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Explicit "ballistic M-model": a refinement of the implicit "modified point mass trajectory model"

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Abstract. Various models of a projectile in a resisting medium are used. Some are very simple, like the "point mass trajectory model", others, like the "rigid body trajectory model", are complex and hard to use, especially in Fire Control Systems due to the fact of numeric complexity and an excess of less important corrections. There exist intermediate ones – e.g. the "modified point mass trajectory model", which unfortunately is given by an implicitly defined differential equation as Sec. 1 discusses. The main objective of this paper is to present a way to reformulate the model obtaining an easy to solve explicit system having a reasonable complexity yet not being parameter-overloaded. The final form of the M-model, after being carefully derived in Sec. 2, is presented in Subsec. 2.5.

Key words: ballistics, equations of motion, projectile path, modified point mass trajectory model, MPMTM, projectile deflection.

1. Preliminaries: physical assumptions and common models

1.1. Brief discussion of different ballistic models and their physical assumptions. We decided to start our work by discussing physical basics of modelling a projectile's trajectory in a resisting medium. Most of the statements in this section are meant as a preliminary and are made rigorous in progress of this paper. We discuss the general properties of three typical models:

- the point mass trajectory model [1],
- the rigid body trajectory model [1–4],
- the modified point mass trajectory model [5, 6].

Commonly by the "point mass trajectory model" one understands a very simple physical model of the trajectory of a projectile in which the only acting forces are external forces, e.g. gravity, and the head-on drag force. Such a model describes a movement that is planar and does not explain such phenomena as the drift caused by gyroscopic precession and thus induced lift forces.

On the other hand, one can form a highly sophisticated "rigid body trajectory model" of a ballistic projectile, which treats the projectile as a six-degree-of-freedom physical object with axial symmetry and non-zero moments of inertia I_x and I_y, being respectively the moment of inertia along the axis of the projectile and the moment of inertia in the plane perpendicular to the projectile's axis. This model has the capabilities of explaining most physical phenomena happening along the flight trajectory but is unfortunately quite cumbersome. In addition, such "rigid body model" requires as input a huge number of various coefficients which cannot be fitted to poor quality experimental data, or even worse – their fitting might be considered highly inaccurate and ambiguous. To derive equations of motion for aerial objects treated as rigid bodies mostly Newtonian approach is used, i.e. forces, momenta or momentum and angular momentum conservation laws. Examples could be found in [7–9]. Sometimes more involved theoretical mechanics is used, e.g. Lagrangian formulation, see [10], or Boltzmann-Hamel equations like in [11]. In this paper, as it is described, we use a four degree of freedom model and to keep things simple – restrict ourselves to forces and momenta.

To find a moderately complex solution but still explaining the drift of a projectile without having to plunge into the depths of an over-parametrized troublesome theory the "modified point mass model" has been derived. Its physical foundation is to add one degree of freedom to the "point mass model" and at the same time assume that the moment of inertia $I_y = 0$. This makes the time of acquiring dynamic equilibrium infinitesimally short. There are various presentations of this type of models. The common problem is that mostly the "modified point mass models" are defined by an implicit ODE (ordinary differential equation). Implicit ODEs define the derivative as an implicit function, i.e. the zero point of a complicated expression. That makes such models hard to solve and questions may arise whether the numerical algorithm is rigorous and does not "fall into traps". The main objective of this paper is to present a way to reformulate the model obtaining an explicit system of ODEs, which is solvable using basic numerical methods. As a start we present an implicit formulation of a "modified point mass model".

1.2. Exemplary implicit modified point mass model. After [5] and [6] for informative purposes we present a version

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of the implicit modified point mass model. The system is as follows

$$\dot{\mathbf{x}} = \mathbf{u}, \tag{1a}$$

$$m \cdot \dot{\mathbf{u}} = \mathbf{DF} + \mathbf{LF} + \mathbf{MF} + m\mathbf{g} + m\mathbf{\Lambda}, \tag{1b}$$

$$I_x \cdot \dot{p} = \mathrm{SDM}, \tag{1c}$$

where $\mathbf{x}$ is the three-dimensional position vector, $\mathbf{u}$ is the velocity vector with respect to a ground reference system[1] and p is the angular velocity of the spinning motion, i.e. the rate of roll or axial spin. Let us remark that p is treated as a scalar in this model – we adopt the convention to use bold faced letters for vectors and standard letters for norms. In the system (1) the torque slowing down the spin is

$$\mathrm{SDM} = \frac{\rho v^2}{2} S d \mathrm{C_{spin}} \frac{p\,d}{v}, \tag{2}$$

where $\mathrm{C_{spin}}$ is the spinning drag coefficient, ρ the air density function and d the caliber. Usually S, being the cross sectional area of the projectile, is taken to be $\frac{1}{4}\pi d^2$. The forces changing $\mathbf{u}$ we propose[2] as

$$\mathbf{DF} = -\frac{\rho v^2}{2} S \left(\mathrm{C_{D0}} + \mathrm{C_{D\alpha^2}} \cdot \alpha_{\mathrm{e}}^2\right) \frac{\mathbf{v}}{v}, \tag{3a}$$

$$\mathbf{LF} = \frac{\rho v^2}{2} S \left(\mathrm{C_{L\alpha}} + \mathrm{C_{L\alpha^3}} \cdot \alpha_{\mathrm{e}}^2\right) \boldsymbol{\alpha}_{\mathrm{e}}, \tag{3b}$$

$$\mathbf{MF} = -\frac{\rho v^2}{2} S \frac{p\,d}{v} \mathrm{C_{mag\text{–}f}} \boldsymbol{\alpha}_{\mathrm{e}} \times \frac{\mathbf{v}}{v}. \tag{3c}$$

Respectively the forces (3) are

- the total drag force[3], with $\mathrm{C_{D0}}$, $\mathrm{C_{D\alpha^2}}$ being the zero-yaw drag coefficient and the yaw drag coefficient,
- the lift force, with $\mathrm{C_{L\alpha}}$, $\mathrm{C_{L\alpha^3}}$ being respectively the linear and cubic lift force coefficients,
- the Magnus force, where obviously $\mathrm{C_{mag\text{–}f}}$ is the Magnus force coefficient.

As one can see, only $\mathrm{C_{D0}}$ and $\mathrm{C_{spin}}$ are dimensionless, the other coefficients dimensions depend on the dimension of the angle α_{e}. Later on we shall introduce only dimensionless quantities as we feel it is the more readable way.

Additionally, $\mathbf{g}$ is the average gravitational acceleration and $\mathbf{\Lambda}$ the Coriolis acceleration. These are not important in detail and we shall not elaborate. It turns out that we can divide forces into two groups: aerodynamic forces and external forces. We see in the next section that the concrete form of the external forces is not important to the model formulation.

However, let us comment on the difference between $\mathbf{v}$ and $\mathbf{u}$, respectively the velocity of the projectile with respect to the air and the velocity of the projectile with respect to a ground-fixed reference system. We define

$$\mathbf{v} = \mathbf{u} - \mathbf{w}, \tag{4}$$

where by $\mathbf{w}$ we mean the wind velocity, i.e. the velocity of the air with respect to the chosen ground-fixed reference system.

The last expression that needs to be explained, and turns out to be the most troublesome one, is $\boldsymbol{\alpha}_{\mathrm{e}}$. We notice from the expressions giving the forces (3) that the direction of the vector $\boldsymbol{\alpha}_{\mathrm{e}}$, i.e. $\frac{\boldsymbol{\alpha}_{\mathrm{e}}}{\alpha_{\mathrm{e}}}$, gives the direction of the lift force $\mathbf{LF}$, i.e a direction lying in the plane perpendicular to the direction of the air flow. The magnitude (or norm) α_{e} defines the effective attack angle, or yaw of repose, that is deflecting the projectile's motion. To calculate $\boldsymbol{\alpha}_{\mathrm{e}}$ we notice that it has to satisfy a dynamical equilibrium equation, which in fact means equality between the overturning aerodynamic moment of force and the stabilizing torque of the gyroscopic precession, we write

$$\frac{\rho v^2}{2} S d \left(\mathrm{C_{M\alpha}} + \mathrm{C_{M\alpha^3}} \cdot \alpha_{\mathrm{e}}^2\right) \boldsymbol{\alpha}_{\mathrm{e}} = -I_x p \frac{\mathbf{v} \times \dot{\mathbf{u}}}{v^2}. \tag{5}$$

Remember that due to the assumption that $I_y = 0$ the equilibrium is attained infinitesimally fast, which was one of the physical paradigms in our model.

Let us stop for a moment though at this point our main problem shows itself! The vector $\boldsymbol{\alpha}_{\mathrm{e}}$ depends on $\dot{\mathbf{u}}$, which makes the differential equation being defined by an implicit function. The main purpose of the paper, apart from shedding light on the physics in the "modified point mass trajectory model", is to formulate the equations in an explicit and elegant way. We now move to a general formulation and derivation of the announced system of explicit ODEs.

1.3. Reformulation and general presentation of the implicit system.

Let us reformulate the Eqs. (1) to a general form which we need for our derivation. We write

$$\dot{\mathbf{x}} = \mathbf{u}, \tag{6a}$$

$$m\dot{\mathbf{u}} = \mathbf{AF} + \mathbf{EF}, \tag{6b}$$

$$I_x \dot{p} = \mathrm{SDM}. \tag{6c}$$

We have introduced $\mathbf{AF}$ and $\mathbf{EF}$. By the former acronym we mean "aerodynamic forces", whereas by the latter we mean "external forces". The aerodynamic forces are all the forces depending additionally on $\boldsymbol{\alpha}_{\mathrm{e}}$. Using (3) we define

$$\mathbf{AF}\left(\mathbf{x},\, \mathbf{u},\, p;\, \mathbf{w},\, \boldsymbol{\alpha}_{\mathrm{e}}\right) = \mathbf{DF} + \mathbf{LF} + \mathbf{MF}, \tag{7}$$

whereas the external forces are all the forces that do not depend on $\boldsymbol{\alpha}_{\mathrm{e}}$, i.e.

$$\mathbf{EF} = \mathbf{EF}\left(\mathbf{x},\, \mathbf{u},\, p;\, \mathbf{w}\right). \tag{8}$$

The concrete choice of the forces adding to $\mathbf{EF}$ is irrelevant from the point of view of this derivation, what is important is the independence from $\boldsymbol{\alpha}_{\mathrm{e}}$.

We have commented earlier that the equilibrium angle $\boldsymbol{\alpha}_{\mathrm{e}}$ depends on $\dot{\mathbf{u}}$ which makes our system implicitly defined. Let us write in full extent

[1]The particular choice of the reference system is irrelevant to our vector-oriented presentation. Hence, we do not provide any more details on this matter.
[2]There might be many points of view on the matter of signs, which are strictly conventional as one can always redefine the coefficients.
[3]By "total" we mean head-on drag plus induced drag.

$$\dot{\mathbf{x}} = \mathbf{u}, \tag{9a}$$

$$m\dot{\mathbf{u}} = \Big(\mathbf{AF}\left(\mathbf{x},\, \mathbf{u},\, p;\, \mathbf{w},\, \boldsymbol{\alpha}_{\mathrm{e}}(\mathbf{x},\, \mathbf{u},\, p,\, \dot{\mathbf{u}};\, \mathbf{w})\right) + \mathbf{EF}(\mathbf{x},\, \mathbf{u},\, p;\, \mathbf{w})\Big), \tag{9b}$$

$$I_x \dot{p} = \mathrm{SDM}. \tag{9c}$$

The crucial problem is hidden in the second equation. Even though the system is only valid as a whole[4] we shall forget about the other two equations in (9) and direct our attention toward the equation (9b) giving the implicit function $\dot{\mathbf{u}}$.

2. Derivation of the explicit ballistic M-model

2.1. Further assumptions: order of truncation and wind homogeneity. To start the process of deriving the explicit function $\dot{\mathbf{u}}(\mathbf{x},\, \mathbf{u},\, p;\, \mathbf{w})$ we will make some further simplifications. These are minor simplifications which are to make the derivation less cumbersome. We will comment on them extensively. First let us note that the expressions for the forces (3) are already written under the assumption that $\|\boldsymbol{\alpha}_{\mathrm{e}}\|$ is small[5]. Thus most of the trigonometric functions and drag coefficients are presented as truncated Taylor series. Some of the forces (3) are written up to $\mathcal{O}(\alpha_{\mathrm{e}}^3)$ order, other, like the Magnus force, are considered only up to $\mathcal{O}(\alpha_{\mathrm{e}})$ terms. Zeros of a nonlinear polynomial system of degree three are possible to find analytically but the zeros might not be unique! The uniqueness and numerical properties would depend on all the coefficients in this system and therefore are hard to control, not to mention not worth to control in some cases as the effects could be small. For a clear presentation we reduce ourselves up to $\mathcal{O}(\alpha^2)$ terms.

According to the above mentioned we assume

$$\mathrm{C}_{\mathrm{L}\alpha^3} \equiv \mathrm{C}_{\mathrm{M}\alpha^3} \equiv 0. \tag{10}$$

Additionally we state that

$$\dot{\mathbf{w}} \equiv 0, \tag{11}$$

which means that at least in the interval of integration the wind is constant and translation invariant. As a result of this (4) yields

$$\dot{\mathbf{u}} = \frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{v} + \mathbf{w}) = \dot{\mathbf{v}} + \dot{\mathbf{w}} = \dot{\mathbf{v}}. \tag{12}$$

The property (12) allows us to present our derivation in a more simple way, nevertheless we discuss inhomogeneous wind in a rigorous way later, after we have dealt with the less complex derivation.

Now the forces in the system (9) simplify a little, we use the variables $\mathbf{x}$, $\mathbf{v}$, p and write

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{w}, \tag{13a}$$

$$m\dot{\mathbf{v}} = \mathbf{AF} + \mathbf{EF}\left(\mathbf{x},\, \mathbf{v},\, p;\, \mathbf{w}\right), \tag{13b}$$

$$I_x \dot{p} = \mathrm{SDM}, \tag{13c}$$

where the summands in $\mathbf{AF}$ are

$$\mathbf{DF} = -\frac{\rho v^2}{2} S \left(\mathrm{C}_{\mathrm{D}0} + \widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}\left(\mathrm{C}_{\mathrm{M}\alpha} \cdot \alpha_{\mathrm{e}}\right)^2\right) \frac{\mathbf{v}}{v}, \tag{14a}$$

$$\mathbf{LF} = \frac{\rho v^2}{2} S \widehat{\mathrm{C}}_{\mathrm{L}\alpha} \mathrm{C}_{\mathrm{M}\alpha} \cdot \boldsymbol{\alpha}_{\mathrm{e}}, \tag{14b}$$

$$\mathbf{MF} = -\frac{\rho v^2}{2} S \frac{p\,d}{v} \widehat{\mathrm{C}}_{\mathrm{mag-f}} \mathrm{C}_{\mathrm{M}\alpha} \cdot \boldsymbol{\alpha}_{\mathrm{e}} \times \frac{\mathbf{v}}{v}. \tag{14c}$$

Above in (14) we have introduced dimensionless coefficients $\widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}$, $\widehat{\mathrm{C}}_{\mathrm{L}\alpha}$ and $\widehat{\mathrm{C}}_{\mathrm{mag-f}}$ given by

$$\widehat{\mathrm{C}}_{\mathrm{D}\alpha^2} = \frac{\mathrm{C}_{\mathrm{D}\alpha^2}}{(\mathrm{C}_{\mathrm{M}\alpha})^2},\ \widehat{\mathrm{C}}_{\mathrm{L}\alpha} = \frac{\mathrm{C}_{\mathrm{L}\alpha}}{\mathrm{C}_{\mathrm{M}\alpha}} \text{ and } \widehat{\mathrm{C}}_{\mathrm{mag-f}} = \frac{\mathrm{C}_{\mathrm{mag-f}}}{\mathrm{C}_{\mathrm{M}\alpha}}. \tag{15}$$

Additionally, the dynamic equilibrium equation simplifies to the dimensionless

$$\mathrm{C}_{\mathrm{M}\alpha} \cdot \boldsymbol{\alpha}_{\mathrm{e}} = -\frac{2I_x\, p}{d\,\rho v^2\, S} \frac{\mathbf{v}}{v} \times \frac{\dot{\mathbf{u}}}{v} = -\frac{2I_x\, p}{d\,\rho v^2\, S} \frac{\mathbf{v}}{v} \times \frac{\dot{\mathbf{v}}}{v}, \tag{16}$$

where the last equation comes from assumption (12), i.e. $\dot{\mathbf{u}} = \dot{\mathbf{v}}$. The main task placed before us is to eliminate $\mathrm{C}_{\mathrm{M}\alpha} \cdot \boldsymbol{\alpha}_{\mathrm{e}}$ from the differential system. To handle that problem we need to introduce a well-adapted coordinate system.

2.2. Phase space reference frame: air-flow reference system. Our phase space is seven dimensional as one can clearly see from the set of variables $\mathbf{x}$, $\mathbf{v}$, p. We will not abandon the Cartesian coordinate chart for $\mathbf{x}$ nor will we change anything in p. The only modification is to use sort-of spherical coordinates for the velocity vector part of our phase space.

We define

$$\mathbf{v} = v \cdot \widehat{\mathbf{e}}_{\boldsymbol{v}} \tag{17}$$

and three time-dependent orthonormal vectors

$$\widehat{\mathbf{e}}_{\boldsymbol{v}} = \begin{bmatrix} \cos\gamma_{\mathrm{a}}\, \cos\chi_{\mathrm{a}} \\ \cos\gamma_{\mathrm{a}}\, \sin\chi_{\mathrm{a}} \\ \sin\gamma_{\mathrm{a}} \end{bmatrix}, \tag{18a}$$

$$\frac{1}{\cos\gamma_{\mathrm{a}}} \frac{\partial\, \widehat{\mathbf{e}}_{\boldsymbol{v}}}{\partial\, \chi_{\mathrm{a}}} = \widehat{\mathbf{e}}_{\boldsymbol{\chi}} = \begin{bmatrix} -\sin\chi_{\mathrm{a}} \\ \cos\chi_{\mathrm{a}} \\ 0 \end{bmatrix}, \tag{18b}$$

$$\frac{\partial\, \widehat{\mathbf{e}}_{\boldsymbol{v}}}{\partial\, \gamma_{\mathrm{a}}} = \widehat{\mathbf{e}}_{\boldsymbol{\gamma}} = \begin{bmatrix} -\sin\gamma_{\mathrm{a}}\, \cos\chi_{\mathrm{a}} \\ -\sin\gamma_{\mathrm{a}}\, \sin\chi_{\mathrm{a}} \\ \cos\gamma_{\mathrm{a}} \end{bmatrix}. \tag{18c}$$

From now on, if we write $\mathbf{v}$ we mostly mean (17), especially in the announced systems of ODEs.

[4] In some cases the air density is constant and Eq. (9a) is independent but usually the density is a function of altitude or more, e.g. $\rho(\mathbf{x})$. In addition, the speed of sound also changes with altitude.

[5] Small with respect to a radian. In fact $\boldsymbol{\alpha}_{\mathrm{e}}$ is not dimensionless as it is an angle, more rigorously $\mathrm{C}_{\mathrm{M}\alpha} \cdot \|\boldsymbol{\alpha}_{\mathrm{e}}\|$ is the correct dimensionless parameter.

[6] In agreement with the commonly used symbols, γ_{a} and χ_{a} may be the air-path inclination angle and the air-path azimuth angle. Naturally, one could choose a different convention and everything will still hold.

The frame (18) is right-oriented. To prove this let us notice[6] that for $\gamma_{\mathrm{a}}=0$, $\chi_{\mathrm{a}}=0$ it holds that $\widehat{\mathbf{e}}_{\boldsymbol{v}}=\widehat{\mathbf{e}}_{\boldsymbol{x}}$, $\widehat{\mathbf{e}}_{\boldsymbol{\chi}}=\widehat{\mathbf{e}}_{\boldsymbol{y}}$ and $\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}=\widehat{\mathbf{e}}_{\boldsymbol{z}}$. Thus we deduce that

$$\widehat{\mathbf{e}}_{\boldsymbol{v}} \times \widehat{\mathbf{e}}_{\boldsymbol{\chi}} = \widehat{\mathbf{e}}_{\boldsymbol{\gamma}}, \tag{19a}$$
$$\widehat{\mathbf{e}}_{\boldsymbol{\chi}} \times \widehat{\mathbf{e}}_{\boldsymbol{\gamma}} = \widehat{\mathbf{e}}_{\boldsymbol{v}}, \tag{19b}$$
$$\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} \times \widehat{\mathbf{e}}_{\boldsymbol{v}} = \widehat{\mathbf{e}}_{\boldsymbol{\chi}}. \tag{19c}$$

We use (19) many times in the process of the derivation which explains the purpose of this seemingly obvious statement. From the above one notices that γ_{a} is the elevation angle of $\mathbf{v}$ measured from the horizontal direction, i.e. the air-path inclination angle and χ_{a} is the azimuth angle of $\mathbf{v}$, i.e. the air-path azimuth angle[7]. Also let us stress that if we use a 3-vector column notation we write the coordinates in the order x, y, z starting from the top, this again is merely a convention. In short, in future derivations, we call (18) the air-flow frame.

When analyzing trajectories one would rather want to study

$$v_x = \mathbf{v} \circ \widehat{\mathbf{e}}_{\boldsymbol{x}}, \qquad v_y = \mathbf{v} \circ \widehat{\mathbf{e}}_{\boldsymbol{y}}, \qquad v_z = \mathbf{v} \circ \widehat{\mathbf{e}}_{\boldsymbol{z}}. \tag{20}$$

For the convenience of the reader we also provide the inverse span. It holds that

$$\widehat{\mathbf{e}}_{\boldsymbol{x}} = \cos\gamma_{\mathrm{a}}\,\cos\chi_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{v}} - \sin\gamma_{\mathrm{a}}\,\cos\chi_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} - \sin\chi_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\chi}}, \tag{21a}$$
$$\widehat{\mathbf{e}}_{\boldsymbol{y}} = \cos\gamma_{\mathrm{a}}\,\sin\chi_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{v}} - \sin\gamma_{\mathrm{a}}\,\sin\chi_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \cos\chi_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\chi}}, \tag{21b}$$
$$\widehat{\mathbf{e}}_{\boldsymbol{z}} = \sin\gamma_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{v}} + \cos\gamma_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}. \tag{21c}$$

Finally, let us note the simple derivation

$$\begin{aligned}\dot{\mathbf{v}} &= \frac{\mathrm{d}}{\mathrm{d}\,t}(v\,\widehat{\mathbf{e}}_{\boldsymbol{v}}) = \dot{v}\widehat{\mathbf{e}}_{\boldsymbol{v}} + v\frac{\mathrm{d}\,\widehat{\mathbf{e}}_{\boldsymbol{v}}}{\mathrm{d}\,t} \\ &= \dot{v}\widehat{\mathbf{e}}_{\boldsymbol{v}} + v\left(\dot{\gamma}_{\mathrm{a}}\frac{\partial\,\widehat{\mathbf{e}}_{\boldsymbol{v}}}{\partial\,\gamma_{\mathrm{a}}} + \dot{\chi}_{\mathrm{a}}\frac{\partial\,\widehat{\mathbf{e}}_{\boldsymbol{v}}}{\partial\,\chi_{\mathrm{a}}}\right) \\ &= \dot{v}\widehat{\mathbf{e}}_{\boldsymbol{v}} + v\,(\dot{\gamma}_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \dot{\chi}_{\mathrm{a}}\,\cos\gamma_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\chi}}).\end{aligned} \tag{22}$$

The result of (22) will be the key ingredient to an elegant presentation and explicit derivation of the differential system defining the M-model.

2.3. Dimensionless physical parameters $\widehat{p}$, $\widehat{I}_x$. We introduce two dimensionless parameters

$$\widehat{I}_x = \frac{I_x}{m\,d^2}, \tag{23a}$$
$$\widehat{p} = \frac{p\,d}{v}. \tag{23b}$$

The number $\widehat{I}_x$ is a quotient of moments of inertia and it holds that

$$\widehat{I}_x \le \frac{1}{4}, \tag{24}$$

where the equality holds for example for an empty cylinder[8] of mass m and diameter d.

The fraction $\widehat{p}$ can be interpreted as two times the quotient of the speed of a point on the surface of the projectile and the velocity of the air-flow and is a dynamic parameter. We shall call it the dimensionless rotational speed of the projectile. The idea to make the presentation more elegant is to use the most dimensionless parameters possible. Hence, the remaining dimensional parameters will be

- velocity – v,
- force – $\frac{\rho v^2}{2}S$,
- mass – m,
- length (caliber) – d.

One might add, that the value of $\widehat{p}$ is rather considerably smaller than one and obviously depends on the rotational velocity of the projectile and its caliber. However, it might be coming close to one tenth for various types of fast ammo, especially sub-caliber ammo.

2.4. Derivation of the M-model. Let us recall that

$$m\dot{\mathbf{v}} = \mathbf{AF} + \mathbf{EF}. \tag{25}$$

We will now make use of the dimensionless parameters (23) and the air-flow reference system (18). After simple computations (16) yields

$$\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\alpha}_{\mathrm{e}} = -\frac{2\,m}{\rho v^2 S}\widehat{I}_x\,\widehat{p}\,\widehat{\mathbf{e}}_{\boldsymbol{v}} \times \dot{\mathbf{v}}. \tag{26}$$

After using the expression (22) for $\dot{\mathbf{v}}$ we recover

$$\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\alpha}_{\mathrm{e}} = \frac{2\,m\,v}{\rho\,v^2\,S}\widehat{I}_x\,\widehat{p}\,(\dot{\gamma}_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\chi}} - \dot{\chi}_{\mathrm{a}}\,\cos\gamma_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}) \tag{27}$$

and

$$\begin{aligned}\|\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\alpha}_{\mathrm{e}}\|^2 &= (\mathrm{C}_{\mathrm{M}\alpha}\,\alpha_{\mathrm{e}})^2 \\ &= \left(\frac{2\,m}{\rho\,v\,S}\widehat{I}_x\widehat{p}\right)^2\left(\dot{\gamma}_{\mathrm{a}}^2 + \dot{\chi}_{\mathrm{a}}{}^2\cos^2\gamma_{\mathrm{a}}\right).\end{aligned} \tag{28}$$

Above we keep a certain form because we anticipate the fact that the term containing the cross-sectional area and the dynamic pressure will cancel in the next steps.

Let us write down the forces (14). Using introduced notation, we have

$$\mathbf{DF} = -\frac{\rho v^2}{2}S\cdot\Bigg(\mathrm{C}_{\mathrm{D}0} + \tag{29a}$$
$$+\,\widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}\left(\frac{2m}{\rho\,v\,S}\widehat{I}_x\widehat{p}\right)^2\left(\dot{\gamma}_{\mathrm{a}}^2 + \dot{\chi}_{\mathrm{a}}{}^2\cos^2\gamma_{\mathrm{a}}\right)\Bigg)\widehat{\mathbf{e}}_{\boldsymbol{v}},$$

$$\mathbf{LF} = m\,v\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\,\widehat{I}_x\,\widehat{p}\,(-\dot{\chi}_{\mathrm{a}}\,\cos\gamma_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \dot{\gamma}_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\chi}}), \tag{29b}$$

$$\begin{aligned}\mathbf{MF} &= -\frac{\rho v^2}{2}\,S\,\frac{p\,d}{v}\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\cdot\mathrm{C}_{\mathrm{M}\alpha}\,\boldsymbol{\alpha}_{\mathrm{e}} \times \widehat{\mathbf{e}}_{\boldsymbol{v}} \\ &= m\,v\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\,\widehat{I}_x\,\widehat{p}^2\,(\dot{\gamma}_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \dot{\chi}_{\mathrm{a}}\,\cos\gamma_{\mathrm{a}}\,\widehat{\mathbf{e}}_{\boldsymbol{\chi}}).\end{aligned} \tag{29c}$$

[7]In fact, the choice of axes is unimportant to this work, sometimes literature chooses a version where the z-axis points down and the y-axis points to the right. We find an upward z-axis more natural but as said – it does not influence the derivation. Moreover, changing signs of both angles will simply flip the mentioned axes.

[8]Rigorously, equation $\widehat{I}_x = \frac{1}{4}$ holds for any rigid body, which points are at distance $\frac{d}{2}$ from the x-axis. On the other hand, $\widehat{I}_x = \frac{1}{8}$ means e.g. a uniform full cylinder.

When looking closer on the forces above one notices the obvious – the force $\mathbf{DF}$ is the only force tangent to the air-flow thus it is parallel to $\widehat{\mathbf{e}}_{\boldsymbol{v}}$. For now, let us focus more on $\mathbf{LF}$ and $\mathbf{MF}$. We easily deduce, that the forces $\mathbf{LF}$ and $\mathbf{MF}$ are perpendicular to the air-flow following which, they are spanned by the time dependent orthonormal pair $\widehat{\mathbf{e}}_{\boldsymbol{\chi}}$, $\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}$. Let us note that $\mathbf{MF}$ depends on $\widehat{p}^2$ which makes it invariant with respect to sign of p, whereas $\mathbf{LF}$, which is mainly responsible for the drift of the projectile, depends on $\widehat{p}$. Hence, the sign of the initial rotational speed p governs whether the drift will be left or right.

Now it is clear what we should do. Recall,

$$\begin{aligned}\dot{\mathbf{v}} &= \dot{v}\widehat{\mathbf{e}}_{\boldsymbol{v}} + v\,\dot{\gamma}_{\mathrm{a}}\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + v\dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\widehat{\mathbf{e}}_{\boldsymbol{\chi}} \\ &= \frac{1}{m}\Big(\mathbf{DF}+\mathbf{LF}+\mathbf{MF}+\mathbf{EF}\Big).\end{aligned} \tag{30}$$

The air-flow frame is orthonormal, hence by taking scalar product of (30) with $\widehat{\mathbf{e}}_{\boldsymbol{v}}$ we recover that

$$\begin{aligned}\dot{v} = \frac{1}{m}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{v}} - \frac{\rho v^2}{2\,m}S\cdot\Bigg(&\mathrm{C}_{\mathrm{D0}}+ \\ &+\widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}\left(\frac{2m}{\rho\,v\,S}\widehat{I}_x\widehat{p}\right)^2\left(\dot{\gamma}_{\mathrm{a}}^2+\dot{\chi}_{\mathrm{a}}{}^2\cos^2\gamma_{\mathrm{a}}\right)\Bigg).\end{aligned} \tag{31}$$

The expression for $\dot{v}$ still depends on $\dot{\gamma}_{\mathrm{a}}$ and $\dot{\chi}_{\mathrm{a}}$. But then, when taking scalar products of (30) with $\frac{1}{v}\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}$ and $\frac{1}{v}\widehat{\mathbf{e}}_{\boldsymbol{\chi}}$ respectively we obtain

$$\begin{aligned}\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\end{bmatrix} &= \frac{1}{m\,v}\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix}+ \\ &+\widehat{I}_x\widehat{p}\begin{bmatrix}\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{mag-f}} & -\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\\ \widehat{\mathrm{C}}_{\mathrm{L}\alpha} & \widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\end{bmatrix}\cdot\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\end{bmatrix}.\end{aligned} \tag{32}$$

The Eq. (32) is just a linear equation! Let us define the mixing matrix

$$\mathbf{M} = \begin{bmatrix}\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{mag-f}} & -\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\\ \widehat{\mathrm{C}}_{\mathrm{L}\alpha} & \widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\end{bmatrix}, \tag{33}$$

then it is obvious that

$$\Big(1-\widehat{I}_x\,\widehat{p}\,\mathbf{M}\Big)\cdot\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\end{bmatrix} = \frac{1}{m\,v}\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix}. \tag{34}$$

Finally, we will use

$$\begin{aligned}\mathbf{K} &= 1-\widehat{I}_x\,\widehat{p}\,\mathbf{M} \\ &= \begin{bmatrix}1-\widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}} & \widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\\ -\widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha} & 1-\widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\end{bmatrix},\end{aligned} \tag{35}$$

then by elementary linear algebra[9]

$$\mathbf{K}^{-1} = \frac{1}{\det\mathbf{K}}\mathbf{K}^{\mathrm{T}}, \tag{36}$$

where

$$\det\mathbf{K} = \big(1-\widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\big)^2 + \big(\widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\big)^2. \tag{37}$$

For mathematical rigor, let us discuss more closely the singularity of $\mathbf{K}$. Note that

$$\det\mathbf{K} = 0 \iff \mathbf{K} = 0, \tag{38}$$

which gives a contradiction if any external forces act perpendicular to the air-flow and corresponds to a physically trivial case. Finally

$$\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\end{bmatrix} = \frac{1}{m\,v}\frac{1}{\det\mathbf{K}}\mathbf{K}^{\mathrm{T}}\cdot\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix}. \tag{39}$$

We are almost done, in fact we could stop here, but let us look at Eq. (30) for $\dot{v}$, which is dependent on $\dot{\gamma}_{\mathrm{a}}$ and $\dot{\chi}_{\mathrm{a}}$, which we have calculated explicitly. Nevertheless, we can further simplify the expression

$$\left(\dot{\gamma}_{\mathrm{a}}^2+\dot{\chi}_{\mathrm{a}}{}^2\cos^2\gamma_{\mathrm{a}}\right). \tag{40}$$

To do that in a clever way let us note that by (36) the matrix

$$\mathbf{O} = \frac{1}{\sqrt{\det\mathbf{K}}}\cdot\mathbf{K} \tag{41}$$

is orthogonal, i.e.

$$\mathbf{O}^{-1} = \sqrt{\det\mathbf{K}}\cdot\mathbf{K}^{-1} = \frac{1}{\sqrt{\det\mathbf{K}}}\cdot\mathbf{K}^{\mathrm{T}} = \mathbf{O}^{\mathrm{T}}. \tag{42}$$

Hence, it does not change the norm of vectors.

Now we can rewrite (39) as

$$\sqrt{\det\mathbf{K}}\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\end{bmatrix} = \frac{1}{m\,v}\mathbf{O}^{\mathrm{T}}\cdot\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix}. \tag{43}$$

The norm of the right-hand-side is not changed under the action of an orthogonal matrix, thus

$$\det\mathbf{K}\left\|\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\cos\gamma_{\mathrm{a}}\end{bmatrix}\right\|^2 = \frac{1}{(m\,v)^2}\left\|\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix}\right\|^2. \tag{44}$$

Thanks to our observation we write

$$\begin{aligned}&\dot{\gamma}_{\mathrm{a}}^2+\dot{\chi}_{\mathrm{a}}{}^2\cos^2\gamma_{\mathrm{a}} = \\ &\quad= \frac{1}{(m\,v)^2}\frac{1}{\det\mathbf{K}}\left((\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}})^2+(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}})^2\right)\end{aligned} \tag{45}$$

and then substituting (45) into (28) we derive that

$$\|\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\alpha}_{\mathrm{e}}\|^2 = \left(\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\right)^2(m\,v)^2\left(\dot{\gamma}_{\mathrm{a}}^2+\dot{\chi}_{\mathrm{a}}{}^2\cos^2\gamma_{\mathrm{a}}\right) \tag{46}$$

$$= \left(\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\right)^2\frac{1}{\det\mathbf{K}}\left((\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}})^2+(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}})^2\right). \tag{47}$$

Remark. Let us stress here, that the above is part of the main goal of this small work – the norm of the vector $\boldsymbol{\alpha}_{\mathrm{e}}$ is explicitly determined by the external forces not depending on $\boldsymbol{\alpha}_{\mathrm{e}}$, e.g. the gravitational force, the Coriolis force, etc. In a short moment we will summarize the results in one paragraph.

[9] This matrix is in fact equivalent to a multiplication by a complex number. Hence, it has a simple formula for the inverse.

By putting (45) into (31) the tangent acceleration takes its final form. We have

$$\dot{v} = \frac{1}{m}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{v}} - \frac{\rho v^2}{2\,m}S\cdot\Bigg(\mathrm{C}_{\mathrm{D}0}+ + \widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}\left(\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\right)^2\frac{(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}})^2+(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}})^2}{\det\mathbf{K}}\Bigg). \tag{48}$$

In the next subsection we shall sum-up all that has been done so far to achieve a clean presentation of the explicit M-model.

2.5. Final form of the M-model. We have derived the explicit system of ODEs, which is equivalent to the "modified point mass model". No implicit functions are involved and any external forces $\mathbf{EF}\,(\mathbf{x},\,\mathbf{v},\,p;\,\mathbf{w})$ not depending on the attack angle are allowed. In the box below we present the final form of our explicit ODE system describing the motion of a spinning projectile in a resistive medium. For the convenience of the reader we "copy-paste" our previous results into one condensed subsection.

By summing up the previous work, e.g. (39), (47), (48), the differential equations for the M-model are as follows:

$$\dot{\mathbf{x}} = \mathbf{v}+\mathbf{w}, \tag{49a}$$

$$\dot{p} = \frac{\rho\,v^2}{2\,I_x}\,S\,d\,\mathrm{C}_{\mathrm{spin}}\cdot\widehat{p}, \tag{49b}$$

$$\dot{v} = \frac{1}{m}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{v}} - \frac{\rho v^2}{2\,m}S\cdot\Bigg(\mathrm{C}_{\mathrm{D}0}+ + \widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}\left(\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\right)^2\frac{(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}})^2+(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}})^2}{\det\mathbf{K}}\Bigg) \tag{49c}$$

and

$$\begin{bmatrix}\dot{\gamma}_{\mathrm{a}}\\ \dot{\chi}_{\mathrm{a}}\,\cos\gamma_{\mathrm{a}}\end{bmatrix} = \frac{1}{m\,v}\,\frac{1}{\det\mathbf{K}}\,\mathbf{K}^{\mathrm{T}}\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix}. \tag{49d}$$

Recalling (35), to define the system (49) we have used

$$\mathbf{K}^{\mathrm{T}} = \begin{bmatrix}1-\widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}} & -\widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\\ \widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha} & 1-\widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\end{bmatrix} \tag{50}$$

and the obvious consequence that

$$\det\mathbf{K} = \big(1-\widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\big)^2+\big(\widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha}\big)^2. \tag{51}$$

Moreover, for convenience, recall the dimensionless coefficients defined in Subsec. 2.3, i.e.

$$\widehat{I}_x = \frac{I_x}{m\,d^2} \qquad \text{and} \qquad \widehat{p} = \frac{p\,d}{v}. \tag{52}$$

Also, we have assumed that in the interval of integration the wind is homogeneous, i.e. $\dot{\mathbf{w}} \equiv 0$[10]. The dimensionless, i.e. "hatted", coefficients were given as

$$\widehat{\mathrm{C}}_{\mathrm{D}\alpha^2} = \frac{\mathrm{C}_{\mathrm{D}\alpha^2}}{(\mathrm{C}_{\mathrm{M}\alpha})^2},\quad \widehat{\mathrm{C}}_{\mathrm{L}\alpha} = \frac{\mathrm{C}_{\mathrm{L}\alpha}}{\mathrm{C}_{\mathrm{M}\alpha}},\quad \widehat{\mathrm{C}}_{\mathrm{mag-f}} = \frac{\mathrm{C}_{\mathrm{mag-f}}}{\mathrm{C}_{\mathrm{M}\alpha}}. \tag{53}$$

Also recall that $\mathbf{v}$ is the velocity of the air with respect to the projectile and $\mathbf{v} = \mathbf{u}-\mathbf{w}$. Thus one should not forget to subtract the wind from the initial velocity of the projectile when preparing the initial condition to the system (49).

Remark. Let us state that the case of general wind, depending both on position and time, is not a problem in an already prepared sequel, which we shall publish soon after this work, we will introduce the enhanced M-model incorporating a general space-time wind distribution. We decided to start with the homogeneous-wind-case as the annoying computations of the general case might spoil the elegance of this particular derivation. Nevertheless, as a logical consequence, they are worthy of a sequel.

Now we firmly state the fact that the system (49) shows no signs of $\boldsymbol{\alpha}_{\mathrm{e}}$. Nevertheless, obviously, having a solution one can – by combining (39) and (27) – calculate

$$\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\alpha}_{\mathrm{e}} = = \frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\,\frac{1}{\det\mathbf{K}}\begin{bmatrix}\widehat{\mathbf{e}}_{\boldsymbol{\chi}} & -\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\end{bmatrix}\mathbf{K}^{\mathrm{T}}\begin{bmatrix}\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\\ \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}\end{bmatrix} \tag{54}$$

and

$$\|\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\alpha}_{\mathrm{e}}\|^2 = \left(\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\right)^2\frac{(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}})^2+(\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}})^2}{\det\mathbf{K}}. \tag{55}$$

Remark. Let us stress here, that the above is exactly the main goal of our work. The vector $\boldsymbol{\alpha}_{\mathrm{e}}$ is explicitly determined by the external forces $\mathbf{EF}(\mathbf{x},\,\mathbf{v},\,p;\,\mathbf{w})$. Into $\mathbf{EF}$ one might incorporate e.g. the gravitational force, the Coriolis force, etc.

Corollary. The system (49) is of course exactly equivalent to the implicit system – "modified point mass trajectory model" – presented at the beginning. The explicit form derived in this paper is both: elegant and rigorous. In addition, easily dealt within numerical applications, which we consider to be a great advantage.

2.6. Example of an M-model: projectile motion influenced by constant gravitational force with constant wind. For convenience of the reader, let us provide a less general example, and perform simplifications to the evolution system (49). We need to postulate what external forces $\mathbf{EF}$ are going to be used. To talk about concrete forces we need to fix our coordinate system. We choose[11] that the z-axis, along $\widehat{\mathbf{e}}_{\boldsymbol{z}}$, corresponds to the altitude, and that the x-axis, along $\widehat{\mathbf{e}}_{\boldsymbol{x}}$ roughly describe the "forward" direction of motion as summarized in Fig. 1. Then $\widehat{\mathbf{e}}_{\boldsymbol{y}}$ is such, that together $\widehat{\mathbf{e}}_{\boldsymbol{x}}$, $\widehat{\mathbf{e}}_{\boldsymbol{y}}$, $\widehat{\mathbf{e}}_{\boldsymbol{z}}$ form a right-oriented orthonormal frame.

[10] The notion is slightly more general than a globally constant wind as result of the simple equation $\dot{\mathbf{w}} = (\boldsymbol{\nabla}_{\mathbf{x}}\,\mathbf{w})\cdot\dot{\mathbf{x}}+\frac{\partial\,\mathbf{w}}{\partial\,t} = 0$, which might have non-trivial, however unphysical solutions.

[11] This choice is fairly obvious but, of course, there might be other preferences.

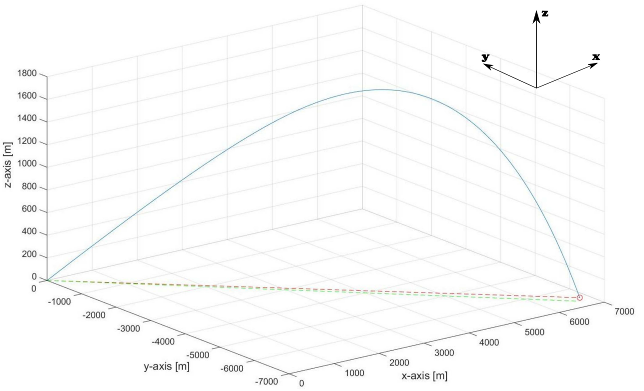

Fig. 1. An example of a trajectory with chosen axis alignment

The system of explicit ODEs.

We put a constant wind[12]

$$\mathbf{w}(t) \equiv \mathbf{w}_0 \tag{56}$$

and constant average gravitational acceleration, which yields external forces

$$\mathbf{EF} = -m\, g\, \widehat{\mathbf{e}}_{\boldsymbol{z}}. \tag{57}$$

Then using (21) the external forces are

$$\mathbf{EF} = -m\, g\, \left(\sin\gamma_{\mathrm{a}}\, \widehat{\mathbf{e}}_{\boldsymbol{v}} + \cos\gamma_{\mathrm{a}}\, \widehat{\mathbf{e}}_{\boldsymbol{\gamma}}\right). \tag{58}$$

Then

$$\mathbf{EF} \circ \widehat{\mathbf{e}}_{\boldsymbol{v}} = -m\, g\, \sin\gamma_{\mathrm{a}}, \tag{59}$$

$$\mathbf{EF} \circ \widehat{\mathbf{e}}_{\boldsymbol{\gamma}} = -m\, g\, \cos\gamma_{\mathrm{a}}, \tag{60}$$

$$\mathbf{EF} \circ \widehat{\mathbf{e}}_{\boldsymbol{\chi}} = 0. \tag{61}$$

Finally, when the above-mentioned are applied to the M-model defined in (49) we get

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{w}_0, \tag{62a}$$

$$\dot{p} = \frac{\rho\, v^2}{2\, I_x}\, S\, d\, \mathrm{C}_{\mathrm{spin}} \cdot \widehat{p}, \tag{62b}$$

$$\dot{v} = -g\, \sin\gamma_{\mathrm{a}} - \frac{\rho v^2}{2\, m}\, S \cdot \Bigg(\mathrm{C}_{\mathrm{D0}} + + \widehat{\mathrm{C}}_{\mathrm{D}\alpha^2} \left(\frac{2\, m\, g}{\rho\, v^2\, S}\right)^2 \frac{\widehat{I}_x^2\, \widehat{p}^2\, \cos^2\gamma_{\mathrm{a}}}{\left(1 - \widehat{I}_x\, \widehat{p}^2\, \widehat{\mathrm{C}}_{\mathrm{mag-f}}\right)^2 + \left(\widehat{I}_x\, \widehat{p}\, \widehat{\mathrm{C}}_{\mathrm{L}\alpha}\right)^2}\Bigg) \tag{62c}$$

and

$$\begin{bmatrix} \dot{\gamma}_{\mathrm{a}} \\ \dot{\chi}_{\mathrm{a}} \cos\gamma_{\mathrm{a}} \end{bmatrix} = = \frac{-\frac{g}{v} \cdot \cos\gamma_{\mathrm{a}}}{\left(1 - \widehat{I}_x\, \widehat{p}^2\, \widehat{\mathrm{C}}_{\mathrm{mag-f}}\right)^2 + \left(\widehat{I}_x\, \widehat{p}\, \widehat{\mathrm{C}}_{\mathrm{L}\alpha}\right)^2} \cdot \begin{bmatrix} 1 - \widehat{I}_x\, \widehat{p}^2\, \widehat{\mathrm{C}}_{\mathrm{mag-f}} \\ \widehat{I}_x\, \widehat{p}\, \widehat{\mathrm{C}}_{\mathrm{L}\alpha} \end{bmatrix}. \tag{62d}$$

The Eqs. (62) form a complete differential system describing, by means of seven scalar variables, the evolution of a spinning projectile in a resistive medium. Recall that

$$\widehat{I}_x = \frac{I_x}{m\, d^2}, \qquad \widehat{p} = \frac{p\, d}{v}. \tag{63}$$

Hence, $\widehat{p}$ is a dynamic variable. Let us also stress, that thanks to using the air-flow direction $\widehat{\mathbf{e}}_{\boldsymbol{v}} = \frac{\mathbf{v}}{v}$ to define our coordinate system, the wind, i.e. $\mathbf{w}_0$, does not appear in the differential equations for $\dot{\mathbf{v}}$. Instead, it simply adds to the position integration. However, when preparing initial condition one should remember that $\mathbf{v} = \mathbf{u} - \mathbf{w}$. As announced let us focus a bit on initial conditions to finalize the presentation of this example.

The initial conditions.

The initial conditions and their interpretation are not entirely trivial. Let us assume for start that we initiate our integration at the moment t_0. Naturally $\mathbf{x}(t_0) = \mathbf{x}_0$ can be chosen as one wishes or needs, e.g. $\mathbf{x}_0 = \mathbf{0}$. We need to focus on the initial velocity vector, we have

$$\mathbf{u}(t_0) = \mathbf{u}_0 = u_0 \begin{bmatrix} \cos\gamma_0\, \cos\chi_0 \\ \cos\gamma_0\, \sin\chi_0 \\ \sin\gamma_0 \end{bmatrix}, \tag{64}$$

where by $\mathbf{u}_0$ we understand the initial velocity vector of the projectile, by γ_0 the inclination angle of the initial velocity and by χ_0 the azimuth angle of the initial velocity. For simplicity we could always rotate the azimuth angle χ_0 to zero. Recall that

$$\mathbf{v}(t_0) = \mathbf{v}_0 = \mathbf{u}_0 - \mathbf{w}_0 = \begin{bmatrix} u_0\, \cos\gamma_0\, \cos\chi_0 - w_{0x} \\ u_0\, \cos\gamma_0\, \sin\chi_0 - w_{0y} \\ u_0\, \sin\gamma_0 - w_{0z} \end{bmatrix}. \tag{65}$$

Thus we have the initial condition

$$v_0 = \|\mathbf{v}_0\| = \|\mathbf{u}_0 - \mathbf{w}_0\|. \tag{66}$$

Then

$$v_0 \sin\gamma_{\mathrm{a}0} = \mathbf{v}_0 \circ \widehat{\mathbf{e}}_{\boldsymbol{z}} = u_0\, \sin\gamma_0 - w_{0z}, \tag{67a}$$

$$\tan\chi_{\mathrm{a}0} = \frac{\mathbf{v}_0 \circ \widehat{\mathbf{e}}_{\boldsymbol{y}}}{\mathbf{v}_0 \circ \widehat{\mathbf{e}}_{\boldsymbol{x}}} = \frac{u_0\, \cos\gamma_0\, \sin\chi_0 - w_{0y}}{u_0\, \cos\gamma_0\, \cos\chi_0 - w_{0x}}. \tag{67b}$$

From the Eqs. (66) and (67) the initial conditions v_0, $\gamma_{\mathrm{a}0}$ and $\chi_{\mathrm{a}0}$ can easily be retrieved.

The last initial condition p_0 can either be known a priori or obtained using a very good approximation[13]

$$p_0 = 2\, \pi\, u_0\, \frac{1}{\eta\, d}. \tag{68}$$

Above the dimensionless parameter η is the twist rate of the rifling at the end of the gun barrel expressed in calibers per full revolution.

Note. Finally, just for visualization purposes we present a numerically generated trajectory in Fig. 1.

[12] Of course one can use a step-function for $\mathbf{w}$ and sew the solutions on the subintervals

[13] The rate of roll (axial spin) of the projectile is determined by 2π times the number of twists per time unit at the end of the barrel.

2.7. Complex number interpretation of the mixing matrix. The matrices $\mathbf{M}$ (33) and $\mathbf{K}$ (35) have a very special property – they are proportional to 2-by-2 orthogonal matrices. Thus, they are equivalent to multiplication by complex numbers[14]. Following this, the determinants of matrices $\mathbf{M}$ and $\mathbf{K}$ are just the squares of absolute values of the corresponding complex numbers, and the inverses correspond to complex number division. We shall not elaborate on this as from the point of view of algebra it is a rather simple and well-know isomorphism. Instead, let us focus on the details below as they should be self-explanatory.

To begin, let us define complex numbers

$$\boldsymbol{\mathcal{M}} = \widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{mag-f}} + \mathbf{i}\cdot\widehat{\mathrm{C}}_{\mathrm{L}\alpha} \tag{69}$$

and

$$\begin{aligned}\boldsymbol{\mathcal{K}} &= 1 - \widehat{I}_x\,\widehat{p}\,\boldsymbol{\mathcal{M}}\\ &= \left(1 - \widehat{I}_x\,\widehat{p}^2\,\widehat{\mathrm{C}}_{\mathrm{mag-f}}\right) - \mathbf{i}\,\widehat{I}_x\,\widehat{p}\,\widehat{\mathrm{C}}_{\mathrm{L}\alpha}.\end{aligned} \tag{70}$$

We shall call the above the ***complex deflection coefficient***. In addition the complex perpendicular force is

$$\boldsymbol{\mathcal{F}}^{\perp} = \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \mathbf{i}\cdot\mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}. \tag{71}$$

Above we see, that the $\widehat{\mathbf{e}}_{\boldsymbol{\gamma}}$ direction has been identified with the real axis, whereas the $\widehat{\mathbf{e}}_{\boldsymbol{\chi}}$ direction with the imaginary axis. Now the plane perpendicular to the air-flow, i.e. perpendicular to $\mathbf{v}$, is treated as a complex plane. Hence, the vector $\boldsymbol{\alpha}_{\mathrm{e}}$, as it is always orthogonal to $\mathbf{v}$ can also be interpreted as a complex number. Let us define

$$\boldsymbol{\mathcal{Z}}_{\boldsymbol{\alpha}_{\mathrm{e}}} = \boldsymbol{\alpha}_{\mathrm{e}}\circ\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \mathbf{i}\,\boldsymbol{\alpha}_{\mathrm{e}}\circ\widehat{\mathbf{e}}_{\boldsymbol{\chi}}. \tag{72}$$

Then the inverse is

$$\boldsymbol{\alpha}_{\mathrm{e}} = \mathrm{Re}\boldsymbol{\mathcal{Z}}_{\boldsymbol{\alpha}_{\mathrm{e}}}\cdot\widehat{\mathbf{e}}_{\boldsymbol{\gamma}} + \mathrm{Im}\boldsymbol{\mathcal{Z}}_{\boldsymbol{\alpha}_{\mathrm{e}}}\cdot\widehat{\mathbf{e}}_{\boldsymbol{\chi}}. \tag{73}$$

Making use of our conventions we can rewrite (54) and define its complex equivalent which is extremely elegant. Note that using complex numbers $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{F}}^{\perp}$ the explicit formula (54) for the equilibrium angle takes the form

$$\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\mathcal{Z}}_{\boldsymbol{\alpha}_{\mathrm{e}}} = \mathbf{i}\,\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\,\frac{\boldsymbol{\mathcal{F}}^{\perp}}{\boldsymbol{\mathcal{K}}}. \tag{74}$$

Recall, we have named the complex number $\boldsymbol{\mathcal{K}}$ the complex deflection coefficient – let us interpret this by rewriting the complex equilibrium equation defined in (74) as

$$\boldsymbol{\mathcal{F}}^{\perp} = -\mathbf{i}\,\frac{\rho\,v^2}{2}\,S\,\frac{\boldsymbol{\mathcal{K}}}{\widehat{I}_x\,\widehat{p}}\,\mathrm{C}_{\mathrm{M}\alpha}\,\boldsymbol{\mathcal{Z}}_{\boldsymbol{\alpha}_{\mathrm{e}}}. \tag{75}$$

We now have all the components needed for presenting the summary of the M-model in complex notation.

Final form of the M-model in complex notation.

Collecting all results from the previous subsection, we summarize that the complexified differential equations of the M-model are as follows

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{w}, \tag{76a}$$

$$\dot{p} = \frac{\rho\,v^2}{2\,I_x}\,S\,d\,\mathrm{C}_{\mathrm{spin}}\cdot\widehat{p}, \tag{76b}$$

$$\dot{v} = -\frac{\rho v^2}{2\,m}S\left(\mathrm{C}_{\mathrm{D0}} + \widehat{\mathrm{C}}_{\mathrm{D}\alpha^2}\left(\frac{2\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\right)^2\left|\frac{\boldsymbol{\mathcal{F}}^{\perp}}{\boldsymbol{\mathcal{K}}}\right|^2\right) + \mathbf{EF}\circ\widehat{\mathbf{e}}_{\boldsymbol{v}} \tag{76c}$$

and

$$\mathbf{d\Psi} = (\dot{\gamma}_{\mathrm{a}} + \mathbf{i}\,\dot{\chi}_{\mathrm{a}}\,\cos\gamma_{\mathrm{a}}) = \frac{1}{m\,v}\,\frac{\boldsymbol{\mathcal{F}}^{\perp}}{\boldsymbol{\mathcal{K}}}. \tag{76d}$$

Then, by depicting the time-dependent complex function

$$\mathrm{C}_{\mathrm{M}\alpha}\cdot\boldsymbol{\mathcal{Z}}_{\boldsymbol{\alpha}_{\mathrm{e}}} = \mathbf{i}\,\frac{2\,\widehat{I}_x\,\widehat{p}}{\rho\,v^2\,S}\,\frac{\boldsymbol{\mathcal{F}}^{\perp}}{\boldsymbol{\mathcal{K}}} \tag{77}$$

on the plane one retrieves the movement of the projectile's tip and a nice geometric interpretation – the complex plane is to be identified with the plane perpendicular to the air flow, i.e. the plane in which the interesting physics happen!

3. Summary: advantages of the ballistic M-model

We have finished our considerations on the subject of the derivation of the explicit M-model from the implicit "modified point mass trajectory model". We have constructed a model which is explicitly defined, numerically easy to solve and does explain drift phenomena. We advertise this model as the optimal "power/price" solution which ***should be used*** in further analysis like:

- probability calculation via initial condition error propagation,
- use in fire control computers,
- preparing or utilizing meteorologic corrections to idealized shooting tables.

To be honest, there is one disadvantage of this model (and the "modified point mass trajectory model" as well), which is the simplification $I_y = 0$. Without the perpendicular moment of inertia any processes of finding equilibrium happen infinitesimally fast. For projectiles with "heavy" moments of inertia the dynamics governed by $\dot{\boldsymbol{\alpha}}_{\mathrm{e}}$ might be much more important than inhomogeneous wind corrections ($\dot{\mathbf{w}}$-corrections), which we plan to analyze in an already prepared sequel to this work. Please note, that the negligence of I_y is a treat of the "modified point mass trajectory model" in general and not a treat of its explicit form! Let us remind that the explicit form, the M-model is ***equivalent*** to its implicit counterpart. We feel, that the "modified point mass model" in the implicit form ***should not be used in applications***. The M-model presented here –

[14]By the isomorphism $1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{i} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, while $z \mapsto \begin{bmatrix} \mathrm{Re}\,z \\ \mathrm{Im}\,z \end{bmatrix}$.

as it is ***explicit*** – does not need to solve the equilibrium equation in every step of the numeric integration. The equilibrium equation for α_e has been ***exactly solved***. Summarizing, we consider the differential system (49) (or example (62)) as an extremely useful, robust and yet rigorous projectile movement model.

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