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Free Vibration of Structures by Radial Basis Function – Pseudospectral Method

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Abstract

The paper deals with the use of the radial basis function-pseudospectral method in vibration analysis of two-dimensional mechanical structures. The method combines meshless features of radial basis function (RBF) with efficiency and simplicity of the pseudospectral method. In present work the main emphasis is laid on appropriate assumption of the interpolant for the sought function due to the number of the boundary conditions in analysed problem. This interpolation function enables to obtain the weighting coefficients for derivative approximation in a governing equation. The method is applied to free vibration analysis of arbitrarily shaped membrane and plate.

Keywords: meshless methods, radial basis function, pseudospectral methods

1. Introduction

Due to some problems encountered during the application of the mesh discretization numerical techniques, in recent years, some methods that discretize the domain with scattered nodes are strongly developed. Many formulations of these so-called meshless techniques have been applied to solve problems from various disciplines of science. An interesting overview can be found in [1,2]. Some formulations of these methods take advantage of radial basis functions (RBF) [3] to approximate the sought solution of the problem analysed. Since the work of Hardy [4], it is well-known that these types of functions are very useful in scattered data approximation.

An interesting example of the mentioned methods is the approach that combines RBF approximation with pseudospectral method [5,6] (RBF-PS). In this approach, derivatives in the governing equation are approximated by a linear weighted sum of unknown function values from all over domain

$$\frac{\partial^r u(\mathbf{x})}{\partial x^p \, \partial y^q}\Big|_{\mathbf{x}=\mathbf{x}_i} = \sum_{j=1}^{N} a_{ij}^{(r)} u_j \tag{1}$$

where $a_{ij}^{(r)}$ are the weighting coefficients for the rth order derivative and N denotes the number of nodes $\mathbf{x}=(x, y)$. With these coefficients and by the use of collocation technique, the governing equation and boundary conditions are discretized reducing the problem to the set of algebraic equations. Since the method involves all nodal function values to approximate a derivative at a node, the method leads to relatively fast convergence, what has been proved by the examples [5].

2. RBF-PS for lower order equations

To obtain the weighting coefficients one has to start from the approximation of the unknown function u by the use of RBF

$$u(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \varphi\left(\left\|\mathbf{x} - \boldsymbol{\xi}_j\right\|\right) \tag{2}$$

where α_j are the interpolation coefficients and $\varphi\left(\left\|\mathbf{x} - \boldsymbol{\xi}_j\right\|\right)$ denote the radial function.

There are different types of RBS [3,5], but their common feature is the dependence on the distance between a collocation point $\mathbf{x}$ and the point $\boldsymbol{\xi}_j$ called as a center $\mathbf{x}, \boldsymbol{\xi}_j \in \mathbb{R}^n$. In this method, the centers are also considered as the collocation points.

From the interpolation problem one can express the interpolation coefficients in terms of function values, what can be put in the following matrix form

$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^{-1}\mathbf{u} \tag{3}$$

where $\boldsymbol{\alpha}$ denotes the vector of interpolation coefficients, $\mathbf{u}$ is the vector that contains the function values at the nodes and the entries of the interpolation matrix has the form: $\boldsymbol{\Phi}_{ij} = \varphi\left(\left\|\mathbf{x}_i - \boldsymbol{\xi}\right\|\right)\Big|_{\boldsymbol{\xi}=\mathbf{x}_j}$, $i, j = 1,...,N$.

Computing appropriate derivative of the interpolant (2) at each node of the domain and introducing the expression (3) one gets the weighting coefficients for the rth order derivative approximation

$$\mathbf{u}^{(r)} = \boldsymbol{\Phi}^{(r)}\boldsymbol{\Phi}^{-1}\mathbf{u} \tag{4}$$

where the entries of matrix $\boldsymbol{\Phi}^{(r)}$ are as follows: $\boldsymbol{\Phi}_{ij}^{(r)} = \dfrac{\partial^r \varphi(\|\mathbf{x} - \boldsymbol{\xi}\|)}{\partial x^p \, \partial y^q}\Bigg|_{\substack{\boldsymbol{\xi}=\mathbf{x}_j \\ \mathbf{x}=\mathbf{x}_i}}$

Once the weighting coefficients $\mathbf{A} = \boldsymbol{\Phi}^{(r)}\boldsymbol{\Phi}^{-1}$ are determined, the differential equation can be discretized.

In the method, the discretization of the mathematical model of a problem is carried out by the collocation technique. Therefore the approach presented can be directly applied to lower order equations that possess one boundary condition at the edge.

In the present work the method is used to solve eigenvalue problem for pre-stretched uniform membrane, for which the governing equation and boundary condition have the form

$$\Delta W = -\Lambda^2 W, \quad W = 0 \ \text{ for } \mathbf{x} \in \partial\Gamma \tag{5}$$

where Δ is Laplacian operator, W is the mode of vibration and $\Lambda = \omega\sqrt{\rho/T}$ is the wave-number (ω – circular frequency, ρ – mass per unit length, T – uniform tension per unit

length). The membrane of the shape presented in Fig.1 is analysed in the work. Irregularly distributed nodes are applied to discretize the domain – an example of the node distribution is shown in Fig. 1.

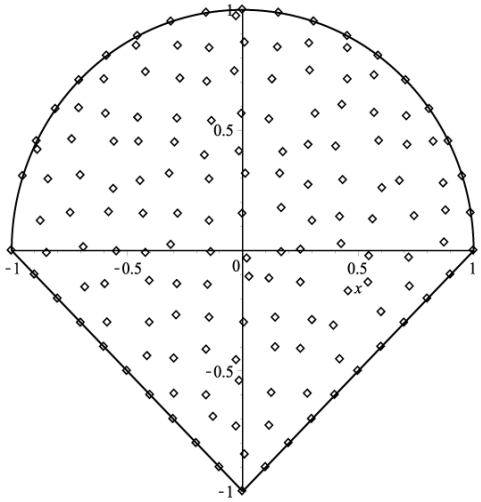

Figure 1. The membrane analysed in the work with an example of the node distribution

Using multiquadrics RBF, the weighting coefficients, described in general way by Eq. (4), for Laplacian operator have been determined. With these coefficients Eq. (5) is reduced to standard eigenvalue problem of the form

$$\mathbf{AW} = -\Lambda^2 \mathbf{W} \tag{6}$$

where vector **W** contains the nodal function values and **A** is the matrix reflecting the discrete form of the Laplacian operator.

The wavenumbers obtained from (6) are presented in Tab. 1 and some modes of vibration are shown in Fig. 2.

Table 1. Wavenumbers of the membrane for various numbers of nodes assumed.

	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
$N = 155$	2.7093	4.2283	4.3580	5.5679	5.9328
$N = 221$	2.7092	4.2263	4.3577	5.5616	5.9340
$N = 314$	2.7099	4.2292	4.3579	5.5676	5.9337
$N = 390$	2.7096	4.2278	4.3579	5.5648	5.9336
Reference results [9]	2.7097	4.2279	4.3579	5.5649	5.9336

The results presented in Tab. 1 are in great agreement with the reference values. The method indicates a proper convergence trend.

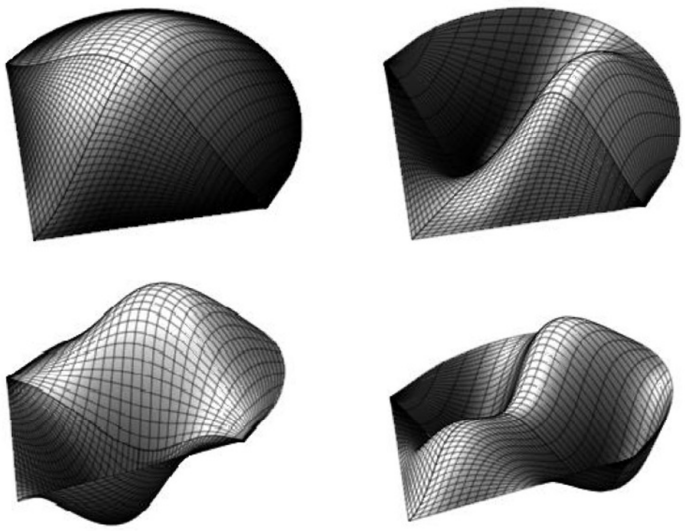

Figure 2. First four modes of vibration of the membrane

3. RBF-PS for higher order equations

For higher order equation, where more than one boundary condition is defined at an edge, one should write more than one discrete equation for each boundary node. It leads to overdetermined system of algebraic equations. Although this system can be solved by least squares technique this approach does not reflect the main idea of the method based on the interpolating function.

To make the method be conveniently applied for higher order equations, one can extend the interpolation formula (2) introducing the additional degrees of freedom. These quantities should correspond to differential operators contained in boundary conditions. The approach can be viewed as a Hermite interpolation problem defined for RBF and in the case of two boundary conditions can be generally written as

$$u(\mathbf{x}) = \sum_{j=1}^{N^I} \alpha_j \varphi\left(\|\mathbf{x}-\boldsymbol{\xi}\|\right)\Big|_{\boldsymbol{\xi}=\mathbf{x}_j^I} + \sum_{j=1}^{N^B} \beta_j \left[B_1^{\boldsymbol{\xi}} \varphi\left(\|\mathbf{x}-\boldsymbol{\xi}\|\right)\right]_{\boldsymbol{\xi}=\mathbf{x}_j^B} + \sum_{j=1}^{N^B} \gamma_j \left[B_2^{\boldsymbol{\xi}} \varphi\left(\|\mathbf{x}-\boldsymbol{\xi}\|\right)\right]_{\boldsymbol{\xi}=\mathbf{x}_j^B} \quad (7)$$

where N^I and N^B denote the numbers of interior ($\mathbf{x}_i^I$) and boundary ($\mathbf{x}_i^B$) nodes, respectively, $B_1^{\boldsymbol{\xi}}$ and $B_2^{\boldsymbol{\xi}}$ are differential operators that act on the radial function treated as a function of $\boldsymbol{\xi}$ variable.

Following the same procedure as previous one can solve interpolation problem (7) and express the interpolation coefficients α_j, β_j, γ_j in terms of function values as well as the values of the derivatives of the function defined at boundaries. Then, by computing

appropriate derivative of the interpolant (7) at each interior node of the domain and introducing the expression for interpolation coefficients one obtains

$$\overline{\mathbf{u}}^{(r)} = \overline{\mathbf{\Phi}}^{(r)} \overline{\mathbf{\Phi}}^{-1} \overline{\mathbf{u}} \tag{8}$$

where the objects from Eq. (8) have the forms

$$\overline{\mathbf{\Phi}}^{(r)} = \left[\mathbf{\Phi}_{L^{\mathbf{x}}} \quad \mathbf{\Phi}_{L^{\mathbf{x}} B_1^{\boldsymbol{\xi}}} \quad \mathbf{\Phi}_{L^{\mathbf{x}} B_2^{\boldsymbol{\xi}}} \right], \overline{\mathbf{\Phi}} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Phi}_{B_1^{\boldsymbol{\xi}}} & \mathbf{\Phi}_{B_2^{\boldsymbol{\xi}}} \\ \mathbf{\Phi}_{B_1^{\mathbf{x}}} & \mathbf{\Phi}_{B_1^{\mathbf{x}} B_1^{\boldsymbol{\xi}}} & \mathbf{\Phi}_{B_1^{\mathbf{x}} B_2^{\boldsymbol{\xi}}} \\ \mathbf{\Phi}_{B_2^{\mathbf{x}}} & \mathbf{\Phi}_{B_2^{\mathbf{x}} B_1^{\boldsymbol{\xi}}} & \mathbf{\Phi}_{B_2^{\mathbf{x}} B_2^{\boldsymbol{\xi}}} \end{bmatrix}, \overline{\mathbf{u}}^{(r)} = \begin{bmatrix} \mathbf{u}_{L^{\mathbf{x}}} \\ \mathbf{u}_{L^{\mathbf{x}} B_1^{\mathbf{x}}} \\ \mathbf{u}_{L^{\mathbf{x}} B_2^{\mathbf{x}}} \end{bmatrix}, \overline{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ \mathbf{u}_{B_1^{\mathbf{x}}} \\ \mathbf{u}_{B_2^{\mathbf{x}}} \end{bmatrix}$$

In the above expressions $L^{\mathbf{x}}$ is the differential operator corresponding to the rth order derivative contained in governing equation, $B_1^{\mathbf{x}}$ and $B_2^{\mathbf{x}}$ denote the same differential operators as $B_1^{\boldsymbol{\xi}}$ and $B_2^{\boldsymbol{\xi}}$, but acting on the radial function viewed as a function of **x** variable. The details of the approach as well as the entries of the objects presented can be found in [7].

Since vector $\overline{\mathbf{u}}$ in Eq. (8) contains the values of the derivatives defined at boundary nodes, all boundary conditions can be directly involved during discretization process.

The approach presented has been used in the work to solve the free vibration problem for thin, isotropic, plate of the shape presented in Fig. 3. Governing equation for this problem is as follows

$$\Delta^2 w = \Omega^2 w \tag{9}$$

where w denotes the form of vibration and Ω is the free vibration parameter related to free vibration frequency by the formula $\Omega = \omega a^2 \sqrt{\rho h / D}$ (ρ – density of the plate material, D – plate stiffness, h – plate thickness, a – characteristic plate dimension).

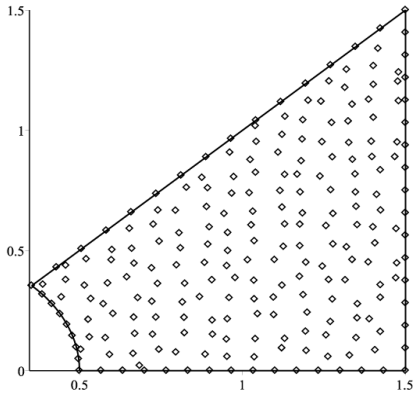

Figure 3. Triangular plate with corner cutout with an example of node distribution

In present paper, the plates with combination of simply supported and clamped boundary conditions are considered

$$w = 0, \quad B^{\mathbf{x}} w = 0 \text{ for } \partial\Gamma \tag{10}$$

For simply supported edge (S) differential operator $B^{\mathbf{x}}$ has the form

$$B^{\mathbf{x}} = \left(\cos^2(\theta) + \nu \sin^2(\theta)\right)\frac{\partial^2}{\partial x^2} + \left(\sin^2(\theta) + \nu \cos^2(\theta)\right)\frac{\partial^2}{\partial y^2} + 2(1-\nu)\cos(\theta)\sin(\theta)\frac{\partial^2}{\partial x \partial y}$$

and for the clamped edge (C) $B^{\mathbf{x}}$ is as follows

$$B^{\mathbf{x}} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$$

where θ is the angle between the normal to the plate boundary and the x-axis.

Adapting the approach presented in this section one can reduce Eq. (9) to algebraic problem of the following form

$$\mathbf{A}\mathbf{w} = \Omega^2 \mathbf{w} \tag{11}$$

where vector **w** contains the nodal function values at the inner nodes and the values of the function as well as the values of derivatives at boundary nodes and **A** is the matrix reflecting the discrete form of the biharmonic operator.

Taking into account that only function values at interior nodes can have non zero values, appropriate columns of the matrix **A** has to be deleted and then standard, algebraic eigenvalue problem can be solved.

The eigenvalues obtained for various configurations of boundary conditions are presented in the Tab. 2 and some chosen form of vibration are shown in Fig. 4.

Table 2. Results for the triangular plate with corner cutout

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS					
N =235, N^I =175	22.262	45.692	58.994	77.843	96.239
N =323, N^I =256	23.198	47.163	60.624	79.699	98.017
Reference results	22.365	47.187	58.968	80.812	97.498
CCCC					
N =235, N^I =175	41.655	71.250	87.967	110.389	130.327
N =323, N^I =256	41.786	71.256	87.915	110.685	130.421
Reference results	41.787	71.256	87.896	110.688	130.415
SCSC					
N =323, N^I =256	28.761	55.707	70.089	89.372	113.577
Reference results	28.869	57.071	69.634	91.974	113.957

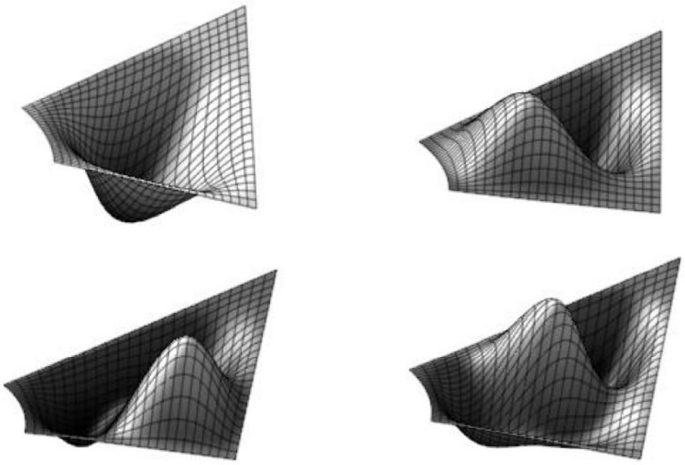

Figure 4. First four modes of vibration of triangular plate with corner cutout

The reference results presented in Tab. 2 have been obtained by the differential quadrature method combined with coordinate transformation. The details of this approach can be found in [8].

The results presented in Tab. 2 show great agreement with reference values. Regardless of the node distribution, the eigenvalues computed are very close to reference results for each configuration of boundary conditions assumed in the work.

4. Conclusions

In the paper the RBF-PS method is applied to free vibration analysis of two-dimensional structures. The basic approach of this method can be easily used for lower order equations, while an extension of this method can be conveniently applied for higher order equations that possess more than one boundary conditions at an edge. Due to the use of RBF, the discretization of the domain can be done by irregularly (randomly) distributed nodes. This feature facilitates the analysis of arbitrarily shaped structures. To show the usefulness of the method, the free vibration analysis for irregularly shaped membrane and plate has been carried out. The results indicate that the method has a potential to become an effective, meshless, numerical technique for wide range of problems.

References

1. T. Belytschko, Y. Krongauz, D. Organ, M. Flrming, P. Krysl, *Meshless methods: an overview and recent developments,* Comp**.** Meth**.** Appl**.** Mech. Eng., **139** (1996) 3-47.
2. G.R. Liu, *Meshlees Methods – Moving beyond the Finite Element Method*, CRC Press, Boca Raton, Florida, 2003.

3. M.D. Buhmann, *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, Cambridge 2003.
4. R.L. Hardy*, Multiquadric equations of topography and other irregular surfaces*, J. Geophysics Res. **176** (1971) 1905-1915
5. G.E. Fasshauer, *Meshfree Approximation Methods with Matlab*, World Scientific Publishing, Singapore, 2007.
6. A.J.M Ferreira, G.E. Fasshauer, *Analysis of natural frequencies of composite plates by an RBF-pseudospectral method*, Compos. Struct., **79** (2007) 202-210.
7. A. Krowiak, *Hermite type RBF-DQ method applied to vibration analysis of plates*, Proceedings of the 12th Conference on Dynamical Systems: Theory and Applications, Łódź 2013, 391-400.
8. M. Malik, C.W. Bert, *Vibration analysis of plates with curvilinear quadrilateral planforms by DQM using blending functions*, J. Sound Vib., **230** (2000) 949-954.
9. S.W. Kang, J.M. Lee, Y.J. Kang, *Vibration analysis of arbitrarily shaped membranes using non-dimensional dynamic influence function*, J. Sound Vib., **221** (1999) 117-132.