) + u(k))\cos(x_2(k)) \\ &x_2(k+1) = (x_1^2(k) + x_2^2(k) + u(k))\sin(x_2(k)) \\ &y(k) = (x_1^2(k) + x_2^2(k)) + \frac{1}{x_1^2(k)+x_2^2(k)-0.25}u(k). \end{aligned} \tag{17}$$

The system (17) is not passive. Considering $V = \frac{1}{2}(x_1^2(k) + x_2^2(k))$ as a storage function, the system can be rendered passive by means of a static state feedback control law, due to the fact that $J(x(k)) = \frac{1}{x_1^2(k)+x_2^2(k)-0.25}$ is invertible and the zero dynamics of system (17) is passive (see Theorem 2.1). Therefore, the passifying control scheme, i.e. $u = \alpha(x) + \beta(x)w$, proposed by equations (8) and (9) is applied to (17). The passified system satisfies the conditions of Theorem 3.1. Consequently, it can be locally asymptotically stabilized by output feedback $w = -\varphi(y)$, where w is the new input of the passified system.

Now, the goal is to find a proper function $\varphi(y)$ in order to minimize the following cost function, related to the passified system:

$$I = \frac{1}{2}\sum_{k=0}^{\infty}(w^2(k) + x(k)^T x(k) + y^2(k)).$$

The choice of this cost function is tightly linked with physical concepts. The term $w^2(k)$ is meant to minimize the consumed energy of control signal. The term $x(k)^T x(k)$ confirms that fast convergence of the states is desired. Finally the term $y^2(k)$ is supposed to make the output signal converge to zero in minimum time and with minimum overshoot and oscillations.

The proposed optimization process has been carried out for three following cases: **Case 1**: Only first term of (16) is considered ($\phi_1(y) = a_1 y$). **Case 2**: First two terms of (16) are considered ($\phi_2(y) = a_1 y + a_2 y^3$). **Case 3**: First three terms of (16) are considered ($\phi_3(y) = a_1 y + a_2 y^3 + a_3 y^5$).

The nonlinear static functions, resulting from the GA optimization procedure are as follows:

$$\begin{aligned} &\varphi_1(y) = 0.02755y \\ &\varphi_2(y) = 0.0245y + 0.0451y^3 \\ &\varphi_3(y) = 0.021413y + 0.0296y^3 + 0.0258y^5. \end{aligned} \qquad (18)$$

The passified dynamics is simulated for the initial condition x_0=[-1, +1] and the control inputs, $w = -\varphi_1(y)$, $w = -\varphi_2(y)$ and $w = -\varphi_3(y)$. Comparison of results is given in Table 1. Also Figs. 2-4 present the response of the output, first and second states of the passified dynamics, respectively.

	$w = -\varphi_1(y)$	$w = -\varphi_2(y)$	$w = -\varphi_3(y)$
$max\,\lvert y\rvert$	0.6178	0.5155	0.6028
I	2.619	2.6097	2.6027

Table 1. The cost functions (I) of control inputs, $w = -\varphi_1(y)$, $w = -\varphi_2(y)$ and $w = -\varphi_3(y)$.

It is worth noting that, considering Table 1 and Figs. 2-4, the proposed controller with $w = -\varphi_3(y)$ has the best performance among the control laws accounted for. Thus, consideration of more terms in (16) may lead to a better performance.

5. Conclusion

In this paper, some properties of nonlinear discrete-time passive systems were studied and a passivity-based optimal control method for a broad class of nonlinear discrete-time systems was proposed. The proposed control law is a static

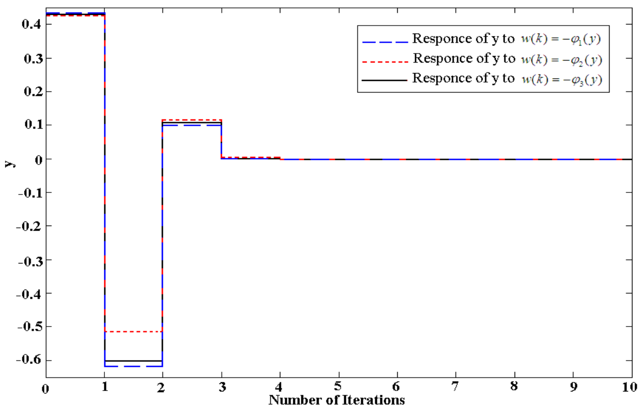

Figure 2. Time-response of the output $y(k)$

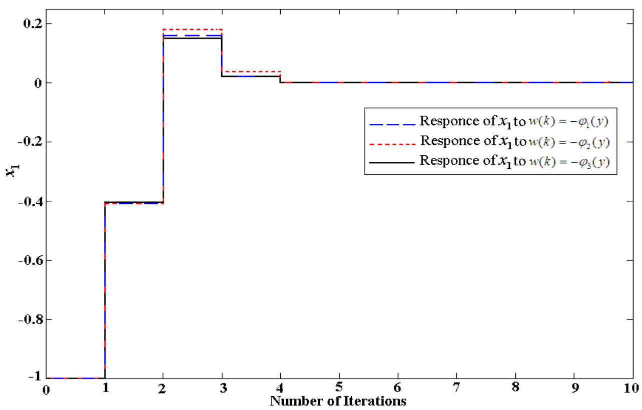

Figure 3. Time-response of the first state $x_1(k)$

output feedback law $u = -\varphi(y)$ (where $\varphi(y)$ is a smooth function belonging to the first-third quadrant sector), which has a general structure with adjustable parameters. These parameters were found by a genetic algorithm, to shape the transient response of the closed-loop system. Effectiveness of the proposed procedure was illustrated by an example.

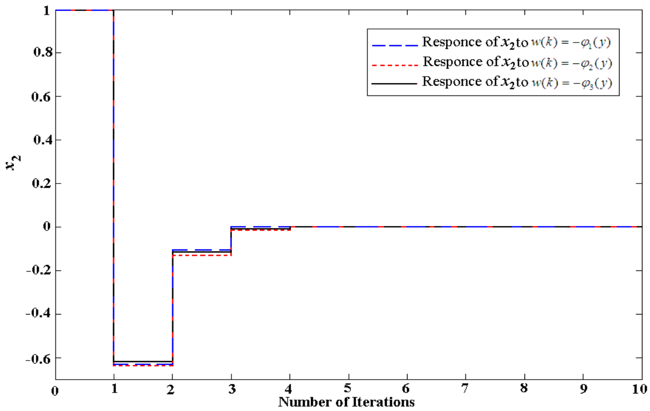

Figure 4. Time-response of the second state $x_2(k)$

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