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RELATIONSHIP BETWEEN THE NUMBER OF MONTE CARLO TRIALS APPLIED FOR MODELLING VOLTAGE OUTPUT ACCELEROMETERS AND THE MAXIMUM DYNAMIC ERRORS

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Abstract: This paper examines the influence of the number of Monte Carlo trials on the maximum dynamic errors. These trials were used for modelling voltage output accelerometers, while the errors were represented by the maximum distance criterion. A weighted least-squares method that is based on a Monte Carlo adaptive procedure was used for accelerometer modelling. The subsequent Monte Carlo trials were provided by the Box-Muller pseudo-random number generator. As an example, modelling of the Endevco87 accelerometer based on the measurement points of its frequency characteristics was presented. Then, the errors corresponding to the input signals with one and two constraints were determined. Finally, the relationship between the number of Monte Carlo trials and the values of maximum dynamic errors was assessed. Modelling of the accelerometer was carried out in MATLAB14, while the maximum errors were determined using MathCad15. For calculations, the following personal computer was used: 64 bits AMD processor 3.4 GHz and 32 GB RAM.

Keywords: Monte Carlo trials, maximum distance criterion, voltage output accelerometer, Box-Muller pseudo-random number generator.

1. INTRODUCTION

Modelling of voltage output accelerometers is most often carried out based on the result of the measurement of amplitude and phase characteristics. Parallel approximation of such characteristics can only be attained by utilising the weighted least-squares (WLS) method that provides both the model parameters and associated uncertainties [1–3]. A vector-matrix equation that employs the diagonal covariance matrix constitutes the kernel for this method. This covariance matrix is determined based on the averages and standard deviations calculated for measurement points of both characteristics, and by utilising the two numbers provided by using the pseudo-random number generator. Annex C.4 for GUM-S1 shows the Box-Muller (B-M) generator that produces normally distributed numbers [4]. The subsequent draw performed by this generator is specified as Monte Carlo (MC) trial in subsection 7.9.4 of GUM-S1. The total number of MC trials is defined as equal at least to 10^4. In this subsection, one can also find the statement: ”The choice of MC trials is arbitrary, but has been found to be suitable in practice”. Taking this into account, the main task of this paper is an assessment of the relationship between the number of MC trials and the maximum dynamic errors. The six values that are contained in the range $<10^4, 10^5>$ are examined here. In order to ensure a sufficient accuracy of modelling, it is reasonable to use the adaptive MC procedure [4]. This procedure is performed by providing such a number of MC trials for which the standard deviations determined on the averages of the parameters and associated uncertainties attain the values resulting from the tolerances referring to the uncertainties. When the adaptive MC procedure is terminated, the averages of the parameters and uncertainties are provided as the result of accelerometer modelling. Such an approach can only ensure that the resulting model exactly reflects the dynamics of the modelled accelerometer. The reliability of such results is verified by performing a chi-squared test that shows the difference between the two complex frequency responses. The first is calculated based on the measurement data, while the second is obtained as a result of accelerometer modelling. When the results of modelling are obtained, it is possible to check the relationship between the number of MC trials and the maximum dynamic errors. The impact of the uncertainties associated with the parameters of the model on the dynamic errors was pointed out in [5]. However, the modelling results were obtained by generating only one set of MC trials and without using the adaptive MC procedure. Taking this into account, this paper addresses the aforementioned issues.

2. MODELLING OF VOLTAGE OUTPUT ACCELEROMETER

The complex frequency response of a voltage output accelerometer is presented by

$$K(\omega)=\frac{S}{1-\left(\frac{\omega}{\omega_0}\right)^2+j\omega\frac{2\beta}{\omega_0}}, \tag{1}$$

where S, β and ω_0 are voltage sensitivity, damping ratio and non-damped natural frequency, respectively.

Let us present this complex frequency response in the equivalent form

$$K(\omega,\mathbf{\theta})=\frac{a_0}{1+ja_1\omega-a_2\omega^2}, \tag{2}$$

where

$$\boldsymbol{\theta} = [a_0, a_1, a_2] \tag{3}$$

is the vector of parameters, while $a_0 = S$, $a_1 = 2\beta/\omega_0$ and $a_2 = 1/\omega_0{}^2$.

The complex frequency response created based on measuring points has the form

$$K(\omega_n) = A(\omega_n)\exp[j\Phi(\omega_n)] = R(\omega_n) + jI(\omega_n) \qquad n = 0,1,\ldots,N-1, \tag{4}$$

where N is the number of measurement points, while $R(\omega_n)$ and $I(\omega_n)$ are the real and imaginary parts associated with this function, respectively.

Let us present the function (2) in relation to the measured frequencies ω_n, as follows

$$K(\omega_n, \boldsymbol{\theta}) = \frac{a_0}{1 - a_2\omega_n{}^2 + ja_1\omega_n}. \tag{5}$$

Let us also now present the difference between (4) and (5) in the following form

$$R(\omega_n) + jI(\omega_n) = e_R(\omega_n) + je_I(\omega_n), \tag{6}$$

where

$$e_R(\omega_n) = a_0 + a_1\omega_n I(\omega_n) + a_2\omega_n{}^2 R(\omega_n), \qquad e_I(\omega_n) = -a_1\omega_n R(\omega_n) + a_2\omega_n{}^2 I(\omega_n) \tag{7}$$

are the errors of modelling.

The kernel of the WLS method that allows estimation of the parameters of vector (3) is represented by

$$\tilde{\boldsymbol{\theta}} = (\boldsymbol{\Psi}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^T\boldsymbol{\Sigma}^{-1}\mathbf{Y}. \tag{8}$$

The $2N$ dimensional vector $\mathbf{Y}$ is formed based on the real and imaginary parts of (4), as follows

$$\mathbf{Y}^T = [R(\omega_0) \;\; I(\omega_0) \;\; R(\omega_1) \;\; I(\omega_1) \;\; \ldots \;\; R(\omega_n) \;\; I(\omega_n)], \tag{9}$$

while the $2N \times 3$ dimensional matrix $\boldsymbol{\Psi}$ has the form

$$\boldsymbol{\Psi}^T = [\boldsymbol{\psi}_0^T \; \boldsymbol{\psi}_1^T \ldots \boldsymbol{\psi}_{N-1}^T], \tag{10}$$

where

$$\boldsymbol{\psi}_n = \begin{bmatrix} 1 & 0 \\ \omega_n I(\omega_n) & -\omega_n R(\omega_n) \\ \omega_n{}^2 R(\omega_n) & \omega_n{}^2 I(\omega_n) \end{bmatrix} \tag{11}$$

is obtained based on (7) according to the order of parameters of vector (3).

In turn, the matrix $\boldsymbol{\Sigma}$ is the diagonal covariance matrix developed by utilising the MC method. This method employs the Box-Muller pseudo-random number generator. Uncertainties $\hat{\boldsymbol{\theta}}$ associated with the estimated parameters $\tilde{\boldsymbol{\theta}}$ are calculated by using of Monte Carlo simulation, as follows

$$\hat{\boldsymbol{\theta}} = \frac{\mathbf{Y} - \boldsymbol{\varepsilon}}{\boldsymbol{\Psi}}, \tag{12}$$

where $\boldsymbol{\varepsilon}$ is the random vector drawn from the three dimensional normal distribution [4]. The covariance matrix that is related to this distribution is calculated as

$$\boldsymbol{\Sigma}_u = \frac{(\boldsymbol{\Psi}\hat{\boldsymbol{\theta}} - \mathbf{Y})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Psi}\hat{\boldsymbol{\theta}} - \mathbf{Y})}{2N-3}(\boldsymbol{\Psi}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\Psi})^{-1}. \tag{13}$$

The uncertainties $\hat{\boldsymbol{\theta}}$ are finally calculated as the root square of the diagonal elements of matrix $\boldsymbol{\Sigma}_u$.

The reliability of the modelling results is confirmed by a two-tailed chi-square test, which is calculated as follows

$$\chi^2_{v,\alpha} \le \sum_{n=0}^{N-1}\frac{|K(\omega_n) - K(\omega_n, \boldsymbol{\theta})|^2}{\sigma^2_{K(\omega_n)}} \le \chi^2_{v,1-\alpha}, \tag{14}$$

where

$$\text{v} = 2N - 3 \tag{15}$$

represents the number of degrees of freedom, α is the significance level that usually has a typical value equal to 0.05, while $\sigma_{K(\omega_n)}$ is the standard deviation calculated based on complex frequency response (4).

The sufficient number of MC trials for each B-M generation is determined by

$$\text{MC} = \max(\lambda, 10^4), \tag{16}$$

where λ is the smallest integer greater than or equal to $100/(1-p)$, and p denotes the required coverage probability. The distribution of the parameters and associated uncertainties is controlled by calculation of the corresponding standard deviations

$$\sigma_{a_i} = \sqrt{\frac{1}{q(q-1)}\sum_{r=1}^{q}(a_i{}^{(r)} - \bar{a})^2}, \quad i = 0,1,2, \tag{17}$$

where

$$\bar{a} = \frac{1}{q}\sum_{r=1}^{q} a_i{}^{(r)} \tag{18}$$

for $q = 1, 2, \ldots, Q$, and Q is the total number of MC generations [4].

3. MAXIMUM DISTANCE CRITERION

The maximum distance criterion always allows one to determine the values of maximum errors in an analytical way. The errors can be calculated for the input signals constrained only in magnitude and simultaneously in magnitude and rate of change [5–7].

In the first case, the error is calculated directly based on relation

$$D_1 = A\Delta\sum_{m=0}^{M-1}|k[m]|, \tag{19}$$

where

$$k[m] = k_a[m] - k_s[m], \;\; m = 0,1,\ldots,M-1 \tag{20}$$

and $k_a[m]$, $k_s[m]$ are the discrete impulse responses of the accelerometer and the standard, while A, M and Δ are the magnitude constraint, the number of samples and the sampling interval, respectively.

The standard is developed by using a low-pass filter whose cut-off frequency is equal to the accelerometer bandwidth.

In the second case, the error is determined using the relation

$$D_2 = \max_m \left(\Delta \sum_{i=0}^{m} k[m-i]\, x_{02}[i] \right) \qquad (21)$$
$$m = 0,1,\ldots,M-1,$$

where

$$x_{02}[m] = \Delta \sum_{i=0}^{m} \mu[i],\; m = 0,1,\ldots,M-1 \qquad (22)$$

and $\mu[\cdot]$ is an auxiliary function obtained in several steps of processing of the function $k[m]$.

The most important property of the maximum dynamic errors expressed by the criterion of maximum distance is a constant value for the number of samples corresponding to the steady-state impulse response.

4. RESULTS OF MODELLING

The modelling of the accelerometer was carried out based on measurements of amplitude and phase characteristics corresponding to 34 frequencies (Table 1).

For all MC trials, the same values of $\widetilde{S}$, $\widetilde{f}_0$ and $\Delta\widetilde{\beta}$ equal $95{,}5\ \mathrm{m}V/(m/s^2)$, $1181\ Hz$ and 0,008 respectively, were obtained. The data reported in Table 2 show that the differences occurred for $\widetilde{\beta}$, $\Delta\widetilde{S}$ and $\Delta\widetilde{f}_0$.

Table 1. Frequency and measurement data (amplitude and phase)

Frequency F [*Hz*]	*Amplitude* A [m*V*/(*m/s*²)]	*Phase* Φ [deg,]	*Frequency* F [*Hz*]	*Amplitude* A [m*V*/(*m/s*²)]	*Phase* Φ [deg,]
10	98	-0,5	400	110	-6,0
15	96	-0,5	500	120	-6,0
20	94	-0,5	600	125	-7,0
30	94	-1,0	700	130	-11,0
40	96	-1,0	800	160	-17,0
50	98	-2,0	900	195	-35,0
60	96	-1,5	1000	200	-62,0
70	98	-1,5	1100	225	-75,0
80	94	-3,0	1200	250	-103,0
90	96	-3,0	1300	180	-127,0
100	94	-4,0	1400	170	-140,0
120	95	-5,0	1500	150	-138,0
140	96	-4,0	1600	100	-144,0
160	95	-5,0	1700	82	-156,0
180	98	-4,0	1800	68	-160,0
200	95	-5,0	1900	57	-163,0
300	100	-5,0	200	49	-165,0

The standard deviations related both to accelerometer parameters and associated uncertainties are reported in Table 3. These statistics are at least twice lower than the numerical tolerances related to the uncertainties associated with the parameters. For the final MC generation, these tolerances are: $\delta_{\Delta\widetilde{S}} = 0{,}05$, $\delta_{\Delta\widetilde{f}_0} = 0{,}5$ and $\delta_{\Delta\widetilde{\beta}} = 0{,}0005$. The two last columns in this table contain the total number Q of MC generation for which stability of the model parameters was attained.

Table 2. Parameters and associated uncertainties

Accelerometer parameter	Associated uncertainties	
$\widetilde{\beta}$ [−]	$\Delta\widetilde{S}$ [m$V/(m/s^2)$]	$\Delta\widetilde{f}_0$ [Hz]
MC trials = 10^4		
0,173	1,5	69
MC trials = $2\cdot10^4$		
0,174	1,5	69
MC trials = $4\cdot10^4$		
0,175	1,4	69
MC trials = $6\cdot10^4$		
0,175	1,5	69
MC trials = $8\cdot10^4$		
0,175	1,4	68
MC trials = 10^5		
0,176	1,4	68

Table 3. Standard deviation related to the parameters and associated uncertainties as well as total number of MC generation

Standard deviation related to accelerometer parameters			Standard deviation related to uncertainties			Total number of MC generations
$\sigma_{\widetilde{S}}$ $\left[\frac{\mathrm{m}V}{m/s^2}\right]$ $\cdot10^{-2}$	$\sigma_{\widetilde{f}_0}$ $[Hz]$ $\cdot10^{-1}$	$\sigma_{\widetilde{\beta}}$ $[-]$ $\cdot10^{-4}$	$\sigma_{\Delta\widetilde{S}}$ $\left[\frac{\mathrm{m}V}{m/s^2}\right]$ $\cdot10^{-2}$	$\sigma_{\Delta\widetilde{f}_0}$ $[Hz]$ $\cdot10^{-1}$	$\sigma_{\Delta\widetilde{\beta}}$ $[-]$ $\cdot10^{-4}$	H $[-]$
MC trials = 10^4						
1,5	1,2	1,8	0,5	2,5	0,1	15150
MC trials = $2\cdot10^4$						
1,5	1,2	2,0	0,5	2,5	0,1	20288
MC trials = $4\cdot10^4$						
1,6	1,2	2,1	0,5	2,5	0,1	22207
MC trials = $6\cdot10^4$						
1,9	1,2	2,0	0,5	2,5	0,1	25128
MC trials = $8\cdot10^4$						
1,5	1,3	2,1	0,4	2,5	0,1	23789
MC trials = 10^5						
1,6	1,3	2,2	0,5	2,5	0,1	24208

Based on Table 3, it is easy to notice that, for all MC trials, the standard deviation $\sigma_{\Delta\widetilde{f}_0}$ has obtained the required value. The minimum and maximum values of Q were obtained for minimum and maximum values of MC trials, respectively.

5. ASSESSMENT OF MONTE CARLO TRIALS

The relationship between the number of MC trials and the maximum dynamic errors was checked for all cases of the accelerometer parameters increase and decrease by the associated uncertainties. The maximum values of dynamic errors were obtained for all MC trials during the following case of change parameters by associated uncertainties: $\widetilde{S} + \Delta\widetilde{S}$, $\widetilde{f}_0 + \Delta\widetilde{f}_0$, $\widetilde{\beta} - \Delta\widetilde{\beta}$. This case was indicated by 1 in Table 5, which contains the values of magnitude and rate of change constraints, the maximum values of errors D_1 and

D_2 as well as the values of the χ^2 test. It is worth noting that the minimum values of errors were obtained for the reverse case of parameter changes.

Table 5. Constraints related to input signal, maximum errors and chi-square test

Case of change parameters	A [mV]	ϑ [m/V]	D_1 [mVs]	D_2 [mVs]	χ^2
MC trials $= 10^4$					
1	97,00	95,85	55,20	40,00	87,1
MC trials $= 2 \cdot 10^4$					
1	97,00	95,73	55,00	39,77	85,1
MC trials $= 4 \cdot 10^4$					
1	96,90	95,51	54,70	39,53	82,9
MC trials $= 6 \cdot 10^4$					
1	97,00	95,60	54,81	39,61	83,0
MC trials $= 8 \cdot 10^4$					
1	96,90	95,43	54,69	39,51	82,3
MC trials $= 10^5$					
1	96,90	95,31	54,50	39,35	79,9

Based on Table 5, one can first of all point out that the χ^2 test has not been met for MC trials equal to 10^4 and $2 \cdot 10^4$. The values of this test should be greater than 47,5 and less than 84,8. This results directly from the chi-squared distribution with the number of degrees of freedom ν equal to 65 in accordance with (17). The values of this test reported in Table 5 have decreased with increasing number of MC trials. This indicates a decrease in the difference between the complex frequency responses (2) and (4). Additionally, it should be noted that the value of both errors decreases with increasing MC trials. However, increasing of the MC trial with a value equal to $2 \cdot 10^4$ entails approximately double the increase in computational time, which for the value of 10^5 was 336 hrs and 35 min. Any further increase of MC trials leads to a considerable increase in calculation time.

6. CONCLUSIONS

It worth be noted, that the minimum number of MC trials determined by the range contained in JCGN 101 standard, should be checked by the value of the chi-square test. Since this test confirms the reliability of the modelling results, the value of maximum errors obtained for the corresponding MC trails can be considered valid. It should also be pointed that for voltage output accelerometers, regardless of the number of MC trials, the maximum values of errors are always obtained for the same case of change parameters by associated uncertainties. Therefore, the calibration of such sensors based on the maximum errors can be performed only for this one case.

7. REFERENCES

1. Soysal A. O., Semlyen A.: Practical Transfer Function Estimation and its Application to Wide Frequency Range Representation of Transformers, IEEE Transactions on Power Delivery, Vol. 8, 1993, pp. 1627–1637.
2. Pintelon R., Schoukens J.: System Identification: A Frequency Domain Approach, IEEE Press, Piscataway, New York, 2001.
3. Link A., Täbner A., Wabinski W., Bruns T., Elster C.: Modelling Accelerometers for Transient Signals using Calibration Measurement upon Sinusoidal Excitation, Measurement, Vol. 40, 2007, pp. 928–935.
4. JCGM 101: Evaluation of Measurement Data. Supplement 1 to the Guide to the Expression of Uncertainty in Measurement - Propagation of Distributions using a Monte Carlo Method, 2008.
5. Tomczyk K.: Impact of Uncertainties in Accelerometer Modelling on the Maximum Values of Absolute Dynamic Error, Measurement, Vol. 49, 2016, pp. 71–78.
6. Layer E., Gawedzki W.: Dynamics of Measurement Systems, Investigation and Estimation, Polish Scientific Publisher, Warsaw, 1991.
7. Layer E., Tomczyk K.: Determination of Non-Standard Input Signal Maximizing the Absolute Error, Metrology and Measurement Systems, Vol. 17, 2009, pp. 199–208.

ZALEŻNOŚĆ POMIĘDZY LICZBĄ PRÓB MONTE CARLO W MODELOWANIU AKCELEROMETRÓW Z WYJŚCIEM NAPIĘCIOWYM A MAKSYMALNYMI BŁĘDAMI DYNAMICZNYMI

Artykuł przedstawia analizę wpływu liczby prób Monte Carlo na wartości maksymalnych błędów dynamicznych. Próby te zastosowano do modelowania akcelerometrów z wyjściem napięciowym, natomiast błędy wyrażono za pomocą kryterium maksymalnej odległości. Do modelowania akcelerometrów zastosowano ważoną metodę najmniejszych kwadratów wykorzystującą adaptacyjną procedurę Monte Carlo. Kolejne próby Monte Carlo zostały wygenerowane za pomocą generatora liczb pseudo-losowych Boxa-Mulllera. W oparciu o punkty pomiarowe charakterystyk częstotliwościowych, przedstawiono procedurę modelowania akcelerometru Endevco87. Następnie wyznaczono błędy odpowiadające sygnałom wejściowym z jednym i z dwoma ograniczeniami. Na końcu oceniono zależność pomiędzy liczbą prób Monte Carlo a wartościami maksymalnych błędów dynamicznych. Modelowanie akcelerometru zrealizowano w programie MATAB14, natomiast maksymalne błędy wyznaczono w programie MathCad15.

Słowa kluczowe: próby Monte Carlo, kryterium maksymalnej odległości, akcelerometr z wyjściem napięciowym, generator liczb pseudo-losowych Boxa-Mullera.