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DIFFUSION APPROXIMATION OF RECURRENT SCHEMES FOR FINANCIAL MARKETS, WITH APPLICATION TO THE ORNSTEIN-UHLENBECK PROCESS

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Abstract. We adapt the general conditions of the weak convergence for the sequence of processes with discrete time to the diffusion process towards the weak convergence for the discrete-time models of a financial market to the continuous-time diffusion model. These results generalize a classical scheme of the weak convergence for discrete-time markets to the Black-Scholes model. We give an explicit and direct method of approximation by a recurrent scheme. As an example, an Ornstein-Uhlenbeck process is considered as a limit model.

Keywords: diffusion approximation, semimartingale, recurrent scheme, financial market, multiplicative scheme, Ornstein-Uhlenbeck process.

Mathematics Subject Classification: 60F17, 60J60, 60G15, 91G80.

1. INTRODUCTION

The paper focuses on the diffusion approximation for the recurrent schemes of financial markets. The problem of convergence of the discrete-time financial models to the models with continuous time is well developed: starting from the central limit theorem for approximation of the Black-Scholes model by the Cox-Ross-Rubinstein model and continuing with more involved models, see, e.g., [1–5, 12]. The evident questions here are: does the weak convergence of the stock price processes imply the convergence of the option price processes or the convergence of the hedging portfolios and the optimal portfolio strategies. The attempts to go ahead from simple binomial schemes were made using the results of weak convergence. These results were generalized with the help of the functional limit theorems in [13] and summarized, e.g., in [19]. The rate of convergence of the option prices in the framework of weak convergence was also widely discussed, e.g. in [2, 11]. However, for the case when the limiting stock price process is a semimartingale of a more general structure, for example, if it is a solution of the diffusion stochastic differential equation, the results are formulated under comparatively restricted conditions. For instance, the approximation by Markov chains is studied and the conditions using analytical terms are formulated in [19]. The binomial and trinomial models are also widely used (see e.g., [19, 24]). For example, the approximation described in Chapter 2 of [19] is based on the binomial model and the sequence of real numbers that one needs to define.

The goal of this paper is to adapt the theorems of the diffusion approximation from [18] and [13] to the multiplicative financial models which are natural for the prelimit market. Moreover, we construct the recurrent schemes for the prelimit market that are even more natural, to our opinion, than the trinomial schemes since they are constructed based on the binomial scheme with the help of the scheme similar to the Euler approximation. For this we proceed with three steps: first, we consider the Euler approximation scheme for the solution of a stochastic differential equation; second, we replace the increments of a Wiener process with binomial random summands; and, third, we take into account the adjusting term that appears when we pass from the multiplicative financial schemes to the additive mathematical ones. To some extent, these ideas were realized in [19], however our approach is more explicit, direct and general.

The paper is organized as follows. In Section 2 we present the general results from [18] concerning the conditions of the weak convergence for the discrete-time processes to the general diffusion process, i.e., we formulate the general functional limit theorem for the diffusion approximation. In Section 3 we adapt these results to the additive and, that is even more important, multiplicative schemes. Then we apply the results to the market for which the limit price process is modeled by the geometric Ornstein-Uhlenbeck process. The recurrent scheme for the diffusion approximation for the case when the limiting process is represented by the geometric Ornstein-Uhlenbeck process is constructed. We discuss the applicability of the geometric Ornstein-Uhlenbeck process in the sense that the corresponding financial model is arbitrage-free and complete. The conditions of convergence for the option prices including the joint convergence of the stock prices and the Radon-Nikodym derivatives are established as well. Note that another type of approximation of the financial market driven by the geometric Ornstein-Uhlenbeck process, and even by the geometric Ornstein-Uhlenbeck-Lévy process, was studied in [21].

2. GENERAL FUNCTIONAL LIMIT THEOREM FOR THE DIFFUSION APPROXIMATION

Recall some notions from the classical semimartingale theory (see, for example, [18]). Let the set $\mathbb{T} = [0, T]$ and $\mathbf{\Omega}_{\mathcal{F}} = (\Omega, \mathcal{F}, (\mathcal{F}_t, t \in \mathbb{T}), \mathsf{P})$ be a complete filtered probability space satisfying the standard assumptions. Denote as $\mathbb{D}(\mathbb{T})$ the set of all real-valued functions on $\mathbb{T}$ that have left-hand limits and are continuous on the right (*cadlag* functions). In what follows we consider only *cadlag* processes (processes with *cadlag* trajectories). A real-valued process $X = \{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ considered on the probability space $\mathbf{\Omega}_{\mathcal{F}}$ is called a semimartingale if it admits the decomposition of the form

$$X_t = X_0 + M_t + A_t,$$

where M is a local martingale with $M_0 = 0$ and A is a process of locally bounded variation. A semimartingale $\{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ is called a special semimartingale if it admits the decomposition mentioned above and the process A is predictable.

Denote as $\langle M \rangle = \{\langle M \rangle_t, t \in \mathbb{T}\}$ the quadratic characteristic of the locally square-integrable martingale $\{M_t, \mathcal{F}_t, t \in \mathbb{T}\}$. It is a predictable increasing process for which the process $M_t^2 - \langle M \rangle_t$ is a local martingale. Also, denote $\Delta X_t = X_t - X_{t-}$.

Theorem 2.1 ([18])**.** *If $X = \{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ is such a semimartingale that for some $a > 0$ and all $t \in \mathbb{T}$ we have $|\Delta X_t| \le a$, then X is a special semimartingale.*

Let $\{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ be a semimartingale. For each $a > 0$ we denote

$$X_t^a = \sum_{0 < s \le t} \Delta X_s \mathbb{1}(|\Delta X_s| > a) = \int_0^t \int_{|x| > a} x d\mu, \quad t \in \mathbb{T},$$

and $Y_t^a = X_t - X_t^a, t \in \mathbb{T}$, where μ is the measure of jumps of the process X. The jumps of Y^a are bounded, $|\Delta Y_t^a| \le a$, so Y^a is a special semimartingale according to the Theorem 2.1. It means that there exists a local martingale M^a and such a predictable process $B^a(X)$ of locally bounded variation that

$$Y_t^a = X_0 + B_t^a(X) + M_t^a, \quad t \in \mathbb{T}, \quad B_0^a = M_0^a = 0.$$

Thus

$$X_t = X_0 + B_t^a(X) + M_t^a + X_t^a, \quad t \in \mathbb{T}.$$

In turn, a local martingale $\{M_t^a, \mathcal{F}_t, t \in \mathbb{T}\}$ admits a decomposition

$$M_t^a = M_t^{ac} + M_t^{ad}$$

into continuous and purely discontinuous parts, where the continuous component M^{ac} does not depend on a, and the purely discontinuous component M^{ad} can be presented as

$$M_t^{ad} = \int_0^t \int_{|x| \le a} x d(\mu - \nu),$$

where ν is the compensator (a dual predictable projection) of the measure μ. Let us denote $M^c = M^{ac}$ and $C_t(X) = \langle M^c \rangle_t$. The processes $(B^a(X), C(X), \nu)$ compose the triplet of predictable characteristics for the semimartingale X. Now we introduce the general pre-limit and limit processes participating in the diffusion approximation. In a connection to the limit process, let $\{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ be a continuous semimartingale.

In this case it is obvious that $\nu \equiv 0$, and $B_t^a(X)$ coincide for any $a > 0$. We denote as $B_t(X)$ the common value of all $B_t^a(X)$. Suppose that

$$B_t(X) = \int_0^t b(s, X)ds \tag{2.1}$$

and

$$C_t(X) = \int_0^t c^2(s, X)ds \tag{2.2}$$

for some predictable measurable functions $b(t, x(\cdot)), c(t, x(\cdot)) : \mathbb{T} \times \mathbb{D}(\mathbb{T}) \to \mathbb{R}$. Moreover, we suppose that $c(t, x(\cdot)) > 0$. In this case we can apply the generalized Lévy theorem (see, e.g., [10]) concluding that there exists a Wiener process $W = (W_t, \mathcal{F}_t,\ t \in \mathbb{T})$, adapted to the filtration $(\mathcal{F}_t, t \in \mathbb{T})$ such that M^c admits the representation $M_t^c = \int_0^t c(s, X)dW_s$. Therefore, X is the solution of the stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X)ds + \int_0^t c(s, X)dW_s. \tag{2.3}$$

Assume that the coefficients of the equation (2.3) satisfy the following condition: there exists a function $L : \mathbb{T} \to \mathbb{R}_+$ such that for any $t \in \mathbb{T}$ and any $X \in \mathbb{D}(\mathbb{T})$

$$|b(t, X)| \leq L(t)(1 + \sup_{s \leq t} |X_s|), \tag{2.4}$$

$$c^2(t, X) \leq L(t)(1 + \sup_{s \leq t} (X_s)^2), \tag{2.5}$$

$$\int_0^T L(t)dt < \infty. \tag{2.6}$$

Suppose that we have the sequence of the probability spaces $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n, t \in \mathbb{T}), \mathsf{P}^n)$, $n \geq 1$ with a filtration, and a sequence of semimartingales $X^n = (X_t^n, \mathcal{F}_t^n, t \in \mathbb{T})$ on the corresponding probability space, with trajectories in $\mathbb{D}(\mathbb{T})$ a.s. and with the triplets of the predictable characteristics $(B^{n,a}, C^n, \nu^n)$. Suppose that for any $\varepsilon > 0$ and $a \in (0, 1]$ the following conditions hold

$$\lim_n \mathsf{P}^n \Big(\sup_{t \in \mathbb{T}} |\Delta X_t^n| \geq \varepsilon \Big) = 0, \tag{2.7}$$

$$\lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| B_t^{n,a} - \int_0^t b(s, X^n)ds \Big| \geq \varepsilon \right) = 0, \tag{2.8}$$

$$\lim_n \mathsf{P}^n \left(\sup_{t\in\mathbb{T}} \left| \langle M^{n,a}\rangle_t - \int_0^t c^2(s,X^n)ds \right| \geq \varepsilon \right) = 0. \tag{2.9}$$

Here we denote as Q and $\mathsf{Q}^n, n \geq 1$ the measures that correspond to the processes X and $X^n, n \geq 1$, respectively.

Theorem 2.2 ([18])**.** *Let the conditions* (2.4)–(2.9) *hold. If, in addition,* $X_0^n \xrightarrow{\mathrm{d}} X_0$ *and functions* b, c *determine uniquely the measure* Q*, then we have the weak convergence of probability measures*

$$\mathsf{Q}^n \xrightarrow{\mathrm{W}} \mathsf{Q}\,.$$

3. A FUNCTIONAL LIMIT THEOREM FOR THE DIFFUSION APPROXIMATION OF THE SUMS AND THE PRODUCTS OF RANDOM VARIABLES

3.1. THE DIFFUSION APPROXIMATION FOR AN ADDITIVE SCHEME

To adapt the well known functional limit theorems towards the financial models, suppose now that we consider the semimartingale $X = \{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ from the previous section but in a simplified situation. More precisely, we suppose that the measurable functions b and c have forms $b = b(t,x)$ and $c = c(t,x) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$. We assume also that the coefficients b and c satisfy the conditions for existence and uniqueness of the weak solution of the stochastic differential equation

$$dX_t = b(t,X_t)ds + c(t,X_t)dW_t, \quad t \in \mathbb{T}, \quad X_0 = x_0. \tag{3.1}$$

Let $c(t,x) \geq 0, t \in \mathbb{T}, x \in \mathbb{R}$, and the limit process X be a solution of this equation.

Remark 3.1. The conditions for existence and uniqueness of the weak solution in the case of the homogeneous coefficients b and c were formulated in [17] and the most general conditions were obtained in [6] and [7]. For the inhomogeneous case, we just refer to the classical book [20] containing results on the existence and uniqueness of the solution of a martingale problem.

Now we simplify the prelimit processes introducing the step-wise functions. Let $n \geq 1$. Consider the sequence of the probability spaces $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n, t\in\mathbb{T}), \mathsf{P}^n)$ with a filtration and the sequence of step-wise semimartingales $X^n = \{X_t^n, \mathcal{F}_t^n, t \in \mathbb{T}\}$ defined on a corresponding probability space and admitting a representation

$$X_t^n = X_{\frac{kT}{n}}^n \quad \text{for} \quad \frac{kT}{n} \leq t < \frac{(k+1)T}{n}. \tag{3.2}$$

So, the trajectories of the process X^n have the jumps at the points kT/n, $k = 0,\dots,n$ and are constant in the interior intervals. Denote $\mathcal{F}_k^n = \sigma(X_t^n, t \leq \frac{kT}{n})$ and $Q_k^{(n)} = \Delta X_{\frac{kT}{n}}^n = X_{\frac{kT}{n}}^n - X_{\frac{(k-1)T}{n}}^n$, $k = 1,\dots,n$. Then the random variables $Q_k^{(n)}$ are $\mathcal{F}_k^n$-measurable, $k = 1,\dots,n$ and in what follows we identify $\mathcal{F}_t^n$ with $\mathcal{F}_k^n$ for

$\frac{kT}{n} \le t < \frac{(k+1)T}{n}$. Let the notation $[x]$ stand for the entire part of a number x. It follows from the definition of the triplet of predictable characteristics that in this case $B_t^{n,a} = \sum_{1\le k\le [\frac{nt}{T}]} \mathsf{E}(Q_k^{(n)} \mathbb{1}_{|Q_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n)$. Since X^n is a jump process, we have that $C^n = 0$. Hence

$$\begin{aligned}\langle M^{n,a}\rangle_t &= \int_0^t \int_{|x|\le a} x^2 d\nu^n - \sum_{0<s\le t} \Big(\int_{|x|\le a} x\nu^n(\{s\}, dx)\Big)^2 \\ &= \sum_{1\le k\le [\frac{nt}{T}]} \left(\mathsf{E}\left(\left(Q_k^{(n)}\right)^2 \mathbb{1}_{|Q_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n\right) - \left(\mathsf{E}\left(Q_k^{(n)} \mathbb{1}_{|Q_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n\right)\right)^2\right) \qquad (3.3)\\ &= \sum_{1\le k\le [\frac{nt}{T}]} \mathrm{Var}\left(Q_k^{(n)} \mathbb{1}_{|Q_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n\right).\end{aligned}$$

Denote by Q and Q^n, $n \ge 1$ the measures that correspond to the processes X and X^n, $n \ge 1$, respectively. Throughout the paper, we put $\sum_{k=1}^0 = 0, \prod_{k=1}^0 = 1$. The next result follows immediately from Theorem 2.2.

Theorem 3.2. *Let the following conditions hold.*

$$X_0^n \xrightarrow{\mathrm{d}} x_0. \qquad (3.4)$$

For any $\varepsilon > 0, a \in (0, 1]$,

$$\lim_n \mathsf{P}^n \Big(\sup_{1\le k\le n} |Q_k^{(n)}| \ge \varepsilon \Big) = 0, \qquad (3.5)$$

$$\lim_n \mathsf{P}^n \Big(\sup_{t\in\mathbb{T}} \Big| \sum_{1\le k\le [\frac{nt}{T}]} \mathsf{E}(Q_k^{(n)} \mathbb{1}_{|Q_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n) - \int_0^t b(s, X_s^n) ds \Big| \ge \varepsilon \Big) = 0 \qquad (3.6)$$

and

$$\lim_n \mathsf{P}^n \Big(\sup_{t\in\mathbb{T}} \Big| \sum_{1\le k\le [\frac{nt}{T}]} \mathrm{Var}(Q_k^{(n)} \mathbb{1}_{|Q_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n) - \int_0^t c^2(s, X_s^n) ds \Big| \ge \varepsilon \Big) = 0. \qquad (3.7)$$

Also, let the functions b, c *uniquely determine the measure* Q. *Then we have the weak convergence of the probability measures*

$$\mathsf{Q}^n \xrightarrow{\mathrm{W}} \mathsf{Q}\,.$$

3.2. A DISCRETE APPROXIMATION SCHEME FOR THE PRODUCT-PROCESSES IN THE FINANCIAL MARKET

Consider the sequence of discrete-time financial markets consisting of two assets, a bond and a stock. We suppose that the bond admits the representation

$$B_t^n = B_0^n \prod_{1\le k\le [\frac{nt}{T}]} (1 + r_k^{(n)}),$$

where $\{r_k^{(n)} > -1, n \geq 1, 1 \leq k \leq n\}$ are real numbers. Let the stock admit the representation

$$S_t^n = S_0^n \prod_{1 \leq k \leq [\frac{nt}{T}]} \big(1 + R_k^{(n)}\big),$$

where $\{R_k^{(n)} > -1, 1 \leq k \leq n\}$ are random variables on the probability space $(\Omega^n, \mathcal{F}^n, \mathsf{P}^n)$, $n \geq 1$. We introduce the σ-fields $\mathcal{F}_0^n = \{\emptyset, \Omega\}$, $\mathcal{F}_k^n = \sigma\{R_i^{(n)}, 1 \leq i \leq k\}$. We are in a position to present the conditions of the weak convergence of this product model to the limit model of the form

$$B_t = B_0 \exp\left\{\int_0^t r(s)ds\right\}, \quad S_t = \exp\left\{X_t - \frac{1}{2}\int_0^t c^2(s, X_s)ds\right\},$$

where the process X is the unique weak solution of the equation (3.1). Let's introduce the processes

$$X_t^n = \sum_{1 \leq k \leq [\frac{nt}{T}]} R_k^{(n)}$$

and

$$Y_t^n = \sum_{1 \leq k \leq [\frac{nt}{T}]} \left(R_k^{(n)} - \frac{1}{2}\left(R_k^{(n)}\right)^2\right).$$

We denote as Q and $\mathsf{Q}^n, n \geq 1$ the measures that correspond to the processes S and $S^n, n \geq 1$, respectively.

Theorem 3.3.
1) *Let the condition* (A) *hold:*

(A) (i) $B_0^n \to B_0$ *and* $\sup_{0 \leq k \leq n} |r_k^{(n)}| \to 0, n \to \infty$;

(ii) $\sum_{1 \leq k \leq [\frac{nt}{T}]} \big(r_k^{(n)} - \frac{1}{2}\big(r_k^{(n)}\big)^2\big) \to \int_0^t r(s)ds, n \to \infty$;

(iii) $\limsup_{n \to \infty} \sum_{1 \leq k \leq n} \left(r_k^{(n)}\right)^2 < \infty$.

Then the point-wise convergence holds: $B_t^n \to B_t, n \to \infty$.
2) *Let the conditions* (B) *and* (C) *hold:*

(B) (i) $S_0^n \to \exp\{x_0\}$ *and* $\sup_{1 \leq k \leq n} |R_k^{(n)}| \xrightarrow{\mathrm{P}} 0, n \to \infty$;

(ii) *for any* $a \in (0, 1]$

$$\lim_{C \to \infty} \limsup_{n \to \infty} \mathsf{P}^n \Big(\sum_{1 \leq k \leq n} \mathsf{E}((R_k^{(n)})^2 \mathbb{1}_{|R_k^{(n)}| \leq a} | \mathcal{F}_{k-1}^n) \geq C\Big) = 0;$$

(iii) *for any* $a \in (0, 1]$

$$\lim_{C \to \infty} \limsup_{n \to \infty} \mathsf{P}^n \Big(\sum_{1 \leq k \leq n} |\,\mathsf{E}(R_k^{(n)} \mathbb{1}_{|R_k^{(n)}| \leq a} | \mathcal{F}_{k-1}^n)| \geq C\Big) = 0;$$

(iv) *for any* $\varepsilon > 0,\ a \in (0,1]$

$$\lim_n \mathsf{P}^n\Big(\sup_{t\in\mathbb{T}}\Big|\sum_{1\le k\le[\frac{nt}{T}]}\mathsf{E}(R_k^{(n)}\mathbb{1}_{|R_k^{(n)}|\le a}|\mathcal{F}_{k-1}^n) - \int_0^t b(s,X_s^n)ds\Big| \ge \varepsilon\Big) = 0;$$

(v) *for any* $\varepsilon > 0,\ a \in (0,1]$

$$\lim_n \mathsf{P}^n\Big(\sup_{t\in\mathbb{T}}\Big|\sum_{1\le k\le[\frac{nt}{T}]}\mathsf{E}(R_k^{(n)})^2\mathbb{1}_{|R_k^{(n)}|\le a}|\mathcal{F}_{k-1}^n) - \int_0^t c^2(s,X_s^n)ds\Big| \ge \varepsilon\Big) = 0;$$

(C) *Functions* b *and* c *uniquely determine the measure* Q.

Then we have weak convergence of the probability measures

$$\mathsf{Q}^n \overset{\mathrm{W}}{\to} \mathsf{Q}\,.$$

Proof. In connection with the convergence of B^n, let $0 < a < 1$ be fixed. Due to condition (A), (i) we can consider such n_0 that for $n \ge n_0$ we have that $\sup_{1\le k\le n}|r_k^{(n)}| < a$. For such n we present $\log(B_t^n)$ as

$$\log(B_t^n) = \log(B_0^n) + \sum_{1\le k\le[\frac{nt}{T}]}\log\Big(1+r_k^{(n)}\Big) = \log(B_0^n) + \sum_{1\le k\le[\frac{nt}{T}]}\Big(r_k^{(n)} - \frac{1}{2}(r_k^{(n)})^2\Big)$$
$$+\alpha\big(a, r_k^{(n)}, 0\le k\le n\big)\sum_{1\le k\le[\frac{nt}{T}]}\Big(r_k^{(n)}\Big)^2,$$

where $|\alpha(a, r_k^{(n)}, 0\le k\le n)|$ does not exceed $\frac{a}{3(1-a)^3}$. Then the convergence of B^n follows immediately from conditions (A), (i) and (ii). Consider the weak convergence of Q^n. Due to condition (B), (i), we can fix any $0 < a < 1$ and it is enough to establish the corresponding convergence for

$$S_t^{n,a} = S_0^n \prod_{1\le k\le[\frac{nt}{T}]}\Big(1+R_k^{(n,a)}\Big),$$

where $R_k^{(n,a)} = R_k^{(n)}\mathbb{1}_{|R_k^{(n)}|\le a}$. We have that

$$\log S_t^{n,a} = \log S_0^n + \sum_{1\le k\le[\frac{nt}{T}]}\log\Big(1+R_k^{(n,a)}\Big)$$
$$= \log S_0^n + \sum_{1\le k\le[\frac{nt}{T}]}\Big(R_k^{(n,a)} - \frac{1}{2}\Big(R_k^{(n,a)}\Big)^2\Big)$$
$$+\alpha\Big(a, R_k^{(n,a)}, 0\le k\le n\Big)\sum_{1\le k\le[\frac{nt}{T}]}\Big(R_k^{(n,a)}\Big)^2,$$

where $|\alpha(a, R_k^{(n,a)}, 0 \le k \le n)|$ does not exceed $\frac{a}{3(1-a)^3}$ a.s. It follows from condition (B), (ii) and the Lenglart inequality ([18, Chapter 1, p. 66]) that for any $a \in (0,1]$

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathsf{P}^n \Big(\sum_{0\le k\le n} \Big(R_k^{(n,a)}\Big)^2 \ge C \Big) = 0. \tag{3.8}$$

We fix an arbitrary $\delta > 0$, apply condition (B), (ii) and find such $C > 0$ and $n(\delta, C)$ that for $n \ge n(\delta, C)$

$$\mathsf{P}^n \Big(\sum_{0\le k\le n} \Big(R_k^{(n,1)}\Big)^2 \ge C \Big) < \delta.$$

Therefore, with probability P^n, exceeding $1-\delta$,

$$\Big|\alpha\Big(a, R_k^{(n,a)}, 0 \le k \le n\Big)\Big| \sum_{1\le k\le [\frac{nt}{T}]} \Big(R_k^{(n,a)}\Big)^2 < \frac{Ca}{3(1-a)^3}.$$

Using the calculations above and conditions (B), (i) we can conclude that the weak convergence of the measures corresponding to the processes $\{Y^n,\ n \ge 1\}$, $\{Y^{n,a} = \sum_{k:0\le \frac{Tk}{n}\le \bullet}(R_k^{(n,a)} - \frac{1}{2}(R_k^{(n,a)})^2)\}$, $\{\log S^n,\ n \ge 1\}$ and $\{\log S^{n,a},\ n \ge 1\}$ holds simultaneously and implies the weak convergence of $\{S^n,\ n \ge 1\}$. At first we consider $\{X^n,\ n \ge 1\}$ and apply Theorem 3.2 with $Q_k^{(n)} = R_k^{(n)}$. It follows from condition (B), (iv) that (3.6) holds. Furthermore, it follows from conditions (B), (i) and (iii) that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathsf{P}^n \Big(\sum_{1\le k\le n} (\mathsf{E}(R_k^{(n)} \mathbb{1}_{|R_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n))^2 \ge \varepsilon \Big) = 0.$$

Therefore, it follows from condition (B), (v) that (3.7) holds, and we get that $X^n \overset{\mathrm{W}}{\to} X$. Now we can apply Theorem 6.26 from [13] and deduce from the weak convergence above and condition (B), (iii) that $\{X^n, [X^n]\} \overset{\mathrm{W}}{\to} \{X, [X]\}$, where $[\cdot]$ means the quadratic variation, X is a weak solution to SDE (2.3). As to $[X]$, it equals $\int_0^t c^2(s, X_s)ds$. Therefore, $\sum_{1\le k\le [\frac{n\cdot}{T}]} (R_k^{(n)})^2 \overset{\mathrm{W}}{\to} \int_0^t c^2(s, X_s)ds$ and we conclude that $Y^n \overset{\mathrm{W}}{\to} X - \frac{1}{2}\int_0^t c^2(s, X_s)ds$ whence the proof follows. □

4. A RECURRENT SCHEME FOR THE DIFFUSION APPROXIMATION WHEN THE LIMIT PROCESS IS A GEOMETRIC ORNSTEIN-UHLENBECK PROCESS

Let $\mathbf{\Omega}_{\mathcal{F}} = (\Omega, \mathcal{F}, (\mathcal{F}_t, t \in \mathbb{T}), \mathsf{P})$ be a complete filtered probability space satisfying the standard assumptions, and consider the adapted Ornstein-Uhlenbeck process with constant parameters on this space

$$dX_t = (\mu - X_t)dt + \sigma dW_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \in \mathbb{T}, \tag{4.1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

We are in a position to construct a discrete scheme that weakly converges to the geometric Ornstein-Uhlenbeck process which is given for technical convenience by the formula $S_t = \exp\{X_t - \frac{\sigma^2}{2}t\}$. In what follows we denote as C constant values of which are not so important, and their values can be different from line to line. Consider the following discrete approximation scheme. Assume we have a sequence of the probability spaces $(\Omega^n, \mathcal{F}^n, \mathsf{P}^n)$, $n \ge 1$ and let $\{q_k^{(n)}, n \ge 1, 0 \le k \le n\}$ be the sequence of iid random variables in the corresponding probability space, each with two possible values $\pm\sqrt{\frac{T}{n}}$, $n \to \infty$ and $\mathsf{P}^n(q_k^{(n)} = \pm\sqrt{\frac{T}{n}}) = \frac{1}{2}$. Let $n > T$. We introduce the recurrent scheme:

$$x_0^{(n)} \in \mathbb{R}, \quad R_k^{(n)} := x_k^{(n)} - x_{k-1}^{(n)} = \frac{(\mu - x_{k-1}^{(n)})T}{n} + \sigma q_k^{(n)}, \quad 1 \le k \le n. \tag{4.2}$$

Let $\mathcal{F}_0^n = \{\emptyset, \Omega\}$ and $\mathcal{F}_k^n = \sigma\{R_i^{(n)}, 1 \le i \le k\}$. Denote $X_t^n = \sum_{1 \le k \le [\frac{nt}{T}]} R_k^{(n)} = x^{(n)}_{[\frac{tn}{T}]} \mathbb{1}_{t \ge \frac{T}{n}}$, let Q^n be the measure corresponding to the process

$$S_t^n = \exp\{x_0^{(n)}\} \prod_{1 \le k \le [\frac{nt}{T}]} \left(1 + R_k^{(n)}\right), \quad t \in \mathbb{T}$$

and Q be the measure that corresponds to the process $S_t = \exp\{X_t - \frac{\sigma^2}{2}t\}$.

Theorem 4.1. *Let $x_0^{(n)} \to x_0$, $n \to \infty$. Then the weak convergence $\mathsf{Q}^n \xrightarrow{\mathrm{W}} \mathsf{Q}$ holds.*

Proof. According to Theorem 3.3, we need to check conditions (B) and (C). However, (C) is evident, so we need to check only (B). At first, we mention that the random variables $x_k^{(n)}$ can be presented as

$$x_k^{(n)} = x_0^{(n)} \left(1 - \frac{T}{n}\right)^k + \mu \left(1 - \left(1 - \frac{T}{n}\right)^k\right) + \sigma \sum_{i=1}^{k} q_i^{(n)} \left(1 - \frac{T}{n}\right)^{k-i}, \tag{4.3}$$

whence there exists a constant $C > 0$ such that $\sup_{0 \le k \le n} |x_k^{(n)}| \le C\sqrt{n}$ a.s. Therefore, $\sup_{0 \le k \le n} |R_k^{(n)}| \le \frac{C}{\sqrt{n}}$ a.s. and it means that the condition (B), (i) holds. Furthermore, in order to establish (B), (ii), we consider any fixed $a \in (0, 1]$ and such n_0 that $\frac{C}{\sqrt{n_0}} \le a$. Then for any $n \ge n_0$ we have that

$$\sum_{0 \le k \le n} \mathsf{E}((R_k^{(n)})^2 \mathbb{1}_{|R_k^{(n)}| \le a} / \mathcal{F}_{k-1}^n) \le C$$

for some $C > 0$ whence condition (B), (ii) follows. To establish (B), (iii), we should note that in our case for any $\varepsilon > 0$, $a \in (0,1]$ and $n \geq n_0$

$$\begin{aligned}
&\lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \sum_{1 \leq k \leq [\frac{nt}{T}]} \mathsf{E}(R_k^{(n)} \mathbb{1}_{|R_k^{(n)}| \leq a} | \mathcal{F}_{k-1}^n) - \int_0^t b(s, X_s^n) ds \Big| \geq \varepsilon \right) \\
&= \lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \sum_{1 \leq k \leq [\frac{nt}{T}]} \mathsf{E}(R_k^{(n)} | \mathcal{F}_{k-1}^n) - \int_0^t (\mu - X_s^n) ds \Big| \geq \varepsilon \right) \\
&= \lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \sum_{1 \leq k \leq [\frac{nt}{T}]} \frac{(\mu - x_{k-1}^{(n)})T}{n} - \sum_{1 \leq k \leq [\frac{nt}{T}]-1} \frac{(\mu - x_k^{(n)})T}{n} \right. \\
&\qquad\qquad \left. - (\mu - x_{[\frac{nt}{T}]}^n) \left(t - \frac{[\frac{nt}{T}]T}{n} \right) \Big| \geq \varepsilon \right) \\
&= \lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \frac{(\mu - x_0^{(n)})T}{n} - (\mu - x_{[\frac{nt}{T}]}^n) \left(t - \frac{[\frac{nt}{T}]T}{n} \right) \Big| \geq \varepsilon \right) = 0.
\end{aligned}$$

Now let us check the condition (B), (iv). At first we shall prove that

$$\lim_{C \to \infty} \limsup_n \mathsf{P}^n \left(\max_{1 \leq k \leq n} |x_k^{(n)}| \geq C \right) = 0.$$

Due to representation (4.3), it is enough to prove that

$$A := \lim_{C \to \infty} \limsup_n \mathsf{P}^n \left(\max_{1 \leq k \leq n} \Big| \sum_{i=1}^k q_i^{(n)} \left(1 - \frac{T}{n} \right)^{k-i} \Big| \geq C \right) = 0.$$

But the last assertion follows immediately from the Kolmogorov's inequality for the sums of iid random variables: $A \leq C^{-2}T = 0$. Now, we have that for any $\varepsilon > 0$ and $a \in (0,1]$

$$\begin{aligned}
&\lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \sum_{1 \leq k \leq [\frac{nt}{T}]} \mathsf{E} \left(\left(R_k^{(n)} \right)^2 \mathbb{1}_{|R_k^{(n)}| \leq a} | \mathcal{F}_{k-1}^n \right) - \int_0^t c^2(s, X_s^n) ds \Big| \geq \varepsilon \right) \\
&= \lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \sum_{1 \leq k \leq [\frac{nt}{T}]} \mathsf{E} \left(\left(R_k^{(n)} \right)^2 | \mathcal{F}_{k-1}^n \right) - \sigma^2 t \Big| \geq \varepsilon \right) \\
&= \lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \Big| \sum_{1 \leq k \leq [\frac{nt}{T}]} \left(\frac{(\mu - x_k^{(n)})T}{n} \right)^2 + \sigma^2 \left[\frac{nt}{T} \right] \frac{T}{n} - \sigma^2 t \Big| \geq \varepsilon \right) \\
&\leq \lim_{C \to \infty} \limsup_n \mathsf{P}^n \left(\max_{1 \leq k \leq n} |x_k^{(n)}| \geq C \right) \\
&\quad + \lim_n \mathsf{P}^n \left(\sup_{t \in \mathbb{T}} \left(\sigma^2 t - \sigma^2 \left[\frac{nt}{T} \right] \frac{T}{n} + \frac{(|\mu| + C)^2 T^2}{n} \right) \geq \varepsilon \right) = 0,
\end{aligned}$$

and (B), (iv) holds. The condition (B), (v): for any $\varepsilon > 0$ and $a \in (0,1]$

$$\lim_n \mathsf{P}^n \Big(\sum_{1\le k\le n} (\mathsf{E}(R_k^{(n)} \mathbb{1}_{|R_k^{(n)}|\le a} | \mathcal{F}_{k-1}^n))^2 \ge \varepsilon \Big)$$
$$= \lim_n \mathsf{P}^n \left(\sum_{1\le k\le n} \left(\frac{(\mu - x_k^{(n)})T}{n} \right)^2 \ge \varepsilon \right)$$
$$\le \lim_{C\to\infty} \limsup_n \mathsf{P}^n(\max_{1\le k\le n} |x_k^{(n)}| \ge C) + \lim_n \mathsf{P}^n \left(\frac{(|\mu| + C)^2 T^2}{n} \ge \varepsilon \right) = 0.$$

The theorem is proved. □

5. THE PRELIMIT AND LIMIT ORNSTEIN-UHLENBECK MARKETS ARE ARBITRAGE-FREE AND COMPLETE

In this section we consider the prelimit discrete discounted Ornstein-Uhlenbeck market

$$Y_t^n = \frac{S_t^n}{B_t^n} = \exp\{x_0^{(n)}\} \prod_{1\le k\le [\frac{nt}{T}]} \frac{1 + R_k^{(n)}}{1 + r_k^{(n)}},$$

where $R_k^{(n)}$ are defined via (4.2), $\{r_k^{(n)} | 1 \le k \le n, n \ge 1\}$ is the non-random interest rate,

$$B_t^n = \prod_{1\le k\le [\frac{nt}{T}]} \big(1 + r_k^{(n)}\big),$$

$r_k^{(n)}$ satisfy the assumptions $\sup_{1\le k\le n} r_k^{(n)} \to 0$ and $B_t^n \to e^{rt}$, $n \to \infty$.

Theorem 5.1. *Let $r_k^{(n)} = o(\frac{1}{\sqrt{n}})$ and the sequence $x_0^{(n)}$ be bounded. Then there exists n_0 such that for any $n > n_0$ the prelimit market (B_t^n, Y_t^n) is arbitrage free and complete.*

Proof. We look for such probability measures $\mathsf{P}^{n,*}$ that $\mathsf{P}^{n,*} \sim \mathsf{P}^n$ and for which

$$\mathsf{E}_{\mathsf{P}^{n,*}}(Y_t^n | \mathcal{F}_s^n) = Y_s^n, \tag{5.1}$$

where

$$\mathcal{F}_s^n = \sigma\Big\{ R_i^{(n)}, 1 \le i \le \Big[\frac{ns}{T}\Big] \Big\} = \sigma\Big\{ q_i^{(n)}, 1 \le i \le \Big[\frac{ns}{T}\Big] \Big\}.$$

The relation (5.1) is equivalent to

$$\mathsf{E}_{\mathsf{P}^{n,*}} \big(Y_{\frac{kT}{n}}^{(n)} | \mathcal{F}_{k-1}^{(n)} \big) = Y_{\frac{(k-1)T}{n}}^{(n)}, \quad 1 \le k \le n, \tag{5.2}$$

where $\mathcal{F}_{k-1}^{(n)} = \sigma\{q_i^{(n)}, 1 \le i \le k-1\}$. Let

$$\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} = \prod_{k=1}^{n}(1+\Delta M_k^n),$$

where $\{M_k^n, 1 \le k \le n\}$ is some $\mathcal{F}_k^n$- martingale, $\Delta M_k^{(n)} > -1$. Evidently, the martingale M^n admits the representation

$$M_k^n = \sum_{i=1}^{k} \rho_{i-1}^{(n)} q_i^{(n)},$$

where $\rho_{i-1}^{(n)}$ is $\mathcal{F}_{i-1}^{(n)}$ - adapted. Therefore,

$$\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} = \prod_{k=1}^{n}(1+\rho_{k-1}^{(n)} q_k^{(n)}). \tag{5.3}$$

The equality (5.2) is equivalent to

$$\frac{\mathsf{E}_{\mathsf{P}^n}\left(\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} Y^n_{\frac{kT}{n}} | \mathcal{F}_{k-1}^n\right)}{\mathsf{E}_{\mathsf{P}^n}\left(\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} | \mathcal{F}_{k-1}^n\right)} = Y^n_{\frac{(k-1)T}{n}}, \quad 1 \le k \le n.$$

We have the equivalent relations:

$$\begin{aligned}
\mathsf{E}_{\mathsf{P}^n}\left(\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n}\frac{1+R_k^{(n)}}{1+r_k^{(n)}} | \mathcal{F}_{k-1}^n\right) &= \mathsf{E}_{\mathsf{P}^n}\left(\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} | \mathcal{F}_{k-1}^n\right),\\
\mathsf{E}_{\mathsf{P}^n}\left(\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} R_k^{(n)} | \mathcal{F}_{k-1}^n\right) &= r_k^{(n)}\, \mathsf{E}_{\mathsf{Q}^n}\left(\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} | \mathcal{F}_{k-1}^n\right),\\
\mathsf{E}_{\mathsf{P}^n}\left(\left(1+\rho_{k-1}^{(n)} q_k^{(n)}\right)\left(\left(\mu - x_{k-1}^{(n)}\right) T n^{-1} + \sigma q_k^{(n)}\right) | \mathcal{F}_{k-1}^n\right) &= r_k^{(n)},\\
\left(\mu - x_{k-1}^{(n)}\right) T n^{-1} + \sigma\, \mathsf{E}\left(\left(1+\rho_{k-1}^{(n)} q_k^{(n)}\right) q_k^{(n)} | \mathcal{F}_{k-1}^n\right) &= r_k^{(n)}.
\end{aligned} \tag{5.4}$$

Denote $y_k^{(n)} = r_k^{(n)} - \left(\mu - x_{k-1}^{(n)}\right) T n^{-1}$. Then we immediately get from (5.4) that

$$\rho_{k-1}^{(n)} = \frac{n y_k^{(n)}}{\sigma T} \quad \text{and} \quad \rho_{k-1}^{(n)} q_k^{(n)} = \frac{y_k^{(n)}}{\sigma q_k^{(n)}}.$$

To provide the arbitrage-free property and completeness, we only have to check the inequality $\frac{y_k^{(n)}}{\sigma q_k^{(n)}} > 1$. The latter inequality will follow from the relation

$$|y_k^{(n)}| < \sigma \sqrt{\frac{T}{n}}.$$

Note that it follows from (4.3) that

$$x_k^{(n)} - \mu = (x_0^{(n)} - \mu)\left(1 - \frac{T}{n}\right)^k + \sigma \sum_{i=0}^{k} q_i^{(n)} \left(1 - \frac{T}{n}\right)^{k-i}, \tag{5.5}$$

whence

$$\begin{aligned} r_k^{(n)} + \left(\frac{T}{n}(x_0^{(n)} - \mu) + \sigma\sqrt{\frac{T}{n}}\right)\left(1 - \frac{T}{n}\right)^k - \sigma\sqrt{\frac{T}{n}} \le r_k^{(n)} + \frac{(x_k^{(n)} - \mu)T}{n} \\ \le r_k^{(n)} + \left(\frac{T}{n}(x_0^{(n)} - \mu) - \sigma\sqrt{\frac{T}{n}}\right)\left(1 - \frac{T}{n}\right)^k + \sigma\sqrt{\frac{T}{n}}. \end{aligned} \tag{5.6}$$

Evidently, if $r_k^{(n)} = o(\frac{1}{\sqrt{n}})$ and $x_0^{(n)}$ is bounded, then for sufficiently large n

$$r_k^{(n)} + \left(\frac{T}{n}(x_0^{(n)} - \mu) + \sigma\sqrt{\frac{T}{n}}\right)\left(1 - \frac{T}{n}\right)^k > 0,$$

and

$$r_k^{(n)} + \left(\frac{T}{n}(x_0^{(n)} - \mu) - \sigma\sqrt{\frac{T}{n}}\right)\left(1 - \frac{T}{n}\right)^k < 0.$$

For such n the prelimit market $(B_t^{(n)}, Y_t^n)$ is arbitrage-free and complete. □

As to the limit process, the discounted price process has form

$$\begin{aligned} Y_t &= \exp\left\{X_t - \frac{\sigma^2}{2}t - rt\right\} = \exp\left\{x_0 e^{-t} + \mu\left(1 - e^{-t}\right) + \sigma\int_0^t e^{s-t} dW_s - \frac{\sigma^2}{2}t - rt\right\} \\ &= \exp\left\{\int_0^t (\mu - X_s)\, ds + \sigma W_t + x_0 - \frac{\sigma^2}{2}t - rt\right\}. \end{aligned}$$

We look for a measure $\mathsf{P}^* \sim \mathsf{P}$ that w.r.t. P^* Y is a martingale. Evidently, the Radon-Nikodym derivative equals to $\frac{d\mathsf{P}^*}{d\mathsf{P}}\Big|_T$, where

$$\frac{d\mathsf{P}^*}{d\mathsf{P}}\Big|_t = \exp\left\{-\int_0^t \frac{\mu - r - X_s}{\sigma} dW_s - \frac{1}{2}\int_0^t \left(\frac{(\mu - r - X_s)^2}{\sigma^2} ds\right)\right\}.$$

But this is correct only if the last relation corresponds to the martingale. According to [16], the process $\varphi_t(\beta) = \exp\left\{\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 ds\right\}$ is a martingale on the interval $[0, T]$ if β is a Gaussian process with

$$\sup_{t \le T} E|\beta_t| < \infty$$

and

$$\sup_{t \le T} E\beta_t^2 < \infty.$$

In our case $\beta_t = -\frac{\mu - r - X_t}{\sigma}$ and, up to the constant terms, $E\beta_t^2 \sim EX_t^2 \sim e^t$, whence $\frac{d\,\mathsf{P}^*}{dP}$ defines a new martingale measure on any interval $[0, T]$. Therefore, the limit market is arbitrage-free and complete.

6. CONVERGENCE OF THE BOND PRICES IN THE GEOMETRIC ORNSTEIN-UHLENBECK MODEL

Suppose that we have an option on a stochastic risk-free rate that is governed by the geometric Ornstein-Uhlenbeck (Vasicek) model. The Ornstein-Uhlenbeck process was used by Vasicek in [22] in deriving an equilibrium model of discount bond prices. This Gaussian process has been used extensively by the other scientists in valuing bond options, futures, futures options, and other types of contingent claims. The examples include [8, 9, 14, 15]. The motivation of considering the geometric Ornstein-Uhlenbeck process is in its price recovery effect that is supplied by its mean reverting property. If we want to establish the convergence of option prices, we need the joint convergence of the bond prices and the Radon-Nikodym derivatives. For this we apply the multidimensional functional limit theorem from [13].

Theorem 6.1. *Let $r_k^{(n)} = \frac{rT}{n}, n \ge 1, 1 \le k \le n$ and let $x_0^{(n)} \to x_0$ as $n \to \infty$. Then the joint weak convergence holds:*

$$\left\{ \frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n}\Big|_{\cdot}, Y^n \right\} \overset{\mathrm{W}}{\to} \left\{ \frac{d\,\mathsf{P}^*}{d\,\mathsf{P}}\Big|_{\cdot}, Y \right\},$$

where the prelimit process of the Radon-Nikodym derivative is defined as

$$\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n}\Big|_t = \prod_{1 \le k \le \left[\frac{nt}{T}\right]} (1 + \rho_{k-1}^{(n)} q_k^n).$$

Proof. We can use the same representations and reasonings that have been used in the general case when we proved Theorem 4.1. Namely, we present the Radon-Nikodym derivative as

$$\begin{aligned} \frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n}\Big|_t &= \exp\left\{ \sum_{1 \le k \le \left[\frac{nt}{T}\right]} \log(1 + \rho_{k-1}^{(n)} q_k^n) \right\} \\ &= \exp\left\{ \sum_{1 \le k \le \left[\frac{nt}{T}\right]} \left(\rho_{k-1}^{(n)} q_k^{(n)} - \frac{1}{2} (\rho_{k-1}^{(n)})^2 \frac{T}{n} \right) \right\} + \mathcal{O}_P(n, t) \\ &= \exp\left\{ \sum_{1 \le k \le \left[\frac{nt}{T}\right]} \frac{r - \mu + x_{k-1}^{(n)}}{\sigma} q_k^{(n)} - \frac{1}{2} \Big(\frac{r - \mu + x_{k-1}^{(n)}}{\sigma} \Big)^2 \frac{T}{n} \right\} + \mathcal{O}_P(n, t), \end{aligned} \tag{6.1}$$

where we do not present the exact form of the remainder term $\mathcal{O}_P(n,t)$, but it can be bounded similarly to $\alpha(a, R_k^{(n,a)}, 0 \le k \le n) \sum_{1\le k\le [\frac{nt}{T}]} (R_k^{(n,a)})^2$ from the proof of Theorem 3.2 and $\sup_{0\le t\le T} \mathcal{O}_P(n,t) \to 0$ in probability as $n\to\infty$.

Furthermore,

$$Y_t^{(n)} = \exp\left\{ x_0^{(n)} + \sum_{1\le k\le \left[\frac{nt}{T}\right]} \frac{(\mu - x_{k-1}^{(n)})T}{n} + \sigma \sum_{1\le k\le \left[\frac{nt}{T}\right]} q_k^{(n)} - \frac{\sigma^2 t}{2} - rt \right\} + \mathcal{O}_P(n,t).$$

So, we have to establish the joint weak convergence of the couple of processes

$$\left\{ \sum_{1\le k\le \left[\frac{nt}{T}\right]} \frac{(\mu - x_{k-1}^{(n)})T}{n}, \sum_{1\le k\le \left[\frac{nt}{T}\right]} \Big(\frac{r-\mu+x_{k-1}^{(n)}}{\sigma}\Big)^2 \frac{T}{n} \right\}$$

to the couple of the processes

$$\left\{ \int_0^\cdot \frac{r-\mu+X_s}{\sigma} dW_s, W_\cdot, \int_0^\cdot \frac{(r-\mu+X_s)^2}{\sigma^2} ds, \int_0^\cdot (\mu - X_s) ds \right\}.$$

We can apply Theorem 5.16 (p. 569) from [13] for the multidimensional weak convergence of semi-martingales because the first pair of the components of the prelimit process are martingales and the second two components can be interpreted as the processes of bounded variation. We omit the tedious details of the condition verification of this theorem since in our simple case they hold evidently. According to this theorem, the joint weak convergence holds and the proof follows. □

Remark 6.2. We can assume only that $\max_{1\le k\le n} |nr_k^{(n)} - r\frac{T}{n}| \to 0$ as $n\to\infty$.

Corollary 6.3. *It follows from Theorem 6.1 that the option prices converge for call and put options as well as for any other options whose price is the functional that is continuous in a Skorokhod topology, for example, Asian options.*

Remark 6.4. Returning to the applicability of the geometric Ornstein-Uhlenbeck (Vasicek) model in the finance, it is no doubt that the interest rate is often supposed to follow this model. The paper [23] is one of the recent examples of modeling a stochastic interest rate by the Vasicek model. However, there are some reasons in favor of the Vasicek model even if we consider stock prices. One of the reasons is that in the standard Black-Scholes model the variance of the total profit $\int_0^T \frac{dS_t}{S_t}$ equals $\sigma^2 T \to \infty$ as $T\to\infty$ while in real markets it often tends to a finite value which is true for the Vasicek model.

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