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Optimal ordering quantities for substitutable deteriorating items under joint replenishment*

by

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Abstract: In this paper, we study an inventory system for two substitutable deteriorating items where when an item is out of stock, the demand for it is met by the other item and any part of demand not met due to unavailability of the other item is lost. The level of inventory of both items deplete due to combined effect of demand fulfilment and deterioration. The rates of demand and deterioration are assumed to be deterministic and constant. Items are ordered jointly in each ordering cycle so as to take advantage of joint replenishment. The problem is formulated and a solution procedure is developed to determine the optimal ordering quantities that minimize the total inventory cost. An extensive numerical analysis is carried out to illustrate the parameters of the model. The results indicate that there is substantial improvement in the optimal total cost of the inventory system with substitution over the case without substitution.

Keywords: inventory control, substitutable items, deterioration, optimal ordering quantities, joint replenishment

1. Introduction

In supermarkets or in retail, the occurrence of temporary stock-outs is a very common phenomenon in the categories of frequently purchased items and it is also very common to see, at any retail outlet or supermarket, customers who, willing to purchase certain items, will be willing to purchase the substitute items, if they face the situation of the stock-outs. A survey report of Anupindi (1998) observed this phenomenon, and it was found that 82-88% of buyers would be willing to buy the substitute items if the desired items are out of stock. This phenomenon of demand substitution can happen under a variety of conditions.

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In stochastic conditions, an item is substituted by another item to avoid or minimize the effects of shortages, occurring due to uncertainty in the system, whereas in deterministic cases, a portion of demand is substituted by another item in a planned manner. In both of these cases, it is noted that an item could either be completely substituted or partially substituted by another item. Accordingly, Kim and Bell (2011) categorize the substitution as symmetrical substitution and asymmetrical substitution. Besides cost related reasons, there could, some times, be some marketing motives that involve item substitutions. The substitutable items, in which sufficient deterioration can take place during the normal storage period of the items and consequently losses may occur in significant amount, need to be taken into account when formulating and analyzing the inventory system of substitutable items for determining the accurate optimal ordering quantities. As can be easily understood, the phenomenon of demand substitution would involve multiple items (at least two). Once more than one item is considered, there arises the issue of replenishment policy, meaning: should the items be procured independently, jointly or in a coordinated manner.

This paper formulates a model of the two deteriorating items inventory system with partial substitution, and for substitution we consider asymmetrical stock out based substitution, together with joint replenishment for the development of the inventory model. The system parameters are assumed to be constant.

The inventory modelling of substitutable deteriorating items has not found sufficient place in the literature of inventory modelling of substitutable items. To the best of our knowledge no one has considered the concept of deterioration for the inventory modelling of substitutable items with joint replenishment for asymmetrical stock out based substitution. There is a huge amount of literature available in inventory modelling of substitutable items and deteriorating items separately. Thus, in the subsequent paragraph first we discuss the recent and previous advancements in the inventory modelling of deteriorating items, and then in inventory modelling of substitutable items.

The first inventory model was developed by Harris (1915) in the second decade of the twentieth century and this model was generalized by Wilson (1934) by deriving the formula to obtain the economic order quantity (EOQ). The inventory model for the deteriorating items was first studied by Whitin (1957), who considered the fashion goods as deteriorating items. Further, there have been several researchers, who developed different inventory models of deteriorating items under different realistic situations. The reader may wish to consult the review papers on inventory of deteriorating items, written by Raafat (1991), Goyal and Giri (2003), Li et al. (2010), and Bakker et al. (2012), and Khanlarzade et al.(2014) for a detailed review of the literature on the inventory of deteriorating items. Recently, Taleizadeh (2014a, b, c), Tat et al. (2015), and Taleizadeh (2016) developed the deteriorating items inventory models incorporating some more realistic and market driven situations.

The literature of substitutable items can be categorised as referring to joint replenishment policies with and without substitution. Under joint replenishment policies (JRP) with substitution, Drezner et al. (1995) developed an EOQ model with substitution for two substitutable items and compared the results with no substitution and proved that full substitution is never optimal. Gurnani and Drezner (2000) extended the model of Drezner et al. (1995) for multiple items. The major assumption of their models is one-to-one substitution among items, where unmet demand of one item is fully converted to that of the other. In view of the fact that in realistic situations of substitution, most of the time the demand for an item would only be partially converted to the demand of the other, Salameh et al. (2014) extended the model of Drezner et al. (1995) by considering the phenomenon of partial substitution. Recently, Rasoulia and Nakhai-Kamalabadi (2014) and Krommyda et al. (2015) developed inventory models similar to that of Salameh et al. (2014), considering, however, more realistic cases of demand dependent on price as well as on stock.

This paper makes the model of Krommyda et al. (2015), Salameh et al. (2014), as well as Rasoulia and NakhaiKamalabadi (2014) still more realistic and applicable, by taking into account the effect of deterioration on substitutable items of the inventory. Demand is considered as a constant function for both of the mutually substitutable items. If one of the items is out of stock, then its demand will be fulfilled by the second item, and if any demand is not met by the substitutable item, it will be completely lost. Both items are ordered jointly and replenishment cycle is the same for both items.

Regarding the replenishment policy under multiple item case, plenty of work has been reported under the category of JRP (joint replenishment policy), but without substitution. One may refer to Khouja and Goyal (2008) for a detailed review. They have categorized the work on JRP by including the studies done between 1989 and 2005. In one of the recent developments in the context of JRP formulation, Porras and Dekker (2008) developed a model, in which they introduced a correction factor in cost function for empty replenishment and showed that there is some improvement in the optimal quantity over their own earlier work (Porras and Dekker, 2003), when no such consideration was incorporated. In most of the previous studies, related to JRP, the optimal ordering quantities have been obtained by some heuristic search algorithms. Hong and Kim (2009) developed a closed form formula to obtain the optimal order quantities using unbiased estimator and genetic algorithm. Further, it has been shown by Schulz and Telha (2011) that the complexity of obtaining the optimal quantities increases exponentially (no polynomial time) with respect to the time for deterministic demand.

While the inventory models of substitution with stochastic demand have been studied by many researchers, the major contributions that ought to be mentioned are as follows: Parlar and Goyal (1984), Pasternack and Drezner (1991), Ernst and Kouvelis (1999), Gerchack and Grosfeld (1999), Mishra and Raghunathan (2004). Zhao et al. (2014) have studied the model for two items, while Ye (2014), Huang and Ke (2014), Li et al. (2013), Li and You (2012), Hsieh (2011), Xue and Song (2007) have developed the inventory policy for multiple substitutable items.

The rest of the paper is organized as follows: In the next section we describe the assumptions and notations used in the entire article, Section 3 gives the problem description and mathematical formulation of the model, while Section 4 describes the details of the solution procedure with the proof of pseudo convexity of the total cost function. Then, Section 5 provides a numerical example, accompanied by the sensitivity analysis of the model and the article ends with summary and conclusions from the study.

2. Assumptions and notation

For the mathematical formulation of the proposed inventory model, the following assumptions and notation are used.

Assumptions

1. The two items considered are ordered jointly in every ordering cycle.
2. The demand rates and the deterioration rates are known and constant for both items.
3. The procurement lead time is zero and replenishment rates for both items are infinite.
4. When an item is completely depleted and it subsequently becomes out of stock and there is on-hand inventory of the second item available, then the second item, while being supplied to fulfil its own demand, substitutes for the demand of the first item during stock out period. This substitution need not be the full substitution. It can be limited to a fraction (known as the substitution rate) of the total demand for the first item during the stock out period. The remaining un-substituted demand for the first item is lost.
5. Both items are mutually substitutable, that is, each one can substitute the other in the case of a lack of stock. However, the rates of substitution may differ.

Notation

Notation is grouped into parameters of the system, intermediate variables, derived functions and objective functions, and is accordingly presented in Table 1.

3. Problem description and mathematical formulation

As stated before, we consider an inventory system with two mutually substitutable deteriorating items under the assumptions mentioned in Section 2. The inventory diagrams for the possible situations (item 1 substituted by item 2, item 2 substituted by item 1 and no substitution) are shown in Figs. 1, 2, and 3, respectively. In this inventory system, at the beginning of the replenishment cycle the retailer orders Q_1 and Q_2 units of item 1 and item 2, respectively, whose respective consumption rates are D_1 and D_2. The inventory levels of both items gradually deplete due to deterioration and consumption. There are three possible cases (two with substitution and one without substitution), namely:

Table 1. Notation

Parameters	
D_1, D_2	Demand rates.
θ	Deterioration rate of item 1 and 2.
α_1, α_2	Substitution rate of item 1 by item 2 and vice versa.
Q_1, Q_2	Ordering quantities of item 1 and 2.
A_1, A_2	Fixed ordering cost per order of item 1 and item 2.
h_1, h_2	Holding cost per unit of item 1 and item 2.
C_1, C_2	Item cost per unit of item 1 and item 2.
π_1, π_2	Shortage cost per unit of item 1 and item 2.
Intermediate variables	
P	Portion of time, during which substitution occurs.
t_1	Time, when the level of inventory of the substituted item is zero.
t_2	Time, when the level of inventory of substitute item is completely depleted in the case of no substitution.
Z	Inventory level of an item when the other item is out of stock.
Derived functions	
$I_1^1(t)$	Inventory level of item 1 when Q_1 depletes before Q_2 at time t, $0 \leq t \leq t_1$.
$I_2^1(t)$	Inventory level of item 2 when Q_1 depletes before Q_2 at time t, $0 \leq t \leq t_1$.
$I_3^1(t)$	Inventory level of item 2 when Q_1 depletes before Q_2 and substitution takes place at time t, $0 \leq t \leq t_1 + p$.
$I_1^2(t)$	Inventory level of item 1 when Q_2 depletes before Q_1 at time t, $0 \leq t \leq t_2$.
$I_2^2(t)$	Inventory level of item 2 when Q_2 depletes before Q_1 at time t, $0 \leq t \leq t_2$.
$I_3^2(t)$	Inventory level of item 1 when Q_2 depletes before Q_1 and substitution takes place at time t, $0 \leq t \leq t_2 + p$.
Objective functions	
$TC(Q_1,Q_2)$	Total cost per cycle with substitution for the case when item 1 is substituted by item 2.
$TC1(Q_1,Q_2)$	Total average annual cost for the case when item 1 is substituted by item 2.
$TC2(Q_1,Q_2)$	Total average annual cost for the case when item 2 is substituted by item 1.
$TC_{WS}(Q_1,Q_2)$	Total average annual cost without substitution.

Case 1: Q_1 depletes before Q_2, i.e. if at time t_1 the inventory of item 1 is out of stock, as depicted in Fig. 1, then item 2 partially substitutes item 1 with substitution rate α_1. A portion of unmet demand for item 1 is assumed to be lost with the rate (1-α_1).

Case 2: Q_2 depletes before Q_1, i.e. if at time t_2 the inventory of item 2 is out of stock, as depicted in Fig. 2, then item 1 partially substitutes item 2 with substitution rate α_2. A portion of unmet demand for item 2 is assumed to be lost with the rate (1-α_2).

Case 3: Q_1 and Q_2 deplete simultaneously (as depicted in Fig. 3), i.e. items 1 and 2 can never go out of stock individually, and thus there is no substitution of any of the items.

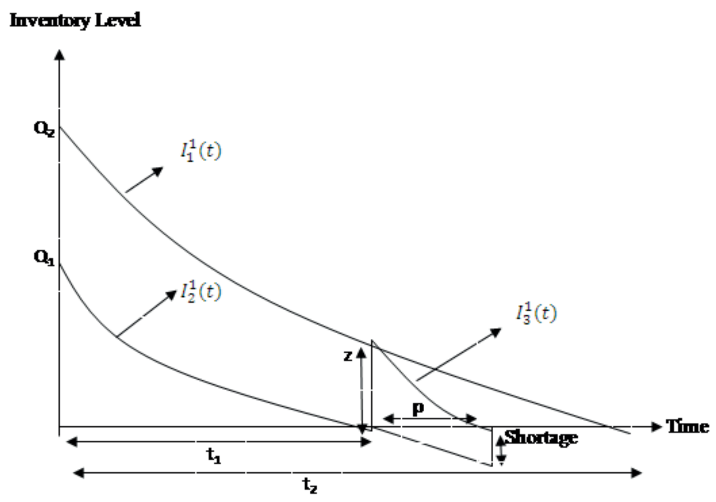


Figure 1. First scenario of the inventory model, when $t_1 < t_2$

The derivations of the total annual costs for the three cases are discussed below.

3.1. Case 1 (Fig. 1): Q_1 depletes before Q_2 (with substitution)

The average total cost is derived in the usual manner i.e. by summing the various cost components per cycle and then by multiplying it by the average number of cycles per year. To determine the various cost components, we determine the inventory level during the cycle time of inventory.

The inventory levels of item 1 and item 2 for this situation is governed by the following differential equations

$$\frac{dI_1^1(t)}{dt} + \theta I_1^1(t) = -D_1 \;; \quad 0 \le t \le t_1$$
with boundary condition $I_1^1(0) = Q_1$ and $I_1^1(t) = 0$. (1)

$$\frac{dI_2^1(t)}{dt} + \theta I_2^1(t) = -D_2 \;; 0 \le t \le t_1$$
with boundary condition $I_2^1(0) = Q_2$ and $I_2^1(t_1) = I_3^1(t_1)$. (2)

$$\frac{dI_3^1(t)}{dt} + \theta I_3^1(t) = -(D_2 + \alpha_1 D_1) \;; t_1 \le t \le t_1 + p$$
with boundary condition $I_3^1(t_1) = I_2^1(t_1)$ and $I_3^1(t_1 + p) = 0$. (3)

The solutions of equations (1), (2) and (3) are:

$$I_1^1(t) = \frac{D_1}{\theta}\left(e^{\theta(t_1 - t)} - 1\right); \; 0 \le t \le t_1 \tag{4}$$

$$I_2^1(t) = \frac{(Q_2\theta + D_2)}{\theta e^{\theta t}} - \frac{D_2}{\theta}; \; 0 \le t \le t_1 \tag{5}$$

$$I_3^1(t) = \frac{D_2 + \alpha_1 D_1}{\theta}\left(e^{\theta(t_1 + p - t)} - 1\right); \; t_1 \le t \le t_1 + p. \tag{6}$$

The cost components per cycle consist of (a) costs related to item 1, (b) costs related to item 2, and (c) the shortage cost.

(a) The total cost associated with item 1 per ordering cycle consists of the fixed ordering cost, and holding cost, and can be expressed as

Total cost associated with item 1 =

$$A_1 + \frac{h_1}{\theta^2}\left(\theta Q_1 - D_1 \ln\left(\frac{\theta Q_1 + D_1}{D_1}\right)\right) \tag{7}$$

(b) The total cost associated with item 2 per ordering cycle consists of the fixed ordering cost and holding cost and can be expressed in terms of a given Q_1 as

Total cost associated with item 2 =

$$\left(\begin{array}{l} A_2 + \frac{h_2}{\theta^2}\left(\theta Q_2 - D_2 \ln\left(\frac{\theta Q_1 + D_1}{D_1}\right)\right) \\ -\frac{h_2(D_1\alpha_1 + D_2)}{\theta^2} \ln\left(\frac{D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)}{(D_1\alpha_1 + D_2)(\theta Q_1 + D_1)}\right) \end{array}\right) \tag{8}$$

(c) The shortage cost/cost of lost sales is incurred due to the demand for item 1, which is not satisfied at a cost of π_1 per unit lost, which can be expressed as

Shortage cost =

$$\frac{\pi_1 D_1}{\theta}\left((1 - \alpha_1) \ln\left(\frac{D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)}{(D_1\alpha_1 + D_2)(\theta Q_1 + D_1)}\right)\right) \tag{9}$$

Thus, the total cost per ordering cycle TC (Q_1, Q_2), from Eqs. (7), (8) and (9) is given as

$$TC(Q_1,Q_2)=\left[\begin{array}{l} A_1+A_2+\frac{h_1}{\theta^2}\left(\theta Q_1-D_1\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)\right)\\ +\frac{h_2}{\theta^2}\left(\theta Q_2-D_2\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)\right)\\ -\frac{h_2(D_1\alpha_1+D_2)}{\theta^2}\ln\left(\frac{D_1(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)}{(D_1\alpha_1+D_2)(\theta Q_1+D_1)}\right)\\ +\frac{\pi_1 D_1}{\theta}\left((1-\alpha_1)\ln\left(\frac{D_1(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)}{(D_1\alpha_1+D_2)(\theta Q_1+D_1)}\right)\right)\end{array}\right] \tag{10}$$

Finally, for case 1 (when $t_1<t_2$), TC1(Q_1,Q_2), the average total cost per unit time (say, a year) is obtained by multiplying the total cost per ordering cycle by the average number of cycles per year, that is

$$\theta\Big/\ln\left(\frac{\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2}{D_1\alpha_1+D_2}\right)$$

and is given as

$$TC1(Q_1,Q_2)=\frac{\theta}{\ln\left(\frac{\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2}{D_1\alpha_1+D_2}\right)}\left(\begin{array}{l} A_1+A_2+\frac{h_1}{\theta^2}\left(\theta Q_1-D_1\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)\right)\\ +\frac{h_2}{\theta^2}\left(\theta Q_2-D_2\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)\right)\\ -\frac{h_2(D_1\alpha_1+D_2)}{\theta^2}\ln\left(\frac{D_1(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)}{(D_1\alpha_1+D_2)(\theta Q_1+D_1)}\right)\\ +\frac{\pi_1 D_1}{\theta}\left((1-\alpha_1)\ln\left(\frac{D_1(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)}{(D_1\alpha_1+D_2)(\theta Q_1+D_1)}\right)\right)\end{array}\right) \tag{11}$$

3.2. Case 2 (Fig. 2): Q_2 depletes before Q_1 (with substitution)

Following the approach analogous to that for case 1, now for case 2 (when $t_1>t_2$), TC2(Q_1, Q_2), we obtain that the average total cost per unit time (say a year) is equal

$$TC2(Q_1,Q_2)=\frac{\theta}{\ln\left(\frac{\alpha_2\theta Q_2+D_2\alpha_2+\theta Q_1+D_1}{D_2\alpha_2+D_1}\right)}\left(\begin{array}{l} A_1+A_2+\frac{h_2}{\theta^2}\left(\theta Q_2-D_2\ln\left(\frac{\theta Q_2+D_2}{D_2}\right)\right)\\ +\frac{h_1}{\theta^2}\left(\theta Q_1-D_1\ln\left(\frac{\theta Q_2+D_2}{D_2}\right)\right)\\ -\frac{h_1(D_2\alpha_2+D_1)}{\theta^2}\ln\left(\frac{D_2(\alpha_2\theta Q_2+D_2\alpha_2+\theta Q_1+D_1)}{(D_2\alpha_2+D_1)(\theta Q_2+D_2)}\right)\\ +\frac{\pi_2 D_2}{\theta}\left((1-\alpha_2)\ln\left(\frac{D_2(\alpha_2\theta Q_2+D_2\alpha_2+\theta Q_1+D_1)}{(D_2\alpha_2+D_1)(\theta Q_2+D_2)}\right)\right)\end{array}\right) \tag{12}$$

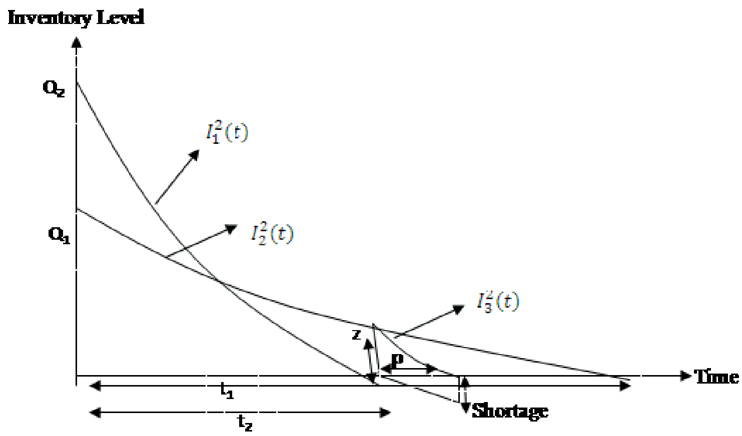


Figure 2. Second scenario of the inventory model, when $t_1 > t_2$

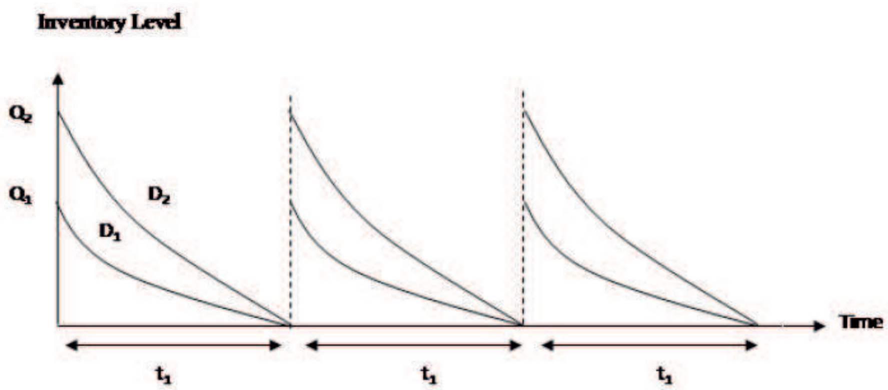


Figure 3. Scenario without substitution under joint replenishment

3.3. Case 3 (Fig. 3): Q_1 and Q_2 deplete simultaneously (no substitution)

Figure 3 illustrates the inventory levels for the here considered case of no substitution. Under a joint replenishment policy, the inventories of both items deplete to zero simultaneously, i.e. $Q_1/D_1 = Q_2/D_2$.

The average total cost per unit time for an inventory system without substitution under joint replenishment, TCWS(Q_1,Q_2), consists only of fixed ordering cost and holding cost, and is given as

$$TC_{WS}(Q_1,Q_2) = \frac{\theta}{\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)}\left(\begin{array}{l} A_1+A_2+\frac{h_1}{\theta^2}\left(\theta Q_1-D_1\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)\right) \\ +\frac{h_2}{\theta^2}\left(\theta Q_2-D_2\ln\left(\frac{\theta Q_1+D_1}{D_1}\right)\right)\end{array}\right). \tag{13}$$

4. Solution procedure

To derive the solution for total cost function in order to obtain the optimal ordering quantities, we use the result on pseudo convexity as provided in Bazaraa et al. (2013), which states as follows "If f: $R^n \to R$ is pseudo convex at x then x is the global minimum if and only if $\nabla f(x) = 0$" and "If f: $R^n \to R$ is twice differentiable at $x$and $\nabla f(x) = 0$, and the Hessian Matrix $H(x)$ is positive definite then x is strict local minimum", where $\nabla f(x)$ is the function gradient vector.

Using these properties we show next that the total cost functions $TC1(Q_1,Q_2)$ and $TC2(Q_1,Q_2)$ are pseudo convex functions with reasonable conditioning and thus they both attain a unique optimal solution.

Theorem 1 *The total cost TC1(Q_1,Q_2) is pseudo convex if* $h_2\left(D_1\alpha_1+D_2\right) > \pi_1 D_1\theta\left(1-\alpha_1\right)$.

Proof See Appendix 1A.

Theorem 2 *The total cost TC2(Q_1,Q_2) is pseudo convex if* $h_1\left(D_2\alpha_2+D_1\right) > \pi_2 D_2\theta\left(1-\alpha_2\right)$.

Proof See Appendix 1B.

Since the total cost functions $TC1(Q_1,Q_2)$ and $TC2(Q_1,Q_2)$ are pseudo convex functions, then with the help of the following algorithm we can obtain the value of unique optimal ordering quantities (Q_1^*, Q_2^*).

Algorithm for obtaining optimal ordering quantities

STEP1: Initialize all constant parameter of the model.

STEP2: Solve the constraint optimization problems

$$\mathrm{P}_1 : \text{Find}(Q_1,Q_2) \text{ that } \min_{Q_1,Q_2} TC1(Q_1,Q_2) \text{ subject to } \frac{Q_1}{D_1} \le \frac{Q_2}{D_2}$$

and

$$P_2 : \text{Find } (Q_1,Q_2) \text{ that } \min_{Q_1,Q_2} TC2(Q_1,Q_2) \text{ subject to } \frac{Q_1}{D_1} \ge \frac{Q_2}{D_2}$$

STEP3: Obtain the optimal ordering quantities(Q_1^*, Q_2^*)as Min(P_1, P_2).

STEP4: Exit from the algorithm.

5. Numerical example and sensitivity analysis

In this section we provide a numerical example, intended to illustrate the proposed model. (Maple mathematical modelling software was used to carry out the illustrative calculations.) In numerical example we use the values of parameter as defined in Table 1, unless otherwise mentioned.

Table 2. Initial parameters used for numerical analysis

Parameter	Item 1	Item 2
Consumption rate(D_1,D_2)	200	100
Deterioration rate (θ)	0.01	0.01
Substitution rate(α_1,α_2)	0.2	0.8
Setup cost (A_1,A_2)	300	300
Holding cost (h_1,h_2)	2	2
Shortage cost(π_1, π_2)	1	4

According the algorithm as defined above; we solve the constraint optimization problem of step 2 by using Maple mathematical software. The solution of the first optimization problem (P1) is $Q_1 = 100.00$, $Q_2 = 260.94$, $TC1(Q_1,Q_2) = 721.89$, and solution of the second optimization problem (P2) is $Q_1 = 283.50$, $Q_2 = 141.75$, $TC2(Q_1,Q_2) = 850.52$. Comparing the obtained total cost of (P1, P2) in step 3 of the algorithm, we can see that P1 attained the optimal solution and corresponding ordering quantities are the optimal ordering quantities. Thus, the optimal ordering quantities in this numerical example are $(Q_1^*, Q_2^*) = (100.00, 260.94)$ with the optimal total cost equal 721.89. The optimal ordering quantities and optimal total cost (using equation (13)) without substitution, under the same environment, is $Q1_{ws} = 283.50$, $Q2_{ws} = 260.94$, $TC_{ws}(Q_1,Q_2) = 850.52$ and the difference in total optimal cost compared to the case with substitution is 128.63 and improvement is 15.12 in percentage points.

In Figs. 5, 6 and 7 we plot the values of total cost function TC1(Q_1, Q_2) with different values of ordering quantities. The behaviour of the total cost function TC1(Q_1, Q_2) is as convex function. These figures verify the result as we derived in theorem 1. Thus total cost function $TC1(Q_1,Q_2)$ always leads to the unique optimal solution.

The sensitivity analysis, whose results are shown in Table 3, regarding the parameters of the model, indicates that the proposed model is quite stable. The optimal total cost of the model is increases as the value of setup cost, substitution rate, deterioration rate increases and optimal total cost of the model is decreases as the value of holding cost, shortage cost increases.

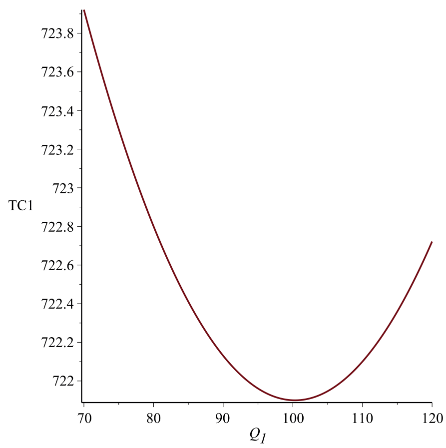


Figure 4. Total cost (TC1) vs. Q_1 at fixed Q_2

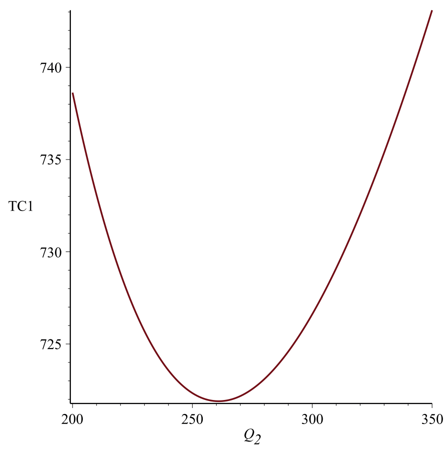


Figure 5. Total cost (TC1) vs. Q_2 at fixed Q_1

Table 3. Sensitivity analysis for optimal total cost and optimal ordering quantities

Parameter	Parameter value	Optimal total cost and optimal ordering quantities with substitution			Optimal total cost and optimal ordering quantities without substitution			Improvement (%) in optimal total cost
		TC1	Q_1^*	Q_2^*	TC_{WS}	Q_{1ws}^*	Q_{2ws}^*	
A_1	300.00	721.89	100.00	260.94	850.52	283.50	141.75	15.12
	400.00	814.93	100.00	307.46	982.45	327.48	163.74	17.05
	500.00	896.43	100.00	348.21	1098.77	366.25	183.12	18.42
	600.00	969.85	100.00	384.92	1203.99	401.33	200.66	19.45
	700.00	1037.24	100.00	418.62	1300.80	433.60	216.80	20.26
h_1	2.00	721.89	100.00	260.94	850.52	283.50	141.75	15.12
	4.00	967.92	49.99	191.98	1201.99	200.33	100.16	19.47
	6.00	1154.73	33.33	159.12	1471.69	163.52	81.76	21.54
	8.00	1311.57	25.00	138.94	1699.05	141.58	70.79	22.81
	10.00	1449.45	20.00	124.94	1899.36	126.62	63.31	23.69
α_1	0.20	721.89	100.00	260.94	850.52	283.50	141.75	15.12
	0.30	743.24	100.00	271.62	850.52	283.50	141.75	12.61
	0.40	762.55	100.00	281.27	850.52	283.50	141.75	10.34
	0.50	780.17	100.00	290.08	850.52	283.50	141.75	8.27
	0.60	796.37	100.00	298.18	850.52	283.50	141.75	6.37
π_1	1.00	721.89	100.00	260.94	850.52	283.50	141.75	15.12
	2.00	818.15	200.00	209.07	850.52	283.50	141.75	3.81
	3.00	850.52	283.50	141.75	850.52	283.50	141.75	0.00
	4.00	850.52	283.50	141.75	850.52	283.50	141.75	0.00
	5.00	850.52	283.50	141.75	850.52	283.50	141.75	0.00
θ	0.01	721.89	100.00	260.94	850.52	283.50	141.75	15.12
	0.05	729.42	100.00	264.71	858.47	286.15	143.07	15.03
	0.10	738.71	100.00	269.35	868.30	289.43	144.71	14.92
	0.15	747.87	100.00	273.93	878.02	292.67	146.33	14.82
	0.20	756.92	100.00	278.46	887.64	295.88	147.94	14.73

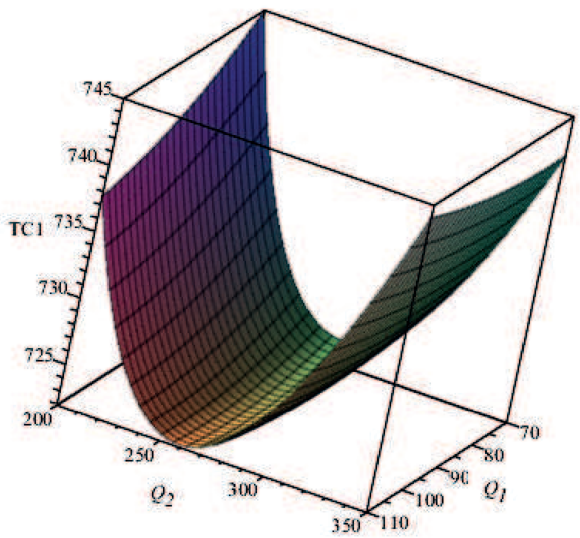


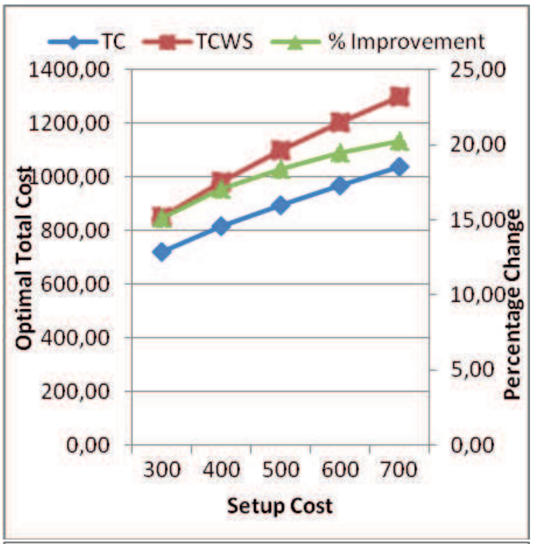
Figure 6. Total cost (TC1) vs. ordering quantities of item 1 and item 2

Now we investigate the improvement in total cost with substitution and without substitution with respect to the various parameters of the system. The results are shown in Figs. 7, 8 and 9.

6. Summary and conclusions

In this paper, we have presented an inventory decision policy for two substitutable deteriorating items under joint replenishment in each replenishment cycle. The inventory decision policy has been mathematically formulated and a solution procedure has been developed for obtaining the optimal ordering quantities. The numerical example has been carried out to illustrate the properties of the model and the optimal total cost, resulting from the model, has also been compared numerically with the model having no substitution. Sensitivity analysis of the model indicated that model is quite stable with respect to the respective parameters. The model should be very useful for the managers of warehouses for the inventory decisions concerning deteriorating substitutable items, as it enables to optimize the outcome of their businesses.

In addition, as this paper only considered two items with joint replenishment in the same replenishment cycle, the future investigations can focus on more than two items, different replenishment cycles for different product having multiple suppliers, trade credit mechanism, supplier-retailer cooperation, and some other realistic market driven situations.



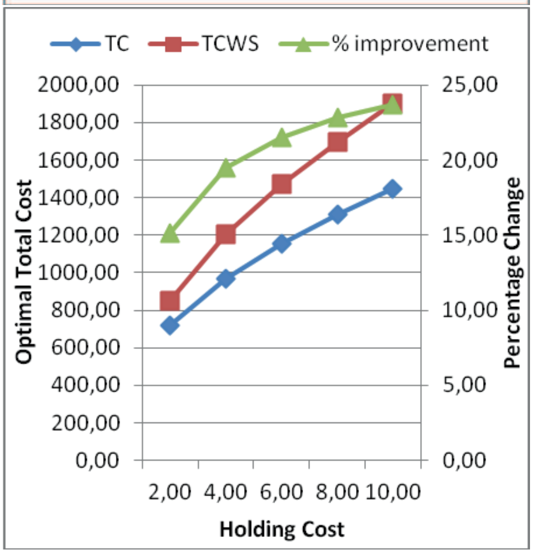
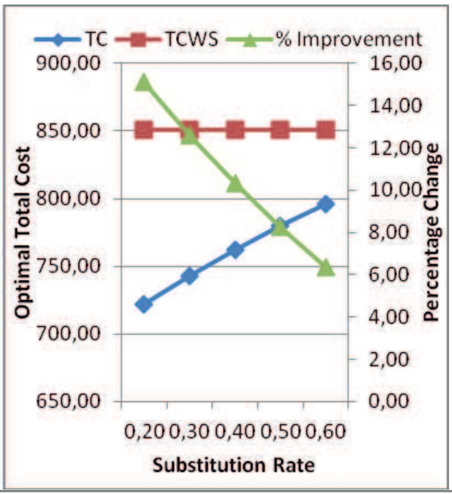


Figure 7. Improvement (%) in TC over TC_{WS} with variation in setup cost (A_1) and holding cost (h_1)

References

Anupindi, R., Dada M., Gupta, M. S. (1998) Estimation of consumer demand with stock out based substitution: an application to vending machine products. *Marketing Science*, 17, 406–423.



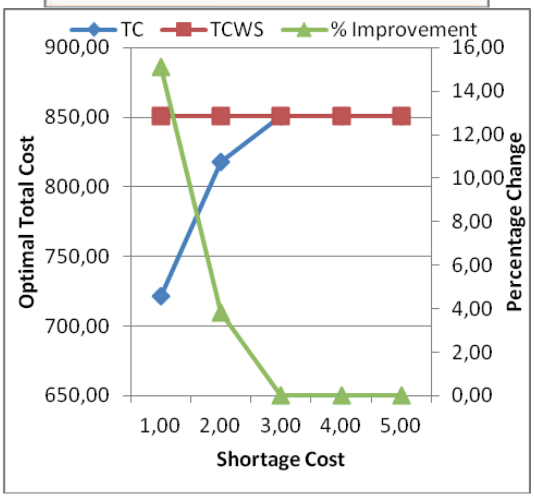


Figure 8. Improvement (%) in TC1 over TC_{WS} with variation in substitution rate ($\pi 1$) and shortage cost ($\alpha 1$)

Bakker, M., Riezebos, J. and Teunter, R.H. (2012) Review of Inventory Systems with Deterioration since 2001. *European Journal of Operational Research*, 221, 275-284.

Bazaraa, M.S., Sherali, H.D., Shetty, C. M. (2013) *Nonlinear Programming: Theory and Algorithms*, 3rd edition. John Wiley and Sons.

Chandra, S. (1972) Strong pseudo convex programming. *Indian Journal of Pure and Applied Mathematics*, 3, 278-282.

Drezner, Z., Gurnani, H., Pasternack, B.A. (1995) EOQ model with substitution between products. *Journal of Operations Research Society*, 46, 887–891.

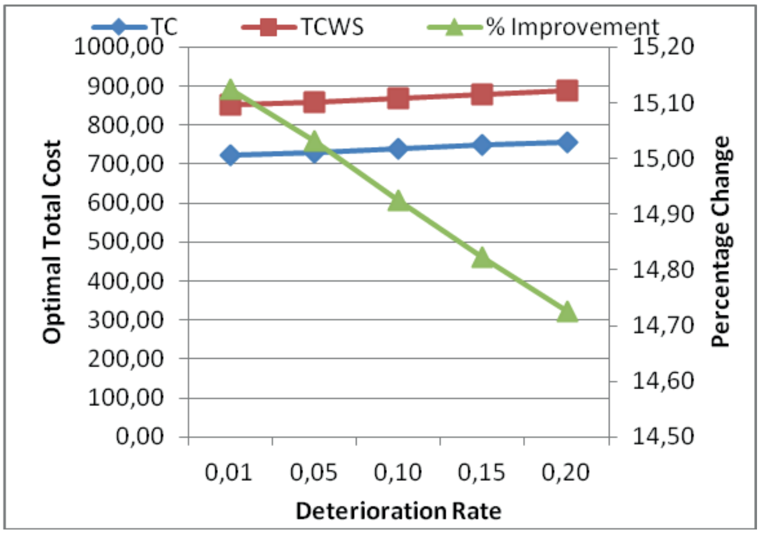


Figure 9. Improvement (%) in TC1 over TC_{WS} with variation in deterioration rate (θ)

Ernst, R., Kouvelis, P. (1999) The effects of selling packaged goods on inventory decisions. *Management Science,* **45**(8), 1142–1155.

Gerchak, Y., Grosfeld-Nir, A. (1999) Lot-sizing for substitutable, production-to-order parts with random functionality yields. *The International Journal of Flexible Manufacturing Systems*, 11, 371–377.

Goyal, S. K. and Giri, B. C. (2003) Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research,* **134**(1), 1–16.

Gurnani, H., Drezner, Z. (2000) Deterministic hierarchical substitution inventory models. *Journal of Operations. Research Society,* 51, 129–133.

Harris, F.W. (1915) *Operations and Cost.* A. W. Shaw Company, Chicago, 48–54.

Hong, S., Kim, Y. (2009) A genetic algorithm for joint replenishment based on the exact inventory cost. *Computers and Operations Research*, **36** (1), 167–175.

Hsieh, Y.-J. (2013) Demand switching criteria for multiple products: An inventory cost analysis. *Omega*, 39, 130–137.

Huang Hu and Ke Hua (2014) Pricing decision problem for substitutable products based on uncertainty theory. *Journal of Intelligent Manufacturing,* 1-12. DOI 10.1007/s10845-014-0991-7.

Khanlarzade, N., Yegane, B. Y., Kamalabadi, I. N. and Farughi, H. (2014) Inventory control with deteriorating items: A state-of-the-art literature review. *International Journal of Industrial Engineering Com-*

putations, 5, 179-198.
Khouja, M. and Goyal, S. (2008) A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research*, **186**(1), 1-16.
Kim, S.-W., Bell, P.C. (2011) Optimal pricing and production decisions in the presence of symmetrical and asymmetrical substitution. *Omega*, **39**(5), 528-538.
Krommyda, I.P., Skouri, K., Konstantaras, I. (2015) Optimal ordering quantities for substitutable products with stock-dependent demand. *Applied Mathematical Modelling*, **39**(1), 147-164.
Li, R., Hongjie, L., Mawhinney, J. R. (2010) A Review on Deteriorating Inventory Study. *Journal of Service Science & Management*, 3, 117-129.
Li, H., You, T. (2012) Capacity commitment and pricing for substitutable products under competition. *Journal of System Science and System Engineering,* **21**(4), 443-460.
Li, X., Nukala, S., Mohebbi, S. (2013) Game theory methodology for optimizing retailers' pricing and shelf-space allocation decisions on competing substitutable products. *International Journal of Advanced Manufacturing Technology,* 68, 375–389.
Mishra, B.K., Raghunathan, S. (2004) Retailer-vs.vendor-managed inventory and brand competition. *Management Science*, **50**(4), 445–457.
Parlar, M., Goyal, S. (1984) Optimal ordering decisions for two substitutable products with stochastic demands. *Opsearch*, 21, 1–15.
Pasternack B., Drezner, Z. (1991) Optimal inventory policies for substitutable commodities with stochastic demand. *Naval Research Logistic*, 38, 221–240.
Porras, E., Dekker, R. (2008) Generalized solutions for the joint replenishment problem with correction factor. *International Journal of Production Economics,* **113** (2), 834–851.
Raafat, F. (1991) Survey of literature on continuously deteriorating inventory model. *Journal of the Operational Research Society*, **42**(1), 27–37.
Rasouli, N., NakhaiKamalabadi, I. (2014) Joint pricing and inventory control for seasonal and substitutable goods mentioning the symmetrical and asymmetrical substitution. *International Journal of Engineering (IJE), TRANSACTIONS C: Aspects*, **27**, 9, 1385-1394.
Salameh, M.K., Yassine, A.A., Maddah, B., Ghaddar, L. (2014) Joint replenishment model with substitution. *Applied Mathematical Modelling,* **38** (14), 3662-3671.
Schulz, A., Telha, C. (2011) Approximation algorithms and hardness results for the joint replenishment problem with constant demands. *Lecture Notes in Computer Science*, **6942**, 628–639.
Taleizadeh, A.A. (2014a) An inventory control problem for deteriorating items with backordering and financial engineering considerations. *Applied Mathematical Modeling,* 38, 93–109.
Taleizadeh, A.A. (2014b) An economic order quantity model with partial

backordering and advance payments for an evaporating item. *International Journal of Production* Economics, 155, 185–193,

Taleizadeh, A.A. (2014c) An economic order quantity model for deteriorating item in a purchasing system with multiple prepayments. *Applied Mathematical Modeling*, 38, 5357–5366.

Taleizadeh, A.A. (2015a) Joint optimization of price, replenishment frequency, replenishment cycle and production rate in vendor managed inventory system with deteriorating items. *International Journal of Production Economics*, 159, 285-295.

Taleizadeh, A.A. (2015b) Joint replenishment policy with backordering and Special Sale. *International Journal of Systems Science,* **46**(7), 1172-1198.

Taleizadeh, A.A. (2016) Pricing and ordering decisions of two competing supply chains with different composite policies: a Stackelberg game-theoretic approach. *International Journal of Production Research,* **54**(9), 2807-2836.

Tat R., Taleizadeh, A.A. and Esmaeili, M. (2015) Developing EOQ model with non-instantaneous deteriorating items in vendor-managed inventory (VMI) system. *International Journal of Systems Science,* **46**(7), 1257-1268.

Whitin, T. M. (1957) *The Theory of Inventory Management*, 2nd ed. Princeton University Press, Princeton, NJ.

Wilson, R.H. (1934) A scientific routine for stock control. *Harvard Business Review*, 13, 116-128.

Xue, Z., Song, J. (2007) Demand management and inventory control for substitutable products. Working paper, The Fuqua School of Business, Duke University, Durham, NC.

Ye, T. (2014) Inventory management with simultaneously horizontal and vertical substitution. *International Journal of Production Economics*, 156, 16-324.

Zhao, J., Wei, J., Li, Y. (2014) Pricing decisions for substitutable products in a two-echelon supply chain with firms different channel powers. *International Journal of Production Economics*, 153, 243–252.

Appendix A. Proof of pseudo convexity of total cost function

Proof of Theorem 1.

The total cost per unit time in per ordering cycle for Case 1 is (from equation (11) :

$$TC1(Q_1, Q_2) = \frac{\theta}{\ln\left(\frac{\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2}{D_1\alpha_1 + D_2}\right)} \times \left(\begin{array}{l} A_1 + A_2 + \frac{h_1}{\theta^2}\left(\theta Q_1 - D_1 \ln\left(\frac{\theta Q_1 + D_1}{D_1}\right)\right) \\ +\frac{h_2}{\theta^2}\left(\theta Q_2 - D_2 \ln\left(\frac{\theta Q_1 + D_1}{D_1}\right)\right) \\ -\frac{h_2(D_1\alpha_1 + D_2)}{\theta^2}\ln\left(\frac{D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)}{(D_1\alpha_1 + D_2)(\theta Q_1 + D_1)}\right) \\ +\frac{\pi_1 D_1}{\theta}\left((1-\alpha_1)\ln\left(\frac{D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)}{(D_1\alpha_1 + D_2)(\theta Q_1 + D_1)}\right)\right) \end{array}\right).$$

This equation can be rewritten as follows:

$$TC1(Q_1, Q_2) = \frac{TC1^*(Q_1, Q_2)}{\frac{\ln\left(\frac{\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2}{D_1\alpha_1 + D_2}\right)}{\theta}}, \text{ where}$$

$$TC1^*(Q_1, Q_2) = \left(\begin{array}{l} A_1 + A_2 + \frac{h_1}{\theta^2}\left(\theta Q_1 - D_1 \ln\left(\frac{\theta Q_1 + D_1}{D_1}\right)\right) \\ +\frac{h_2}{\theta^2}\left(\theta Q_2 - D_2 \ln\left(\frac{\theta Q_1 + D_1}{D_1}\right)\right) \\ -\frac{h_2(D_1\alpha_1 + D_2)}{\theta^2}\ln\left(\frac{D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)}{(D_1\alpha_1 + D_2)(\theta Q_1 + D_1)}\right) \\ +\frac{\pi_1 D_1}{\theta}\left((1-\alpha_1)\ln\left(\frac{D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)}{(D_1\alpha_1 + D_2)(\theta Q_1 + D_1)}\right)\right) \end{array}\right).$$

Clearly, in the above equation

$$\ln\left(\frac{\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2}{D_1\alpha_1 + D_2}\right)\Big/\theta$$

is a positive concave function. Since the ratio of positive convex function over positive concave function is a strong pseudo convex function (Chandra, 1972), so, to prove the pseudo convexity of $TC1(Q_1,Q_2)$, here we prove that $TC1^*(Q_1,Q_2)$ is a positive convex function, and for the convexity of $TC1^*(Q_1,Q_2)$ we prove that all the principal minors of the Hessian matrix of $TC1^*(Q_1,Q_2)$ are non-negative. The H-matrix of function $TC1^*(Q_1,Q_2)$ is defined as

$$H = \begin{bmatrix} \frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_1^2} & \frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_2 \partial Q_1} & \frac{\partial^2 TC1^*(Q_1,Q_2)}{\partial Q_2^2} \end{bmatrix}$$

$$\frac{\partial TC1^{*}(Q_1,Q_2)}{\partial Q_1}=\frac{1}{(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)(\theta Q_1+D_1)}$$
$$\times\begin{pmatrix}\pi_1\left(D_1\alpha_1\theta Q_2+D_1D_2\alpha_2-D_1\theta Q_2-D_1D_2\right)\\ +h_1\left(\alpha_1\theta Q_1^2+D_1\alpha_1Q_1+\theta Q_1Q_2+D_2Q_1\right)\\ +h_2\left(D_1\alpha_1Q_2-D_2\alpha_1Q_1\right)\end{pmatrix}$$

$$\frac{\partial TC1^{*}(Q_1,Q_2)}{\partial Q_2}=\frac{1}{(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)}\begin{pmatrix}\pi_1\left(D_1-D_1\alpha_1\right)\\ +h_2\left(\alpha_1Q_1+Q_2\right)\end{pmatrix}$$

$$\frac{\partial^2 TC1^{*}(Q_1,Q_2)}{\partial Q_1^2}=$$
$$\frac{\begin{pmatrix}\pi_1D_1\theta(1-\alpha_1)(2\alpha_1\theta Q_1+2D_1\alpha_1+\theta Q_2+D_2)(\theta Q_1+D_1)\\ +h_1D_1(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)^2\\ -h_2\alpha_1\begin{pmatrix}\alpha_1D_1^2(2\theta Q_2+D_2)+D_1(2Q_2Q_1\theta^2\alpha_1\\ +(\theta Q_2+D_2)^2)-D_2Q_1^2\theta^2\alpha_1\end{pmatrix}\end{pmatrix}}{(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)^2(\theta Q_1+D_1)^2}>0$$

$$\Rightarrow \text{IF}\quad\begin{pmatrix}\pi_1D_1\theta(1-\alpha_1)(2\alpha_1\theta Q_1+2D_1\alpha_1+\theta Q_2+D_2)(\theta Q_1+D_1)\\ +h_1D_1(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)^2\\ -h_2\alpha_1\begin{pmatrix}\alpha_1D_1^2(2\theta Q_2+D_2)+D_1(2Q_2Q_1\theta^2\alpha_1\\ +(\theta Q_2+D_2)^2)-D_2Q_1^2\theta^2\alpha_1\end{pmatrix}\end{pmatrix}>0\ (A)$$

$$\Rightarrow h_2\left(D_1\alpha_1+D_2\right)>\pi_1D_1\theta\left(1-\alpha_1\right)$$

$$\frac{\partial^2 TC1^{*}(Q_1,Q_2)}{\partial Q_2^2}=\frac{1}{(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)^2}\begin{pmatrix}\pi_1D_1\theta\left(\alpha_1-1\right)\\ +h_2\left(D_1\alpha_1+D_2\right)\end{pmatrix}>0$$
$$\Rightarrow \pi_1D_1\theta\left(\alpha_1-1\right)+h_2\left(D_1\alpha_1+D_2\right)>0$$
$$\Rightarrow h_2\left(D_1\alpha_1+D_2\right)>\pi_1D_1\theta\left(1-\alpha_1\right)$$

$$\frac{\partial^2 TC1^{*}(Q_1,Q_2)}{\partial Q_1Q_2}=\frac{\alpha_1}{(\alpha_1\theta Q_1+D_1\alpha_1+\theta Q_2+D_2)^2}\begin{pmatrix}\pi_1D_1\theta\left(\alpha_1-1\right)\\ +h_2\left(D_1\alpha_1+D_2\right)\end{pmatrix}$$

The determinant of the H-matrix of $TC1^*(Q_1,Q_2)$ is

$$H = \frac{\left(\begin{array}{l} \pi_1 D_1\theta(1-\alpha_1)(2\alpha_1\theta D_1 + 2D_1\alpha_1 \\ +\theta D_2 + D_2)(\theta D_1 + D_1) \\ +h_1 D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)^2 \\ -h_2\alpha_1\left(\begin{array}{l} \alpha_1 D_1^2(2\theta Q_2 + D_2) + D_1(2Q_2Q_1\theta^2\alpha_1 \\ +(\theta Q_2 + D_2)^2) - D_2 Q_1^2\theta^2 y_1 \end{array}\right) \end{array}\right)\left(\begin{array}{l} \pi_1 D_1\theta\,(\alpha_1 - 1) \\ +h_2\,(D_1\alpha_1 + D_2) \end{array}\right)}{(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)^4(\theta Q_1 + D_1)^2}$$

$$-\frac{\alpha_1^2\left(\begin{array}{l} \pi_1 D_1\theta\,(\alpha_1 - 1) \\ +h_2\,(D_1\alpha_1 + D_2) \end{array}\right)^2}{(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)^4} > 0$$

$$\Rightarrow \left(\left(\begin{array}{l} \pi_1 D_1\theta(1-\alpha_1)(2\alpha_1\theta Q_1 + 2D_1\alpha_1 + \theta Q_2 + D_2)(\theta Q_1 + D_1) \\ +h_1 D_1(\alpha_1\theta Q_1 + D_1\alpha_1 + \theta Q_2 + D_2)^2 \\ -h_2\alpha_1\left(\begin{array}{l} \alpha_1 D_1^2(2\theta Q_2 + D_2) + D_1(2Q_2Q_1\theta^2\alpha_1 \\ +(\theta Q_2 + D_2)^2) - D_2 Q_1^2\theta^2\alpha_1 \end{array}\right) \end{array}\right)\right.$$

$$\left. -\alpha_1^2(\theta Q_1 + D_1)^2\left(\begin{array}{l} h_2\,(D_1\alpha_1 + D_2) \\ -\pi_1 D_1\theta\,(1-\alpha_1) \end{array}\right)\right) > 0$$

{from (A)}.

Thus,

$$h_2\,(D_1\alpha_1 + D_2) > \pi_1 D_1\theta\,(1-\alpha_1)$$

This completes the proof. □

Appendix B.

Proof of Theorem 2:

This proof is similar to the proof of Theorem 1.