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RECURSIVE DIFFERENTIATION METHOD: APPLICATION TO THE ANALYSIS OF BEAMS ON TWO PARAMETER FOUNDATIONS

Mohamed Taha Hassan
Cairo University, Faculty of Engineering, Giza, Egypt
e-mail: mtahah@eng.cu.edu.eg; mtaha@alfaconsult.org

Eid Hassan Doha
Cairo University, Faculty of Science, Giza, Egypt; e-mail: eiddoha@sci.cu.edu.eg

The recursive differentiation method (RDM) is introduced and employed to obtain analytical solutions for static and dynamic stability parameters of beams resting on two-parameter foundations in various different end conditions. The present analysis reflects the reliability, efficiency and simplicity of the proposed RDM in tackling boundary value problems. In fact, it is widely common that the critical load accompanied with the first buckling mode is the smallest critical load, and then it is the dominant factor in the static stability analysis. In contrast, the present analysis indicates that such a conclusion is correct only for the case of beams without foundations or in the case of a weak foundation relative to the beam. It is proved that critical loads accompanied with higher buckling modes may be smaller than those accompanied with the lower modes and then it may control the stability analysis. The same phenomenon exists for natural frequencies in the presence of an axial load. Several illustrations are introduced to highlight the effects of both the foundation stiffness and beam slenderness on the critical loads and natural frequencies.

Keywords: critical loads, natural frequencies, recursive differentiation method, beam on elastic foundation

1. Introduction

Numerous analytical and numerical methods have been developed to obtain approximate solutions for Boundary Value Problems (BVP). The most commonly used analytical techniques are: Adomian Decomposition Method (ADM) proposed by Adomian (1994) and used by Taha *et al.* (2012), Bahnasawi *et al.* (2004) and Wazwas (2001). The Variational Iteration Method (VIM) was developed by He (2007) and used by Noor and Mohyud-Din (2008). The Homotopy Perturbation Method (HPM) used by Tan *et al.* and Abbasbandy (2009) and Jin (2008). The Differential Transform (DTM) by Ali (2012) and perturbation techniques by Nayfeh and Nayfeh (1994) and Maccari (1999). On the other hand, numerical methods such as the Finite Element Method (FEM) used by Mullapudi and Ayoub (2010) and Naidu and Rao (1996) and the Differential Quadrature Method (DQM) used by Taha and Nassar (2014) and Chen (2002) offer tractable alternative solutions for many BVPs that involve complicated mathematical formulations.

Analytical methods construct the solution to BVP as a polynomial such that the coefficients of the polynomial are obtained to satisfy both the governing differential equation and the boundary conditions. However, numerical techniques transform the differential equation into a system of algebraic equations either on the boundary of the BVP domain or at discrete points in the BVP domain. Indeed, the degree of success in dealing with mathematical manipulation and accuracy determines the method efficiency.

In addition, techniques for finding approximate solutions for differential equations, based on classical orthogonal polynomials, are popularly known as spectral methods. Approximating functions in spectral methods are related to polynomial solutions of eigenvalue problems in ordinary differential equations, known as Sturm-Liouville problems. In the last few decades, there has been a growing interest in this subject. As a matter of fact, spectral methods provide a competitive alternative to other standard approximate techniques for a large variety of problems.

Initial applications were concerned with investigations of periodic solutions to BVP using trigonometric polynomials. Subsequently, the analysis was extended to algebraic polynomials. The reader interested in spectral method is referred to Shen (1994, 1995), Doha and Abd-Elhameed (2002), Gottlieb and Orszag (1977).

Practically, in static analysis of axially loaded beams (columns), the determination of axial loads at which the beam losses its static stability is the main issue, while in analysis of forced vibration of beams, natural frequencies of the beam are the dominant factor in the avoidance of the resonance phenomenon which leads to unbounded response.

In the present work, the Recursive Differentiation Method (RDM) is introduced and employed to investigate static and dynamic behaviour of beams resting on two-parameter foundations taking into account rotational inertia of the beam. The analysis indicates that the RDM is straightforward in its mathematical formulation and very efficient in achieving accurate solutions. Analytic expressions for the amplitude of the lateral displacement are derived, and then the applications of boundary conditions at the beam ends yield the corresponding characteristic equation in two parameters (P_{cr}, ω_n). The solution to the characteristic equation yields either the critical loads (for $\omega = 0$) or the natural frequencies in the case $(P < P_{cr})$. The effect of different parameters on both the critical loads and natural frequencies will be analysed.

The paper is organized as follows: in Sections 2 and 3, the RDM is introduced and employed to obtain analytical solutions for static and dynamic stability parameters of beams resting on two-parameter foundations in various different end conditions. In Section 4, some numerical results are discussed and a comparison with other algorithms available in the literature is presented. Some concluding remarks are given in Section 5.

2. Recursive Differentiation Method (RDM)

In this Section, we are interested in developing the RDM to solve analytically the n-th order differential equation

$$y^{(n)}(x) = \sum_{i=0}^{n-1} A_{i,0} y^{(i)}(x) + f_0(x) \qquad x_0 \leqslant x \leqslant x_1 \tag{2.1}$$

subject to the boundary conditions

$$g_k(x, y, y^{(1)}, \dots, y^{(n-1)}) = b_k \qquad k = 1, 2, \dots, n \tag{2.2}$$

where $y^{(i)}(x)$ is the i-th derivative, $f_0(x)$ is the source function and $A_{i,0}$, $i = 0, 1, 2, \dots, n-1$ and b_k are known constants.

In the RDM, we seek a solution to Eq. (2.1) subject to Eq. (2.2) in the form

$$y(x) = \sum_{m=0}^{\infty} T_m \frac{(x - x_0)^m}{m!} \tag{2.3}$$

where T_m, $m = 0, 1, \dots$ are unknown coefficients to be determined such that differential equation (2.1) with its boundary conditions (2.2) is to be satisfied.

The coefficients T_m are related to the governing differential equation on the boundary as

$$T_m = y^{(m)}\big|_{x=x_0} \tag{2.4}$$

Now, if we differentiate Eq. (2.1) once, and eliminate $y^{(n)}(x)$ from the resulting equation, and after some little manipulations, we can write

$$y^{(n+1)}(x) = \sum_{i=0}^{n-1} A_{i,1} y^{(i)}(x) + f_1(x) \tag{2.5}$$

where

$$\begin{aligned} &A_{0,1} = A_{0,0}A_{n-1,0} \qquad A_{i,1} = A_{i-1,0} + A_{i,0}A_{n-1,0} \qquad i = 1,2,\dots,n-1 \\ &f_1(x) = f_0^{(1)}(x) + f_0(x)A_{n-1,0} \end{aligned} \tag{2.6}$$

Recursive differentiations of Eq. (2.4) k-times lead to

$$y^{(n+k)}(x) = \sum_{i=0}^{n-1} A_{i,k} y^{(i)}(x) + f_k(x) \qquad x_0 \leqslant x \leqslant x_1 \tag{2.7}$$

where the recurrence formulae for the coefficients $A_{i,k}$ and $f_k(x)$, $i = 1,2,\dots,n-1$ and $k = 1,2,\dots$, may be expressed as

$$\begin{aligned} &A_{0,k} = A_{0,0}A_{n-1,k-1} \qquad A_{i,k} = A_{i-1,k-1} + A_{i,0}A_{n-1,k-1} \\ &f_k(x) = f_{k-1}^{(1)}(x) + A_{n-1,k-1}f_0(x) \end{aligned} \tag{2.8}$$

Making use of Eq. (2.7) enables one to get a recurrence relation for the coefficients T_{n+k}, $k = 0,1,2\dots$ in the form

$$T_{n+k} = \sum_{i=0}^{n-1} A_{i,k}T_i + f_k(x_0) \tag{2.9}$$

and substitution of Eqs. (2.8) and (2.9) into Eq. (2.3) yields the solution to Eq. (2.1) in the form

$$y(x) = \sum_{j=0}^{n-1} T_j R_j(x) + R_f(x) \tag{2.10}$$

where the recursive functions $R_j(x)$ and the force recursive function $R_f(x)$ are

$$\begin{aligned} &R_j(x) = \frac{(x-x_0)^j}{j!} + \sum_{i=0}^{\infty} A_{j,i}\frac{(x-x_0)^{n+i}}{(n+i)!} \qquad j = 0,1,2,\dots,n-1 \\ &R_f(x) = \sum_{i=0}^{\infty} f_i(x_0)\frac{(x-x_0)^{n+i}}{(n+i)!} \end{aligned} \tag{2.11}$$

In fact, the practical solution to Eq. (2.3) is truncated as

$$y(x) = \sum_{m=0}^{M} T_m \frac{(x-x_0)^m}{m!} \tag{2.12}$$

where M is the truncation index selected to achieve the pre-assigned degree of accuracy.

Now, the application of the boundary conditions yields the characteristic equation of the system which may be solved to investigate the significance of different parameters on the system behaviour. It is to be noted that the transformation of the solution domain to $[0,1]$ has great effect on enhancing the convergence of the obtained solutions.

3. Free vibration of beams on two-parameter foundation

The equations of motion of an infinitesimal element of an axially loaded beam resting on two--parameter foundations shown in Fig. 1, taking into consideration the rotational inertia of the beam, are

$$\begin{aligned}&\frac{\partial V}{\partial x}+q(x,t)-k_1y(x,t)+k_2\frac{\partial^2 y}{\partial x^2}=\rho A\frac{\partial^2 y}{\partial t^2}\\&V(x,t)+p\frac{\partial y}{\partial x}-\frac{\partial M}{\partial x}=\rho I\frac{\partial^2\theta}{\partial t^2}\end{aligned} \tag{3.1}$$

while the slope-deflection and force-displacement relations are

$$\theta=\frac{\partial y}{\partial x}\qquad\qquad M(x,t)=-EI\frac{\partial^2 y}{\partial x^2} \tag{3.2}$$

where EI is the flexural stiffness of the beam, ρ is the density, A is the area of the cross section, p is the axially applied load, k_1 and k_2 are the linear and shear foundation stiffness per unit length of the beam, $q(x,t)$ is the lateral excitation, E is the modulus of elasticity, I is the moment of inertia, $\theta(x,t)$ is the rotation, $V(x,t)$ is the shear force, $M(x,t)$ is the bending moment, $y(x,t)$ is the lateral response of the beam, x is the coordinate along the beam and t is time.

Substitution of Eqs. $(3.1)_2$ and (3.2) into Eq. $(3.1)_1$ yields the equation of the beam lateral response in the form

$$EI\frac{\partial^4 y}{\partial x^4}+(p-k_2)\frac{\partial^2 y}{\partial x^2}-\rho I\frac{\partial^4 y}{\partial x^2\partial t^2}+k_1y(x)+\rho A\frac{\partial^2 y}{\partial t^2}=q(x,t) \tag{3.3}$$

Although the proposed RDM algorithm enables finding solutions for forced vibration of beams with different spatial distributions of the excitation function $f_0(x)$, in the present work the numerical analysis is limited to calculate the natural frequencies resulting from free vibration analysis, i.e. $q(x,t)=0$.

Assuming that the solution to Eq. (3.3) is in the form $y(x,t)=Y(x)\exp(\mathrm{i}\omega t)$ and introducing the dimensionless variables $\xi=x/L$ and $w(x)=Y(x)/L$, where ω is the natural frequency, $w(x)$ is the dimensionless amplitude of the lateral displacement and L is the beam length, then Eq. (3.3) may be expressed as

$$\frac{d^4 w}{d\xi^4}+\Big(P-K_2+\frac{\lambda^4}{\eta^2}\Big)\frac{d^2 w}{d\xi^2}+(K_1-\lambda^4)w(\xi)=0 \tag{3.4}$$

where

$$\begin{aligned}&K_1=\frac{k_1L^4}{EI}\qquad &&K_2=\frac{k_2L^2}{EI}\qquad &&P=\frac{pL^2}{EI}\\&\lambda^4=\frac{\rho A\omega^2L^4}{EI}\qquad &&\eta=\frac{L}{r}\qquad &&r=\sqrt{\frac{I}{A}}\end{aligned}$$

The parameters K_1 and K_2 are the foundation linear and shear stiffness parameters, P is the axial load parameter, λ is the frequency parameter, η is the slenderness parameter and r is the radius of gyration of the beam cross section.

3.1. Boundary conditions

For the beam shown in Fig. 1, the boundary conditions in the dimensionless form may be expressed as:

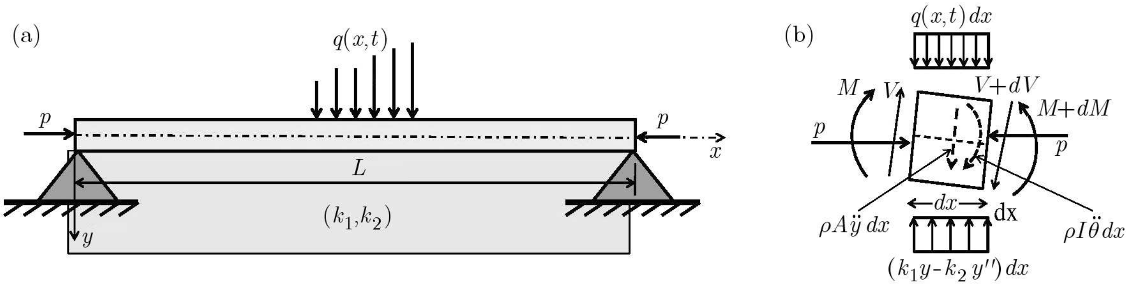

Fig. 1. (a) Beam on a two-parameter foundation, (b) element forces

— in the case of the pinned-pinned (P-P) beam

$$w(0) = w''(0) = 0 \quad \text{at} \quad \xi = 0 \qquad w(1) = w''(1) = 0 \quad \text{at} \quad \xi = 1 \tag{3.5}$$

— in the case of the clamped-pinned (C-P) beam

$$w(0) = w'(0) = 0 \quad \text{at} \quad \xi = 0 \qquad w(1) = w''(1) = 0 \quad \text{at} \quad \xi = 1 \tag{3.6}$$

in the case of the clamped-clamped (C-C) beam

$$w(0) = w'(0) = 0 \quad \text{at} \quad \xi = 0 \qquad w(1) = w'(1) = 0 \quad \text{at} \quad \xi = 1 \tag{3.7}$$

in the case of the clamped-free (C-F) beam

$$w(0) = w'(0) = 0 \quad \text{at} \quad \xi = 0 \qquad w''(1) = 0 \quad w'''(1) = -Pw'(1) \quad \text{at} \quad \xi = 1 \tag{3.8}$$

3.2. Application of the Recursive Differentiation Method

To use the RDM, the governing equation of the beam-foundation system (Eq. (3.4)) is rewritten in the recursive form

$$w^{(4)}(\xi) = A_{0,0}w^{(0)}(\xi) + A_{2,0}w^{(2)}(\xi) \tag{3.9}$$

where the constants $A_{i,0}$, $i = 0, 1, 2, 3$ are

$$A_{0,0} = -K_1 + \lambda^4 \qquad A_{1,0} = 0 \qquad A_{2,0} = -P + K_2 - \frac{\lambda^4}{\eta^2} \qquad A_{3,0} = 0 \tag{3.10}$$

Making use of Eqs. (2.7) and (2.8), the coefficients $A_{i,k}$ and T_{4+k} for $i = 1, 2, 3$ and for $k = 1, 2, \ldots$ can be obtained; hence the amplitude of the lateral displacement may be expressed as

$$w(\xi) = \sum_{j=0}^{3} T_j R_j(\xi) \tag{3.11}$$

where the recursive functions $R_j(\xi)$ are obtained as

$$R_j(x) = \frac{\xi^j}{j!} + \sum_{i=0}^{M} A_{j,i}\frac{\xi^{n+i}}{(n+i)!} \qquad j = 0, 1, 2, 3 \tag{3.12}$$

Substitution of Eq. (3.11) into the boundary conditions (Eqs. (3.5)-(3.8)) leads to the corresponding characteristic (frequency) equations in different cases of end conditions as follows

$$\begin{aligned} &P - P \text{ case:} && R_{10}R_{32} - R_{12}R_{30} = 0 \\ &C - C \text{ case:} && R_{20}R_{31} - R_{30}R_{21} = 0 \end{aligned} \tag{3.13}$$

$$C-P \text{ case:} \qquad R_{20}R_{32}-R_{30}R_{22}=0$$
$$C-F \text{ case:} \qquad R_{22}(R_{33}+PR_{31})-R_{32}(R_{23}+PR_{21})=0$$

where $R_{ij}=R_i^{(j)}(1)$, $i,j=0,1,2,3$.

Using a proper iterative technique, the solution of the corresponding eigenvalue problem with the two parameters (P_{cr},ω_n) can be obtained. However, for $\omega_n=0$, the eigenvalues represent P_{cr} while the eigenvectors represent the corresponding buckling modes. On the other hand, assigning a value for $P<P_{cr}$, then the eigenvalues represent the natural frequencies ω_n while the eigenvectors represent the mode shapes of free vibration.

3.3. Verification

To verify the analytical expressions obtained from the RDM, the critical load parameter P_{cr} and the natural frequency parameter λ_n for a beam resting on a two-parameter foundation calculated from the RDM (Eqs. (3.13)) for the truncation index $M=25$ and those obtained from FEM (Mullapudi and Ayoub, 2010) are presented in Tables 1 and 2. In Table 1, values of P_{cr} are presented for different foundation parameters and different end conditions, while in Table 2, values of λ_n are presented through the effect of the loading ratio γ. It is clearly seen that the RDM results are in close agreement to those calculated from the FEM. The critical load parameter P_{cr}, the natural frequency parameter λ_n and the loading ratio γ are defined as

$$\lambda_n^4=\frac{\rho A\omega_n^2L^4}{EI} \qquad P_{cr}=\frac{p_{cr}L^2}{EI} \qquad \gamma=\frac{P}{P_{cr}} \tag{3.14}$$

Table 1. Critical load parameter for beams on elastic foundations ($\eta=50$)

K_1	K_2	P-P beams FEM	P-P beams RDM	C-C beams FEM	C-C beams RDM
		P_{cr}			
0	0	9.8696	9.8696	39.479	39.4784
	π^2	19.739	19.7392	49.349	49.3480
100	0	20.002	20.0020	47.007	47.0066
	π^2	29.871	29.8713	56.877	56.876
10^4	0	201.41	201.4060	233.82	233.785
	π^2	211.28	211.2751	243.69	243.655

Table 2. Frequency parameter for beams on elastic foundations ($\eta=50$)

K_1	K_2	γ	P-P beams FEM	P-P beams RDM	C-C beams FEM	C-C beams RDM
			λ_n			
0	0	0	3.1415	3.1416	4.7300	4.7286
		0.6	2.5097	2.4984	3.7508	3.7795
	π^2	0	3.7306	3.7360	4.9925	4.9910
		0.6	2.9842	2.9711	3.9618	3.9926
100	0	0	3.7483	3.7484	4.9504	4.9489
		0.6	2.9940	2.9810	3.9323	3.9612
	π^2	0	4.1437	4.1437	5.1823	5.1808
		0.6	3.3095	3.2954	4.1193	4.1503

4. Numerical results

4.1. Values of the foundation stiffness parameters K_1 and K_2

Several models have been proposed to simulate the foundation reactions (Vlasov and Leontev, 1960; Zhaohua and Cook, 1983). The foundation parameters (K_1, K_2) depend on both the beam properties (EI, η) and the foundation linear and shear stiffness factors (k_1, k_2). In the present work, k_1, k_2 are calculated using the expressions proposed by Zhaohua and Cook (1983) for a rectangular beam resting on a two-parameter foundation as

$$k_1 = \frac{E_0 b}{2(1-\nu_0^2)} \frac{\delta}{\chi} \qquad\qquad k_2 = \frac{E_0 b}{4(1-\nu)} \frac{\chi}{\delta} \tag{4.1}$$

where

$$\chi = \sqrt[3]{\frac{2EI(1-\nu_0^2)}{bE_0(1-\nu^2)}} \qquad\qquad E_0 = \frac{E_s}{1-\nu_s^2} \qquad\qquad \nu_0 = \frac{\nu_s}{1-\nu_s}$$

E and ν are the elastic modulus and Poisson's ratio of the beam, E_s and ν_s are the respective quantities of the foundation and δ is a parameter describing the beam-foundation loading configuration (it is a common practice to assume $\delta = 1$). Typical values of the elastic modulus and Poisson's ratio for different types of foundations are given in Table 3.

Table 3. Typical values of modulus of elasticity and Poisson's ratio for soils

Type of foundation	No foundation (NF)	Weak foundation (WF)	Medium foundation (MF)	Stiff foundation (SF)
E_s [N/m^2]	0	1E+07	5E+07	1E+08
Poison ratio ν_s	0	0.40	0.35	0.30

Though the RDM solution expressions are obtained in dimensionless forms to make analysis of different specific configurations possible, in the present investigation however, the properties of the beam are (concrete beam): $b = 0.2\,\mathrm{m}$, $h = 0.5\,\mathrm{m}$, $E = 2.1E + 10\,\mathrm{Pa}$, Poisson's ratio $\nu = 0.15$. A simple MATLAB code has been assembled to obtain the presented results.

4.2. Critical loads

It is widely known that the dominant (fundamental) critical load or natural frequency of a beam corresponds to the first mode and is used as the limiting value for the avoidance of static or dynamic instability. In fact, this is correct only in the cases of beams without foundations or where the supporting foundation has a weak stiffness relative to the beam. In other words, in the case of a beam resting on a stiff foundation, the critical loads and natural frequencies accompanied higher modes may be smaller than those corresponding to the lower modes. This result is crucial for stability analysis to avoid unbounded deformation in a static or dynamic case. The variations of the critical loads and natural frequency parameters with the foundation stiffness and slenderness ratio of the beam for different buckling modes and free vibration mode shapes are obtained and represented in Fig. 2 for the P-P case as an example. It is observed that for a beam on an elastic foundation, as the slenderness ratio increases; i.e. the foundation stiffness increases relative to the beam stiffness, the smallest critical load or natural frequency parameter may correspond to one of higher modes. Thus, to avoid static or dynamic instability, the critical loads and natural frequencies for different modes should be calculated first, and the smaller one is then used as the limiting value for stability analysis.

Variations of the critical loads against the slenderness ratio (η) for different foundation stiffnesses (E_s, μ_s) and different end conditions are presented in Figs. 3 and 4.

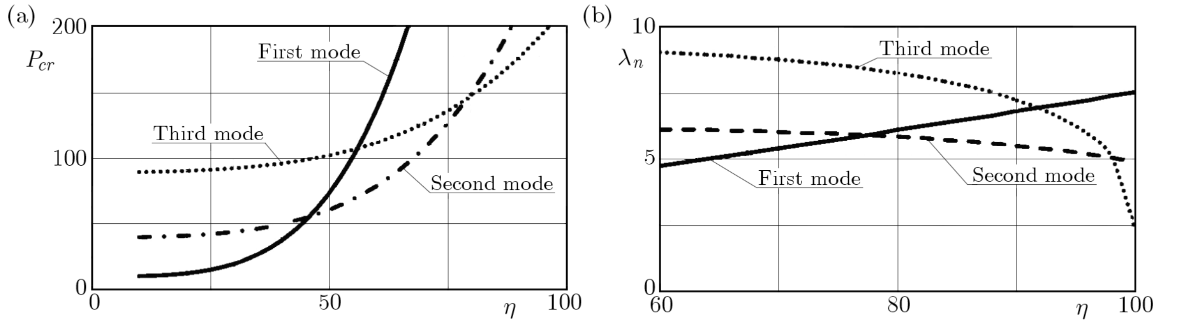

Fig. 2. Critical loads and natural frequencies for different modes (P-P beams); (a) P_{cr} ($E_s = 1E+08$), (b) λ_n ($E_s = 5E+07$)

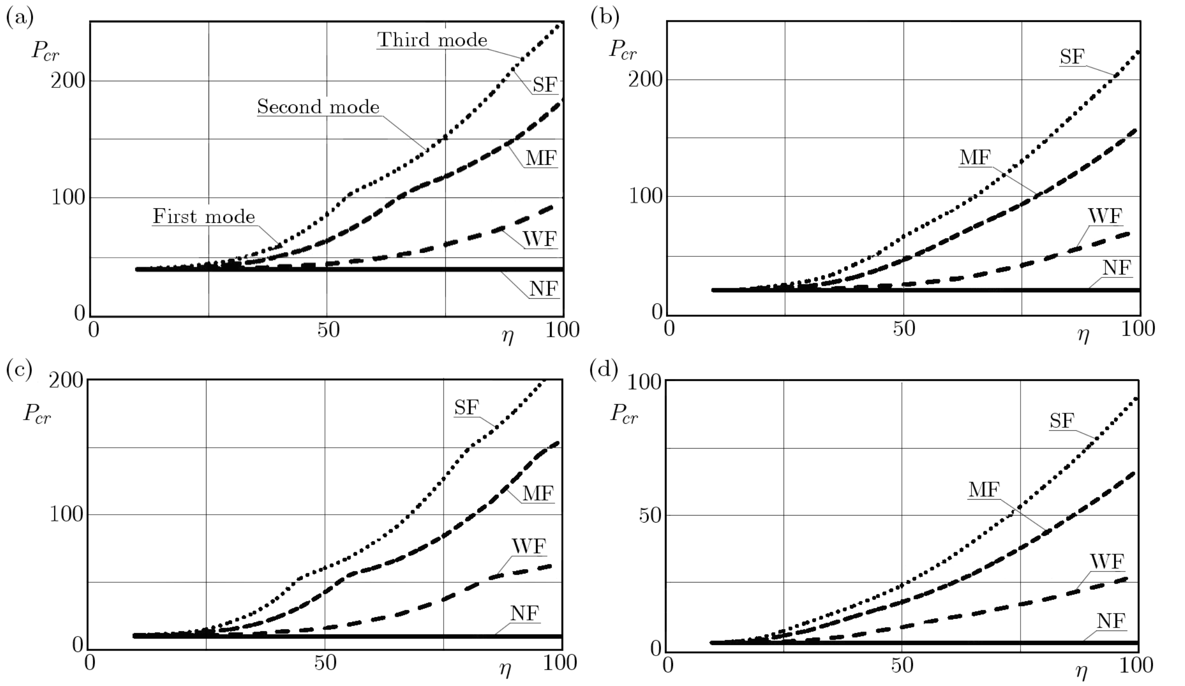

Fig. 3. Variation of P_{cr} with η and E_s; (a) C-C beams, (b) C-P beams, (c) P-P beams, (d) C-F beams

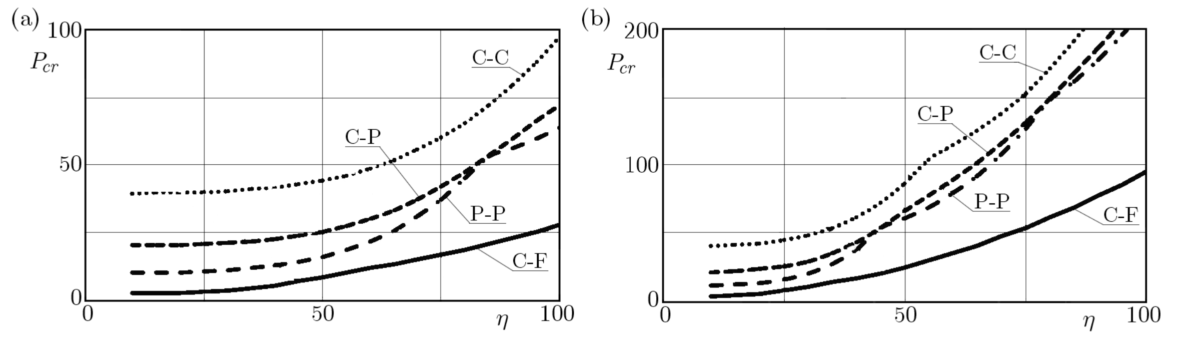

Fig. 4. Effect of the beam end condition on P_{cr}; (a) weak foundation (WF), (b) stiff foundation (SF)

In Fig. 3, case (a) represents the clamped-clamped beams (C-C), case (b) clamped-pinned beams (C-P), case (c) pinned-pinned beams (P-P) and case (d) clamped-free (cantilever) beams (C-F). The influence of the foundation increases as the slenderness ratio increases. The influence of end conditions decreases as the slenderness ratio increases.

The transition between different modes is clear in (P-P) and (C-C) cases for a medium stiffness (MF) and stiff foundations (SF). Actually, the assembled MATLAB code always picks the smallest critical load in spite of the buckling mode.

4.3. Natural frequencies

The variations of the natural frequency parameter λ_n with the foundation stiffness (E_s, μ_s) and slenderness ratio (η) are presented in Figs. 5 and 6 for $\gamma = 0$, while the effect of γ are shown in Figs. 7-10.

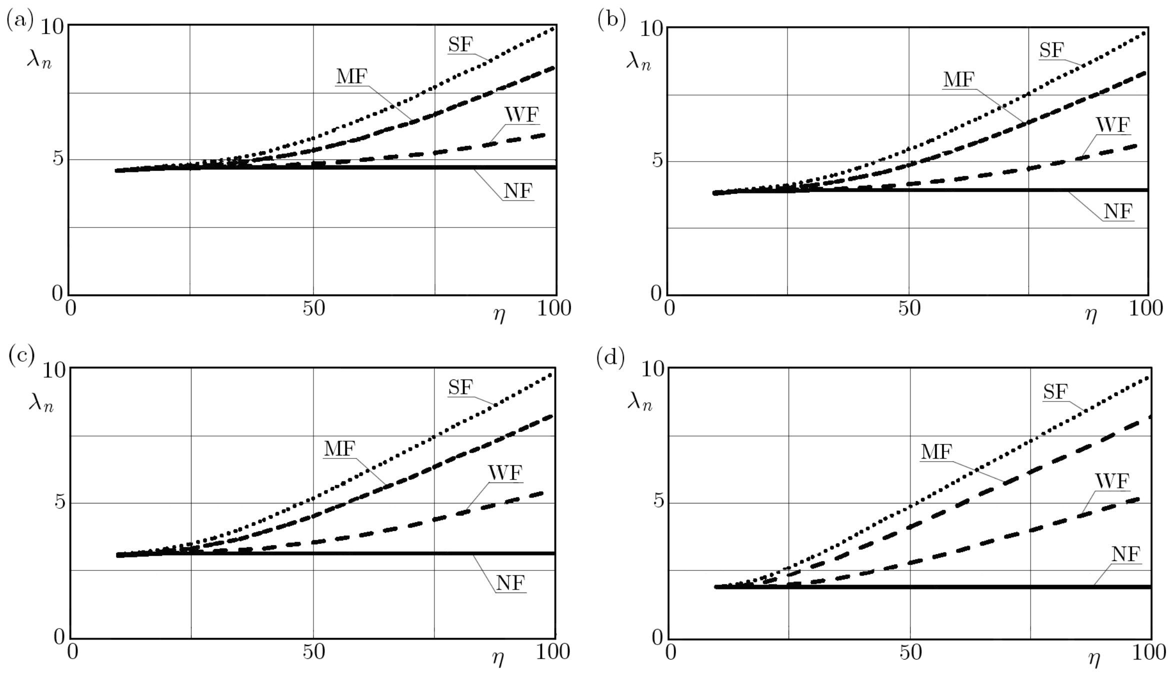

Fig. 5. Varaiations of λ_n with η and E_s; (a) C-C beams, (b) C-P beams, (c) P-P beams, (d) C-F beams

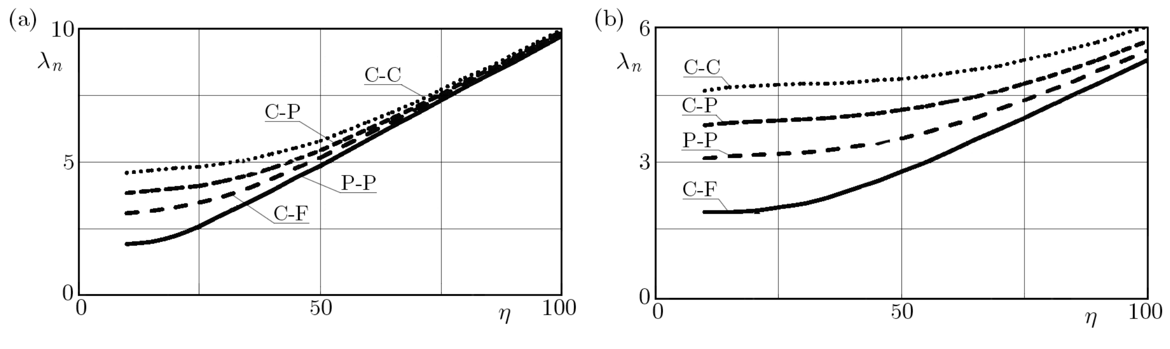

Fig. 6. Effect of the beam end condition on λ_n; (a) weak foundation (WF), (b) stiff foundation (SF)

The influence of the slenderness ratio increases with the increase of the foundation stiffness and vice versa. The effect of end conditions vanishes with the increase in the slenderness ratio for a weak foundation faster than in the case of stiff foundation (SF). The transition between different vibration modes is not obvious in the frequency charts, as in the absence of the axial load, the natural frequency of the first vibration mode is always the smaller one. In the case of axially loaded beams, the transition between the first and second mode is detected for slender beams with (P-P) and (C-C) end conditions.

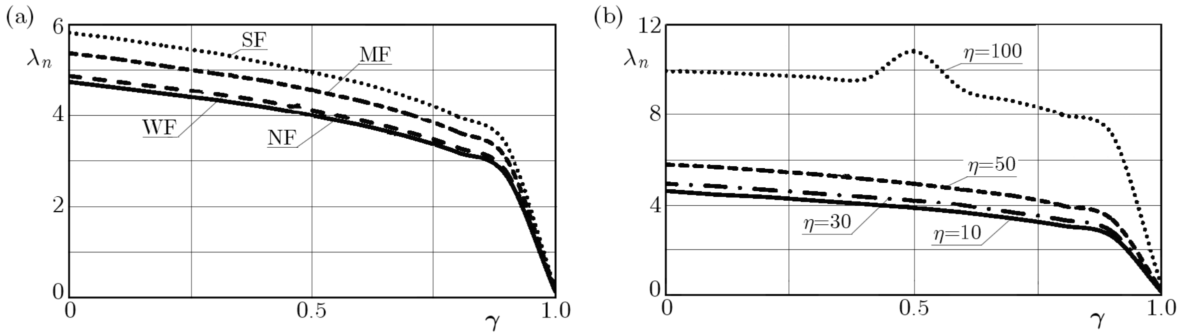

Fig. 7. Variations of λ_n with γ for C-C beams; (a) effect of E_s, (b) effect of η

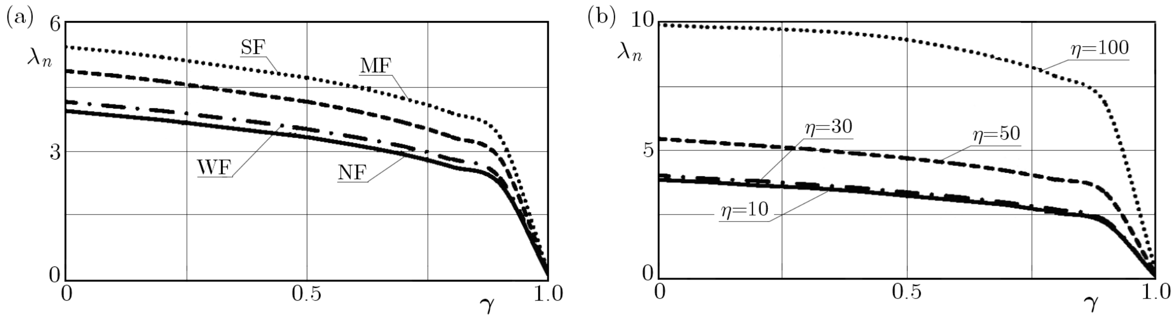

Fig. 8. Variations of λ_n with γ for P-C bemas; (a) effect of E_s, (b) effect of η

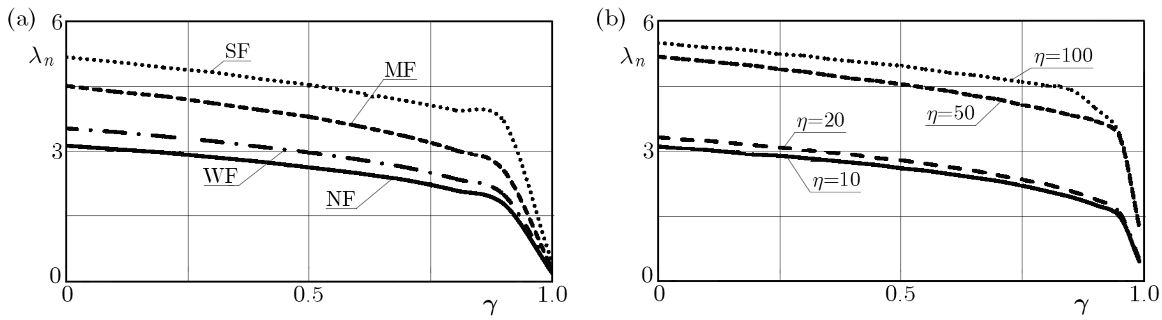

Fig. 9. Variations of λ_n with γ for P-P beams; (a) effect of E_s, (b) effect of η

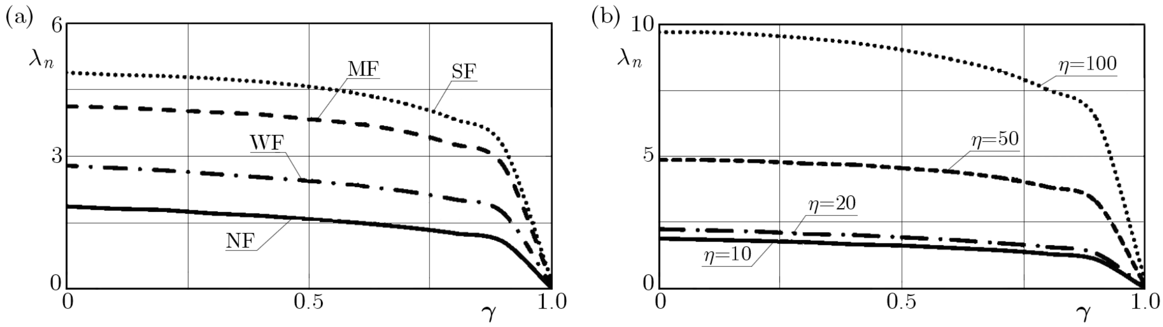

Fig. 10. Variations of λ_n with γ for C-F beams; (a) effect of E_s, (b) effect of η

In the case of axially loaded beams (Figs. 7-10), the effect of the loading ratio γ may be approximated by a linear relation up to $\gamma = 0.5$. Also, the influences of both the slenderness ratio and the foundation stiffness are more noticeable in the (C-F) case.

Furthermore, it is found that taking the rotational inertia of the beam into consideration decreases the natural frequencies of short beams, and the effect may be ignored as the slenderness ratio $\eta > 30$.

5. Conclusions

The RDM is introduced and employed in the investigation of the static and dynamic stability parameters of axially loaded beams resting on two-parameter foundations. Recursive functions of the problem are derived first, and then after applying the end conditions, the frequency equations accompanied with different end conditions are obtained. However, it is found that the accuracy of the obtained RDM expressions is greatly enhanced when the solution domain is transformed to the domain $[0, 1]$.

The critical loads required in static stability analysis and natural frequencies required in dynamic stability analysis are obtained and investigated.

It is observed that in the case of beams resting on elastic foundations, the critical load of the first buckling mode is not always the smallest critical load in contrast to that common fact in the case of beams without foundation. The critical load of a higher mode may be smaller than the critical load of a lower buckling mode. This phenomenon is also observed for the natural frequency, but in the presence of an axial load.

It is also concluded that both the influence of the foundation stiffness and the slenderness ratio are more noticeable for the (C-F) case.

Although the proposed RDM solution is applicable for forced vibration, the numerical results are limited to free vibration to calculate the natural frequencies which are required in stability analysis.

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