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Heating control of a finite rod with a mobile source

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The Green's function approach is applied for studying the exact and approximate null-controllability of a finite rod in finite time by means of a source moving along the rod with controllable trajectory. The intensity of the source remains constant. Applying the recently developed Green's function approach, the analysis of the exact null-controllability is reduced to an infinite system of nonlinear constraints with respect to the control function. A sufficient condition for the approximate null-controllability of the rod is obtained. Since the exact solution of the system of constraints is a long-standing open problem, some heuristic solutions are used instead. The efficiency of these solutions is shown on particular cases of approximate controllability.

Key words: null-controllability, mobile control, nonlinear constraints, triangular wave, rectangular wave, Green's function approach, heuristic control, lack of exact controllability

1. Introduction

The ability of a system to be transmitted from any given state in finite time to any other given state by an external control is referred to as controllability. It is one of the most important properties of control systems, which is being studied for a long time now. Accordingly, there is a great deal of references studying different types and aspects of controllability for various model systems described by all types of state constraints. Some of the recent fundamental works in the area of controllability can be found in [1–6] and references therein.

This paper studies the controllability of a finite rod from any given initial temperature distribution to a state where each point of the rod has a constant uniform temperature by means of a source moving along the rod with a constant

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intensity and controllable trajectory. In general, the controllability analysis of heating process of bodies is a very delicate problem, especially in two- and three-dimensions. Since the heat transfers slower compared to, e.g., sound, unbounded and even some cases of bounded bodies may not be controlled. The case of unbounded bodies has been studied, e.g., in [7–10], and that of bounded bodies in [11, 12]. In these and similar studies, usually, boundary controls or controls distributed over a fixed, bounded region of the body are considered. On the other hand, due to high thermal efficiency, laser heating is widely applied in various areas of contemporary engineering technologies [13]. Laser heater is usually modelled as a point source or a source having a small area of concentration moving over the surface of the heating body. In mathematical terms, the heat transfer induced in a body as a result of laser heating is modelled by means of usual heat equation with a source term represented as a Dirac function of time dependent argument:

$$c_{\mathrm{V}} \frac{\partial \widetilde{\Theta}}{\partial \widetilde{t}} = \kappa \Delta \widetilde{\Theta} + \widetilde{\Theta}_s \delta \left(\widetilde{x} - u_x \left(\widetilde{t}\right)\right) \delta \left(\widetilde{y} - u_y \left(\widetilde{t}\right)\right) \delta \left(\widetilde{z} - u_z \left(\widetilde{t}\right)\right), \qquad (\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \Omega,$$

where c_{V} and κ are the volumetric heat capacity and thermal conductivity of the body occupying the domain $\Omega \subset \mathbb{R}^3$, $\widetilde{\Theta}_s$ is the constant coefficient and $\left(u_x, u_y, u_z\right)$ is the trajectory of the source.

The control problem is in determination of the triple $\left(u_x, u_y, u_z\right)$ such that for any initial temperature distribution $\widetilde{\Theta}\left(\widetilde{x}, \widetilde{y}, \widetilde{z}, 0\right)$, any given distribution $\widetilde{\Theta}(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{T})$ is provided in given finite time $\widetilde{T}$. Such control problems, referred to as mobile control problems, have been first considered by prominent mathematician A.G. Butkovskiy in 1970s [14] and then studied mainly by his students (for a relatively complete list of references, see, [15] and [16]). The main difficulty of mobile control problems is that the determination of the control trajectory is reduced to the solution of an infinite system of *nonlinear* constraints. Analytical solution to that problem has not been obtained yet [17]. Nevertheless, there exist quite efficient methods for numerical solution of mobile control problems [15, 16].

In this paper, a mobile control problem for the one-dimensional heat equation is considered by the Green's function approach [6, 18, 19] efficiently applicable to the analysis of exact and approximate controllability of both linear and nonlinear processes [20–25]. Using the explicit solution of the heat equation in terms of Green's function, explicit representation of the controllability residue in terms of the control trajectory is obtained. As a result, an infinite system of nonlinear constraints is obtained for the exact null-controllability, and an inequality sufficient for the approximate null-controllability is obtained. Some parametric families of control including triangular and rectangular waves and their superpositions are developed using the heuristic method [26, 27]. The efficiency of the heuristic solutions is proved in particular examples of approximate controllability. It is showsn also that for a particular initial distribution of the temperature, the rod is not exactly null-controllable by a source fixed at some point of the rod and having a constant intensity.

2. Controllability problem

The heat transfer in a rod is described by the following one-dimensional heat equation

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} + \Theta_s \delta\left(x - u(t)\right), \qquad x \in (0,1), \quad t > 0, \tag{1}$$

in which all variables and quantities are dimensionless:

$$\Theta = \frac{\widetilde{\Theta} - \widetilde{\Theta}_T^0}{\widetilde{\Theta}_T^0 - \widetilde{\Theta}_0^0}$$

describes the temperature of the rod, where $\widetilde{\Theta}$, $\widetilde{\Theta}_0^0$ and $\widetilde{\Theta}_T^0$ are the actual, initial and desired constant temperature distribution of the rod,

$$\Theta_s = \frac{\widetilde{\Theta}_s l^2}{\kappa\left(\Theta_T^0 - \Theta_0^0\right)}, \quad x = \frac{\widetilde{x}}{l}, \quad t = \frac{\kappa \widetilde{t}}{l^2 c_V},$$

rod has length l, t is the Fourier number. The rod is assumed to be sufficiently thin, so that the temperature distribution over any of its cross section is uniform.

The heat transfer between the rod and the external medium does not happen including both $x = 0$ and $x = 1$ ends:

$$\Theta(0,t) = \Theta(1,t) = 0, \qquad t \geqslant 0. \tag{2}$$

The temperature distribution in the rod for $t = 0$ is given by

$$\Theta\left(x, 0\right) = \Theta_0(x), \qquad x \in [0,1]. \tag{3}$$

Let $\Theta_0 \in L^2[0,1]$, $\Theta_0(0) = 0$, so that the initial and boundary conditions are consistent. Note that since the temperature has been normalized, Θ_0 merely describes the temperature distribution law. Therefore, for the sake of simplicity, it is assumed that $\|\Theta_0\|^2_{L^2[0,1]} \leqslant 1$.

The aim is to study the null-controllability of the rod, i.e., the possibility of derivation of trajectories of the source, such that at a given *finite* value T of the Fourier number, the rod is brought to the null-state

$$\Theta(x,T) \equiv 0, \qquad x \in [0,1]. \tag{4}$$

Apparently, when (4) is ensured for the rod, the source may be turned off to ensure the null-state of the rod for $t \geqslant T$ as well.

Assume that the heat source does not tear off of the rod, but may have some discontinuities. Therefore, the set of admissible controls will be defined as

$$\mathcal{U} = \left\{ u \in C_p[0,T],\ u \in (0,1) \right\}.$$

The problem is to characterize the set of admissible trajectories of the source for which the residue

$$\mathcal{R}_T(u) = \|\Theta(x,T)\|^2_{L^2[0,1]} = \int\limits_0^1 |\Theta(x,T)|^2 \,\mathrm{d}x \tag{5}$$

satisfies either exact or approximate controllability condition

$$\mathcal{R}_T(u) = 0 \quad \text{or} \quad \mathcal{R}_T(u) \leqslant \varepsilon,$$

respectively, where $\varepsilon > 0$ is a given constant. The dependence $\mathcal{R}_T$ on u will be made explicit in the next section.

3. Green's function solution

In order to analyze the controllability of (1), the Green's function approach [6] is involved. Represent the general solution of (1)-(3) in terms of the Green's function [28]:

$$\Theta(x,t) = \int\limits_0^1 G(x,\xi,t)\Theta_0(\xi)\,\mathrm{d}\xi + \Theta_s \int\limits_0^T G\left(x,u(\tau),T-\tau\right)\mathrm{d}\tau,$$
$$x \in [0,1], \qquad t \geqslant 0, \tag{6}$$

where

$$G(x,\xi,t) = 2\sum_{k=1}^{\infty} \sin(\pi k x)\sin(\pi k \xi)\exp\left[-(\pi k)^2 t\right].$$

In order to make the dependence $\mathcal{R}_T = \mathcal{R}_T(u)$ explicit, evaluate (6) at $t = T$ and substitute it into (5):

$$\mathcal{R}_T(u) = \int\limits_0^1 \left| \int\limits_0^1 \Theta_0(\xi)G(x,\xi,T)\,\mathrm{d}\xi + \Theta_s \int\limits_0^T G\left(x,u(\tau),T-\tau\right)\mathrm{d}\tau \right|^2 \mathrm{d}x. \tag{7}$$

4. Exact null-controllability

First, examine the exact null-controllability of (1). Using the methodology of [6], the following result can be established.

Theorem 1 *For the exact null controllability of (1), it is necessary and sufficient that for given Θ_0, Θ_s and T,*

$$\Theta_s \int_0^T \sin[\pi n u(\tau)] \exp\left[-(\pi n)^2(T-\tau)\right] \mathrm{d}\tau = = -\Theta_{0n} \exp\left[-(\pi n)^2 T\right], \qquad n \in \mathbb{Z}. \tag{8}$$

Here, Θ_{0n} are the Fourier sine-coefficients of Θ_0.

Proof. By the definition of norm, (7) is equivalent to

$$\Theta_s \int_0^T G\left(x, u(\tau), T-\tau\right) \mathrm{d}\tau = M_T(x), \quad x \in [0,1], \tag{9}$$

where

$$M_T(x) = -\int_0^1 \Theta_0(\xi) G(x, \xi, T) \mathrm{d}\xi.$$

Expanding (9) into Fourier series, we obtain the following infinite system of integral constraints on u:

$$\Theta_s \int_0^T G_n\left(u(\tau), T-\tau\right) \mathrm{d}\tau = M_{Tn}, \qquad n \in \mathbb{Z}, \tag{10}$$

with

$$G_n(\xi, t) = \int_0^1 G(x, \xi, t) \sin(\pi n x) \mathrm{d}x = \sum_{k=1}^{\infty} \delta_k^n \sin(\pi k \xi) \exp\left[-(\pi k)^2 t\right],$$

$$M_{Tn} = \int_0^1 M_T(x) \sin(\pi n x) \mathrm{d}x = -\sum_{k=1}^{\infty} \delta_k^n \Theta_{0k} \exp\left[-(\pi k)^2 T\right],$$

where δ_k^n is the Kronecker symbol.

Substituting these expressions into (10), infinite system (8) is eventually obtained.

Apparently, (8) is an infinite system of integral constraints on u. Determination of u from (8) explicitly is a long-standing open problem [17]. Usually, it is done numerically [15, 16].

5. Approximate null-controllability

The approximate controllability is studied in a fashion similar to [6].

Theorem 2 *For the approximate null-controllability of (1), it is sufficient that for the required accuracy* $\varepsilon > 0$,

$$2\Theta_s \sum_{k=1}^{\infty} \int_0^T \sin^2 [\pi k u(\tau)] \exp\left[-2(\pi k)^2 (T-\tau)\right] \mathrm{d}\tau \leqslant \varepsilon - C(T) \, \|\Theta_0\|^2_{L^2[0,1]} \,, \tag{11}$$

provided that

$$\varepsilon - C(T) \, \|\Theta_0\|^2_{L^2[0,1]} \geqslant 0. \tag{12}$$

Proof. Making use of Minkowski's inequality, (7) is reduced to

$$\mathcal{R}_t(u) \leqslant \left\| \int_0^1 \Theta_0(\xi) G(x,\xi,T) \mathrm{d}\xi \right\|_{L^2[0,1]} + \Theta_s \left\| \int_0^T G\left(x, u(\tau), T-\tau\right) \mathrm{d}\tau \right\|_{L^2[0,1]} := I_1 + \Theta_s I_2 \,. \tag{13}$$

Jensen's inequality applied to I_1 and I_2 provides

$$I_1 \leqslant \int_0^1 |\Theta_0(\xi)|^2 \, \|G\left(\cdot,\xi,T\right)\|^2_{L^2[0,1]} \, \mathrm{d}\xi,$$

$$I_2 \leqslant \int_0^T \|G\left(\cdot,u(\tau),T-\tau\right)\|^2_{L^2[0,1]} \,.$$

Taking into account that

$$\left[\sum_{k=1}^{\infty} A_k \sin(\pi k x)\right]^2 = \sum_{k=1}^{\infty} A_k^2 \sin^2(\pi k x) + \sum_{k=1}^{\infty} A_k \sin(\pi k x) \sum_{\substack{j=1 \\ j\neq k}}^{\infty} A_j \sin(\pi j x),$$

and that the family $\{\sin(\pi k x)\}_{k=1}^{\infty}$ is orthogonal in [0, 1], we obtain

$$\|G(x,\xi,T)\|_{L^2[0,1]}^2 = 2\sum_{k=1}^{\infty}\sin^2(\pi k\xi)\exp\left[-2(\pi k)^2 T\right].$$

Therefore,

$$I_1 \leqslant 2\sum_{k=1}^{\infty}\exp\left[-2(\pi k)^2 T\right]\int_0^1 |\Theta_0(\xi)|^2\sin^2(\pi k\xi)\,\mathrm{d}\xi \leqslant$$
$$\leqslant 2\,\|\Theta_0\|_{L^2[0,1]}^2\sum_{k=1}^{\infty}\exp\left[-2(\pi k)^2 T\right] =$$
$$= \|\Theta_0\|_{L^2[0,1]}^2\left(\vartheta_3\left(0,\exp\left[-2\pi^2 T\right]\right)-1\right) := C(T)\,\|\Theta_0\|_{L^2[0,1]}^2,$$
$$I_2 \leqslant 2\sum_{k=1}^{\infty}\int_0^T \sin^2\left[\pi k u(\tau)\right]\exp\left[-2(\pi k)^2(T-\tau)\right]\mathrm{d}\tau,$$

where ϑ_3 is the Jacobi theta function.

Substituting these estimates into (13), we will obtain

$$\mathcal{R}_T(u) \leqslant C(T)\,\|\Theta_0\|_{L^2[0,1]}^2 + 2\Theta_s\sum_{k=1}^{\infty}\int_0^T \sin^2\left[\pi k u(\tau)\right]\exp\left[-2(\pi k)^2(T-\tau)\right]\mathrm{d}\tau.$$

Satisfying the approximate null-controllability criterion, we immediately derive (11). □

6. Some heuristic trajectories

As it has been mentioned above, the exact solution to the infinite system (8) is an open problem. Nevertheless, following to [26, 27], it is possible to construct particular solutions based on the physical interpretation of the problem. One of the heuristic solutions appropriate to the control process under study is the triangle wave

$$u(t) = \frac{2}{\pi}\sum_{k=1}^{K} A_k \arcsin\left[\sin\left(\frac{2\pi}{\omega_k}t+\varphi_k\right)\right]. \tag{14}$$

Here, K, A_k, ω_k and φ_k are parameters determined to satisfy (8) or (11), respectively.

Another heuristic trajectory is defined by the rectangular wave as follows:

$$u(t) = \sum_{k=1}^{K} A_k \left[\theta\left(t - t_{1k} + \frac{t_{2k}}{2}\right) - \theta\left(t - t_{1k} - \frac{t_{2k}}{2}\right) \right]. \tag{15}$$

In this case, the control parameters are K, A_k, t_{1k} and t_{2k}. At this, t_{1k} and t_{2k} define the sides of the rectangular wave. In order to ensure that $u \in \mathcal{U}$, it is accepted $\theta(0) = 1/2$.

Superposition of (14) and (15) can be considered as another heuristic trajectory of the sources. Additional constraints on control parameters follow from the restriction $u \in (0, 1)$. Particular heuristic controls are plotted in Fig. 1.

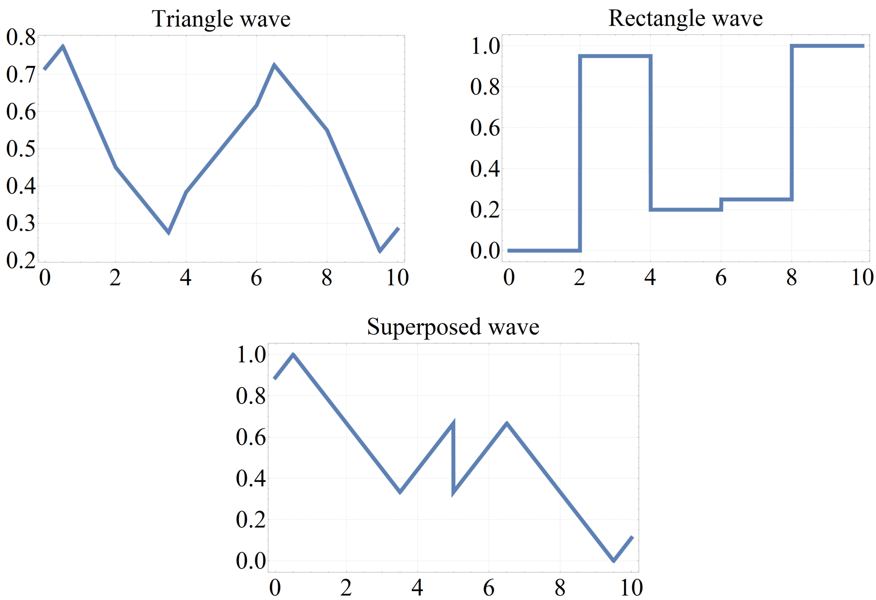

Figure 1: Examples of triangle, rectangle and superposed waves

Note that all three types of controls are easily implementable for machines regulating the laser motion. Other heuristic trajectories of the form of saw-tooth wave or more smooth functions can be used as well.

7. Numerical analysis

In this section, several particular cases are studied numerically to show the practical efficiency of the heuristic trajectories discussed in the previous section especially for the approximate controllability analysis.

7.1. Exact controllability

Since $\Theta_0 \in L^2[0,1]$, Θ_{0n} decays as n increases. Therefore, it is appropriate to consider (8) for some finite N. However, note that, in general, the consideration of the truncated system will lead to the determination of controls providing merely approximate controllability of the rod.

Let, for instance, $\Theta_0(x) = \cos\left((1-2x)/2\right)$ with $\|\Theta_0\|^2_{L^2[0,1]} \approx 0.921$. Then,

$$\Theta_{0n} = \begin{cases} \dfrac{1}{\pi}\dfrac{4n}{4n^2-1}\cos\left(\dfrac{1}{2}\right), & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases}$$

Heuristic considerations provide a physically reasonable solution of the form of a single rectangle

$$u(t) = u_0\left[\theta(t) - \theta(t-T)\right],$$

where u_0 is an unknown constant. Substituting it into (8), leads to:

$$\Theta_s \sin\left[\pi n u_0\right] \frac{1-\exp\left[-(\pi n)^2 T\right]}{(\pi n)^2} = -\left[1+(-1)^n\right]\frac{2n}{n^2-1}\cos\left(\frac{1}{2}\right)\exp\left[-(\pi n)^2 T\right].$$

It becomes more apparent now that when n increases, both sides of the last equality decay to zero. Apparently, $u_0 = 1/2$, corresponding to the case when the laser heats the mid-point of the rod, makes the equality valid for odd values of n. For even values of $n = 2m$, it reduces to

$$\Theta_s \sin[\pi m]\frac{1-\exp\left[-(2\pi m)^2 T\right]}{(2\pi m)^2} = -\frac{8m}{4m^2-1}\cos\left(\frac{1}{2}\right)\exp\left[-(2\pi m)^2 T\right]$$

or

$$-\frac{8m}{4m^2-1}\cos\left(\frac{1}{2}\right)\exp\left[-(2\pi m)^2 T\right] = 0$$

holding for all m only when T is infinite. Therefore, in this case, a lack of exact controllability is encountered.

7.2. Approximate controllability

Numerical analysis shows that C decays very fast with increase of T. Figure 2 shows that for $T > 0.5$, $C(T) \leqslant 10^{-4}$. Therefore, since $\|\Theta_0\|^2_{L^2[0,1]} \leqslant 1$, (12) may hold even for $\varepsilon \sim 10^{-4}$. For most of metals, the thermal conductivity, $\kappa \gg 1$ [W/m K]. Therefore, the consideration is limited to the case $\Theta_s \ll 1$.

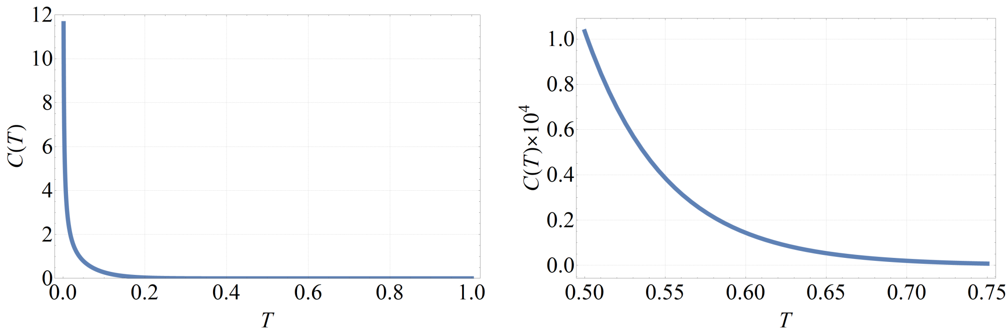

Figure 2: Graph of $C(T)$ against T

Figure 3 shows that the approximate controllability of the rod can be provided by (14) for $K = 1$, $A = 1$, $\omega = 0.05$, $\varphi = \pi/4$, where

$$S(T) = \sum_{k=1}^{\infty} \int_0^T \sin^2 [\pi k u(\tau)] \exp\left[-2(\pi k)^2 (T - \tau)\right] \mathrm{d}\tau.$$

This trajectory corresponds to the motion of the source starting from $x = 0.5$ point (mid-point of the rod) to $x = 1$ end and then back to $x = 0$ end, and repeating the same motion till $t = 1$.

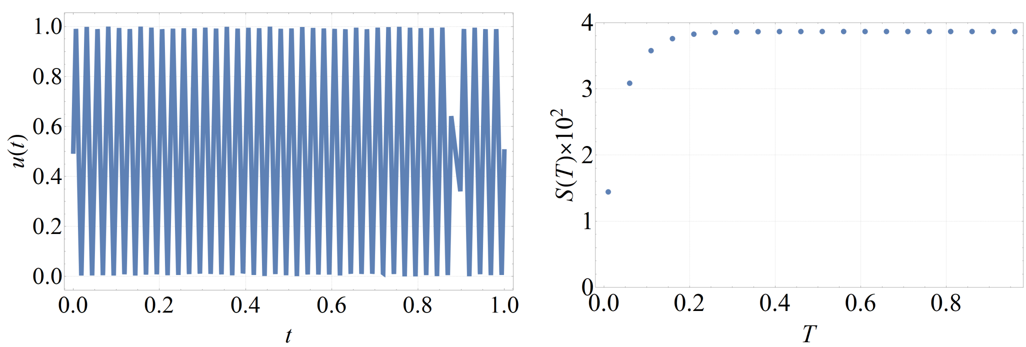

Figure 3: Graph of u and corresponding expression of $S(T)$ for (14)

The plot of $S(T)$ shows that, when the Fourier number T increases, S increases to $\approx 4 \cdot 10^{-2}$ and remains constant. Therefore, as soon as, e.g., $\Theta_s \sim 10^{-3}$, the rod is approximately null-controllable for any Θ_0 and T.

Consider the case of (15). As it is shown in Fig. 4, the approximate controllability of the rod for $\varepsilon \sim 10^{-4}$ can be provided for any $T \geqslant 6$ by

$$u(t) = \left| \sum_{k=1}^{3} A_k \left[\theta\left(t - t_{1k} + \frac{t_{2k}}{2}\right) - \theta\left(t - t_{1k} - \frac{t_{2k}}{2}\right) \right] \right|$$

with $A_1 = -0.5$, $A_2 = 0.75$, $A_3 = -0.1$, $t_{11} = 1$, $2t_{21} = 1.5$, $t_{12} = 3$, $2t_{22} = 3$, $t_{13} = 8$, $2t_{23} = 2$.

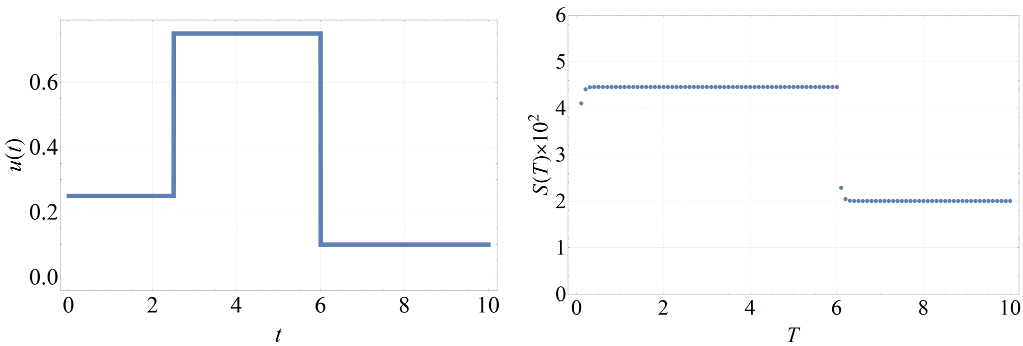

Figure 4: Graph of u and corresponding expression of $S(T)$ for (15)

8. Conclusions

Heuristic trajectories in the form of triangle, rectangle and their superposed waves are derived in this paper to study the exact and approximate controllability of a finite, sufficiently thin rod. The rod is heated by a source moving over its surface and the control process is carried out by the trajectory of the source. Applying the Green's function approach, the solution of the exact controllability problem is reduced to the solution of an infinite system of integral constraints with respect to a single control function. For the approximate controllability, a sufficient condition is derived. Analysis of particular cases shows the efficacy of the heuristic trajectories especially for the approximate controllability of the rod. It is also shown that a lack of exact controllability occurs for finite Fourier numbers.

In our future works, we are going to generalize the developed solution for 2D heat equation for a membrane and for the case when elastic dissipation is accepted. In the latter case, the elastic energy is coupled with the thermal energy, leading to a coupled system of partial differential equations describing the heat transfer in the solid.

References

[1] J. Klamka: *Controllability of Dynamical Systems*. Kluwer Academic, Dordrecht, 1991.

[2] S.A. Avdonin and S.A. Ivanov: *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*. Cambridge University Press, New York, 1995.

[3] A. Fursikov and O.Yu. Imanuvilov: *Controllability of Evolution Equations*. Lecture Notes Series, vol. 34. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[4] E. Zuazua: *Controllability and Observability of Partial Differential Equations: Some Results and Open Problems*. Handbook of Differential Equations: Evolutionary Differential Equations, vol. 3, Elsevier/North-Holland, Amsterdam, 2006.

[5] R. Glowinski, J.-L. Lions and J. He: *Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach*. Cambridge University Press, New York, 2008.

[6] A.S. Avetisyan and As.Zh. Khurshudyan: *Controllability of Dynamic Systems: The Green's Function Approach*. Cambridge Scholars Publishing, Cambridge, 2018.

[7] S. Micu and E. Zuazua: On the lack of null-controllability of the heat equation on the half-line. *Transactions of the American Mathematical Society*, **353**(4), (2001), 1635–1659.

[8] S. Micu and E. Zuazua: *Null Controllability of the Heat Equation in Unbounded Domains*. In "Unsolved Problems in Mathematical Systems and Control Theory", edited by Blondel V.D., Megretski A., Princeton University Press, Princeton, 2004.

[9] V. Barbu: Exact null internal controllability for the heat equation on unbounded convex domain. *ESAIM: Control, Optimisation and Calculus of Variations*, **20** (2014), 222–235, DOI: 10.1051/cocv/2013062.

[10] As.Zh. Khurshudyan: (2019), Distributed controllability of heat equation in un-bounded domains: The Green's function approach. *Archives of Control Sciences*, **29**(1), (2019), 57–71, DOI: 10.24425/acs.2019.127523.

[11] S. Ivanov and L. Pandolfi: Heat equation with memory: Lack of controllability to rest. *Journal of Mathematical Analysis and Applications*, **355** (2009), 1–11, DOI: 10.1016/j.jmaa.2009.01.008.

[12] A. Halanay and L. Pandolfi: Approximate controllability and lack of controllability to zero of the heat equation with memory. *Journal of Mathematical Analysis and Applications*, **425** (2015), 194–211, DOI: 10.1016/j.jmaa.2014.12.021.

[13] B.S. Yilbas: *Laser Heating Applications: Analytical Modelling*. Elsevier, Waltham, 2012.

[14] A.G. Butkovskiy and L.M. Pustylnikov: *Mobile Control of Distributed Parameter Systems*. Chichester, Ellis Horwood, 1987.

[15] V.A. Kubyshkin and V.I. Finyagina: *Moving control of systems with distributed parameters* (in Russian). Moscow: SINTEG, 2005.

[16] Sh.Kh. Arakelyan and As.Zh. Khurshudyan: The Bubnov-Galerkin procedure for solving mobile control problems for systems with distributed parameters. *Mechanics. PNAS Armenia*, **68**(3), (2015), 54–75.

[17] A.G. Butkovskiy: Some problems of control of the distributed-parameter systems. *Automation and Remote Control*, **72** (2011), 1237–1241, DOI: 10.1134/S0005117911060105.

[18] A.S. Avetisyan and As.Zh. Khurshudyan: Green's function approach in approximate controllability problems. *Proceedings of National Academy of Sciences of Armenia. Mechanics*, vol. 69, issue 2, (2016), 3–22, DOI: 10.33018/69.2.1.

[19] A.S. Avetisyan and As.Zh. Khurshudyan: Green's function approach in approximate controllability of nonlinear physical processes. *Modern Physics Letters A*, **32** 1730015, (2017), DOI: 10.1142/S0217732317300154.

[20] As.Zh. Khurshudyan: Resolving controls for the exact and approximate controllabilities of the viscous Burgers' equation: the Green's function approach. *International Journal of Modern Physics C*, **29**(6), 1850045, (2018), DOI: 10.1142/S0129183118500456.

[21] A.S. Avetisyan and As.Zh. Khurshudyan: Exact and approximate controllability of nonlinear dynamic systems in infinite time: The Green's function approach. *ZAMM*, **98**(11), (2018), 1992–2009, DOI: 10.1002/zamm.201800122.

[22] As.Zh. Khurshudyan: Exact and approximate controllability conditions for the micro-swimmers deflection governed by electric field on a plane: The Green's function approach. *Archives of Control Sciences*, **28**(3), (2018), 335–347. DOI: 10.24425/acs.2018.124706.

[23] J. Klamka and As.Zh. Khurshudyan: Averaged controllability of heat equation in unbounded domains with uncertain geometry and location of controls: The Green's function approach. *Archives of Control Sciences*, **29**(4), (2019), 573–584, DOI: DOI: 10.24425/acs.2018.124706.

[24] J. Klamka, A.S. Avetisyan and As.Zh. Khurshudyan: Exact and approximate distributed controllability of the KdV and Boussinesq equations: The Green's function approach. *Archives of Control Sciences*, **30**(1), (2020), 177–193, DOI: 10.24425/acs.2020.132591.

[25] J. Klamka and As.Zh. Khurshudyan: Approximate controllability of second order infinite dimensional systems. *Archives of Control Sciences*, **31**(1), (2021), 165–184, DOI: 0.24425/acs.2021.136885.

[26] As.Zh. Khurshudyan: Heuristic determination of resolving controls for exact and approximate controllability of nonlinear dynamic systems. *Mathematical Problems in Engineering*, (2018), Article ID 9496371, DOI: 10.1155/2018/9496371.

[27] H. Hossain and As.Zh. Khurshudyan: Heuristic control of nonlinear power systems: Application to the infinite bus problem. *Archives of Control Sciences*, **29**(2), (2019), 279–288, DOI: 10.24425/acs.2019.129382.

[28] A.G. Butkovskii and L.M. Pustyl'nikov: *Characteristics of Distributed-Parameter Systems*. Kluwer Academic Publishers, 1993.