The Analysis of the Impact of Different Shape Functions in Tolerance Modeling on Natural Vibrations of the Rectangular Plate with Dense System of the Ribs in Two Directions

Martyna RABENDA
Department of Structural Mechanics, Lodz University of Technology
al. Politechniki 6, 90-924 Łódź, martynarabenda@gmail.com

Abstract

The main concern of this paper are thin rectangular plates with dense system of the ribs in two directions. The aim of the analysis is the examination of the impact of different shape functions in tolerance modeling on natural vibrations of the plates.
The plate is made of two different materials, both for matrix and ribs. The thickness of the plate is comparable to the width of the ribs. This provides a powerful tool for getting a desirable frequency of natural vibrations of the plate. The tolerance averaging approach is the base for the formulation of averaged model equations. The most accurate readings presenting this method are described in Wozniak et al. [1].
By application of the tolerance averaging technique to the known differential equations of considered plates, the averaged equations of the tolerance model have been derived. The general results of the contribution are illustrated using the analysis of natural vibrations. The effect of different shape functions on free vibration frequencies is examined.

Keywords: dynamic, tolerance average technique, thin plates, natural vibrations

1. Introduction

The object of the contribution is thin composite plate with dense system of the ribs. The aim of the analysis is the diagnosis of the impact of different shape functions in tolerance modeling on natural vibrations of the plates.

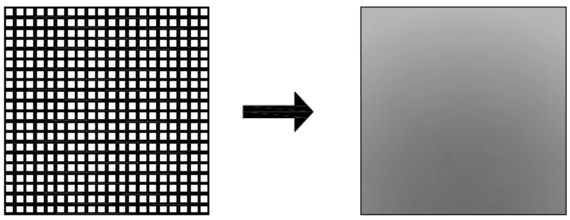

Figure 1. Composite plate at microscopic level and at macroscopic level

The space between the ribs is filled with a homogeneous matrix material (Figure 1). The analogous plate was examined in the paper [2]. The period $l = \sqrt{l_1 l_2}$ of heterogeneity is presumed to be sufficiently small versus the measure of the midplane of the plate. Simultaneously, it is assumed that the microstructure length parameter l is appropriately small in contrast with the minimum characteristic length dimension of the plate. The size of the microstructure l is comparable with the thickness of the plate h ($h \cong l$) (Figure 2). The differential equations of this kind of the plates have discontinuous and rapidly oscillating coefficients. The applications of those equations to engineering problems is not the most efficient tool. Thus, an averaged model has been proposed in which material properties are represented by functional but smooth effective stiffnesses.

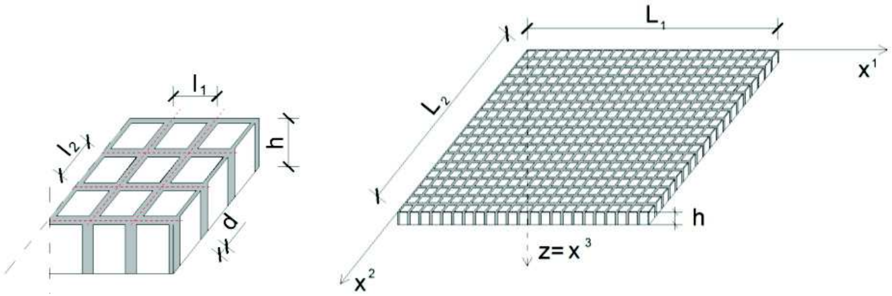

Figure 2. Detailed geometry of the plate

Analogous plate has been described in the paper [3] where it has been considered the influence of initial stress forces on the free vibrations of the plate. In this work the calculations were shown for different geometric and material properties.

The formulation of the averaged mathematical model for the analysis of dynamic behaviour of these plates is based on the tolerance averaging approach. This approach can be find in book Woźniak et al. [1]. This technique was applied in many papers. Some of the following papers can be mentioned here as examples: Baron [4] has analyzed the plates in which the period length is comparable with the thickness of the plate. In the work [5] propagation of harmonic wave in periodically laminated composites was analyzed. Furthermore, in the paper [6] the rectangular composite plate under the plane stress was analyzed. The elastic plate is reinforced by system of periodically distributed parallel ribs. Michalak [7] examined vibrations of thin plates with initial geometrical imperfections as a model of elastic wavy plates. In the contribution [8] the vibrations of periodic three-layered plates with inert core has been analysed.

In contrast to the previous works [9-10], where the gradation only in one direction is described, in the present paper it is analyzed in two directions. What is more, in the majority of above mentioned notes, in which the plates are considered, the thickness h of the plate is essentially smaller compared to the microstructure length parameter $l = \sqrt{l_1 l_2}$ (l_1, l_2 - dimensions of the cell). Baron [4] considered the thickness of the plate similar to the period length which is analogous to the current contribution. The difference is in the geometry of the plate which is reinforced in two directions not just in one (paper [4]). On a microscopic level we deal with the microheterogeneous plate while, after averaging, we deal with a special case of a functionally graded material on the macroscopic level (Figure 1).

2. Direct description and modelling technique

In this contribution the rectangular plates shown in Figure 2 are considered. The orthogonal Cartesian coordinate system is introduced $Ox_1x_2x_3$ and the time coordinate t. In all respects in the note, indices $i,k,l\ldots$ run over $1,2,3$, indices $\alpha,\beta,\gamma,\ldots$ and indices $A,B,C,\ldots$ run over $1,2$. The summation convention holds all aforementioned sub-and superscripts. Adopting $x\equiv(x_1,x_2)$ and $z=x_3$ the undeformed plate occupies the region $\Omega\equiv\{(x,z):-h/2\le z\le h/2, x\in\Pi\}$, where Π is the rectangular plate midplane and h is the plate thickness.

In the framework of a well known theory of thin plates the averaged model equations of the dynamic behavior of microheterogeneous plate are obtained. The displacement field of the arbitrary point of the plate is given in form

$$w_3(x,z)=w_3(x) \qquad w_\alpha(x,z)=w_\alpha^0(x)-\partial_\alpha w_3(x)z \tag{1}$$

Denoting by $p(x,t)$ the external forces, ρ the mass density, $g_{\alpha\beta}$ the metric tensor, $\in_{\alpha\beta}$ a Ricci tensor. Setting $\partial_k=\partial/\partial x^k$ we also introduce gradient operators $\nabla\equiv(\partial_1,\partial_2)$. After application of the linear approximated theory for thin plates we obtain the following system of equations:

(i) strain-displacement relations

$$\varepsilon_{\alpha\beta}(x,z)=\kappa_{\alpha\beta}(x)\ z, \qquad \kappa_{\alpha\beta}=-\nabla_{\alpha\beta}w_3 \tag{2}$$

(ii) strain energy

$$E_z(x,z)=\tfrac{1}{2}C^{\alpha\beta\gamma\delta}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta} \tag{3}$$

(iii) kinetic energy

$$K_z(x,z)=\tfrac{1}{2}\rho(\dot w_3\,\dot w_3+\dot w_\alpha\,\dot w_\beta\,\delta^{\alpha\beta}) \tag{4}$$

for $z\in(-h/2,h/2)$.

The strain energy averaged over the shell thickness is given by

$$E(x)=\tfrac{1}{2}B^{\alpha\beta\gamma\delta}\,\nabla_{\alpha\beta}w_3\,\nabla_{\gamma\delta}w_3 \tag{5}$$

where $B^{\alpha\beta\gamma\eta}=\dfrac{E\,h^3}{12(1-\nu^2)}0.5(\delta^{\alpha\eta}\,\delta^{\beta\gamma}+\delta^{\alpha\gamma}\delta^{\beta\eta}+\nu(e^{\alpha\gamma}e^{\beta\eta}+e^{\alpha\eta}e^{\beta\gamma})$.

The coefficients in the above equations are discontinuous and highly oscillating. The above equations will be used as a starting point of the modeling procedure.

Consequently, going to the modeling technique let us introduce the orthogonal coordinates system $O\xi^1\xi^2$ in the undeformed midplane. The midplane of the plate occupies the region $\Pi\equiv[0,L_1]\times[0,L_2]$ (Figure 2). Assuming that the number of ribs in ξ^1 and ξ^2 directions is respectively n and m $(1/n,1/m<<1)$. Hence $l_1=L_1/n$ and $l_2=L_2/m$ are the dimensions of the cell $\Delta\equiv(-l_1/2,l_1/2)\times(-l_2/2,l_2/2)$. We introduce, for the arbitrary cell $\Delta(\xi^\alpha)\equiv\Delta+\xi^\alpha$ with center situated at point (ξ^1,ξ^2), the orthogonal local coordinate system Oy_1y_2 which is local with its origin at $(\xi^1,\xi^2)\in\overline{\Pi}_\Delta$ where $\Pi_\Delta\equiv(l_1/2,L_1-l_1/2)\times(l_2/2,L_2-l_2/2)\subset\Pi$.

In order to derive averaged model equations for skeletonal plate under consideration we applied tolerance averaging approach [1]. There will be introduced some basic concepts of this technique: an averaging operator, a tolerance parameter, a tolerance periodic function, a slowly varying function and a highly oscillating function.
The starting point of the modeling procedure is a decomposition of displacement fields.

$$\begin{aligned} &w_3(\xi^\alpha,z,t)=V_3(\xi^\alpha,t)\\ &w_\alpha(\xi^\alpha,z,t)=(-\partial_\alpha V_3(\xi^\alpha,t)+h^A(\xi^\alpha)u_\alpha^A(\xi^\alpha,t))\,z \end{aligned}\tag{6}$$

for $\xi^\alpha=\Pi$, $z\in(-h/2,h/2)$, $A=I,II$ and every time t.
The governing equations derived from stationary action principle of the averaged lagragian [2,3] $<\mathrm{L}>=<K>-<E>+<F>$ have the form

$$\begin{aligned} &\nabla_{\alpha\beta}\left(\widetilde{B}^{\alpha\beta\gamma\delta}\nabla_{\gamma\delta}V_3-\widetilde{B}^{\gamma A\alpha\beta}u_\gamma^A\right)+\langle\widetilde{\rho}\rangle\ddot{V}_3-\left\langle f^3\right\rangle=0\\ &\widetilde{B}^{\alpha A\gamma\delta}\nabla_{\gamma\delta}V_3-\widetilde{B}^{\alpha A\gamma B}u_\gamma^B=0 \end{aligned}\tag{7}$$

After simple manipulations we obtain finally the following equation for the averaged displacements $V_3(\xi^\alpha,t)$,

$$\nabla_{\alpha\beta}\left(F^{\alpha\beta\gamma\delta}\nabla_{\gamma\delta}V_3\right)+\langle\widetilde{\rho}\rangle\ddot{V}_3=\left\langle f^3\right\rangle\tag{8}$$

where $\widetilde{\rho}=\rho h$ is mass density related to plate midplane. In contrast to equations in direct description with the discontinuous and highly oscillating coefficients, the coefficients in the above equation are smooth and functional.

3. Applications - fluctuation shape functions

The key point of the tolerance modeling technique is to determine of fluctuation shape function (FSF). In dynamic problems, the system of fluctuation shape function can be taken to represent the principal modes of free vibrations of the cell $\Delta(x_\alpha)$ or a physically reasonable approximation of these modes. Our analysis is to investigate the impact of different shape functions on free vibrations of the plate. We are restricted to the case where we have two fluctuation shape functions, $h^I(x_\alpha,y_\alpha)$ and $h^{II}(x_\alpha,y_\alpha)$ (Figure 3)

$$h^I(x_\alpha,y_\alpha)=S_1(y_1)\cdot\left[1-\left(\frac{2y_2}{b_2}\right)^2\right],\qquad h^{II}(x_\alpha,y_\alpha)=S_2(y_2)\cdot\left[1-\left(\frac{2y_1}{b_1}\right)^2\right]\tag{9}$$

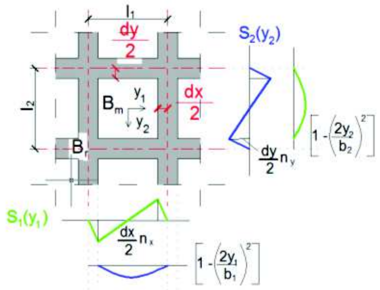

Figure 3. Fluctuation shape functions in the considered cell

$$S_1(y_1)=\begin{cases} -\left(y_1+\dfrac{l_1}{2}\right)\cdot\eta_x & y_1\in\left\langle -\dfrac{l_1}{2},-\dfrac{b_1}{2}\right\rangle \\ \dfrac{dx}{l_1-dx}\cdot y_1\eta_x & y_1\in\left\langle -\dfrac{b_1}{2},\dfrac{b_1}{2}\right\rangle \\ -\left(y_1-\dfrac{l_1}{2}\right)\cdot\eta_x & y_1\in\left\langle \dfrac{b_1}{2},\dfrac{l_1}{2}\right\rangle \end{cases}$$

$$S_2(y_2)=\begin{cases} -\left(y_2+\dfrac{l_2}{2}\right)\cdot\eta_y & y_2\in\left\langle -\dfrac{l_2}{2},-\dfrac{b_2}{2}\right\rangle \\ \dfrac{dy}{l_2-dy}\cdot y_2\eta_y & y_2\in\left\langle -\dfrac{b_2}{2},\dfrac{b_2}{2}\right\rangle \\ -\left(y_2-\dfrac{l_2}{2}\right)\cdot\eta_y & y_2\in\left\langle \dfrac{b_2}{2},\dfrac{l_2}{2}\right\rangle \end{cases} \tag{10}$$

where $b_1=l_1-dx(x_1)$ $\quad b_2=l_2-dy(x_2)$.

We have considered for four different amplitudes of functions $S_1(y_1)$ and $S_2(y_2)$ as following (respectively *versions 1-4*):

1. $\eta_x(x_1)=\dfrac{dx(x_1)}{l_1}$, $\eta_y(x_2)=\dfrac{dy(x_2)}{l_2}$

2. $\eta_x(x_1)=\dfrac{dx(x_1)}{l_1}\cdot\dfrac{2(B_r-B_m)}{(B_r+B_m)}$, $\eta_y(x_2)=\dfrac{dy(x_2)}{l_2}\cdot\dfrac{2(B_r-B_m)}{(B_r+B_m)}$ (10b)

3. $\eta=\eta_x(x_1)=\eta_y(x_2)=\dfrac{1}{l_1\cdot l_2}\sqrt{(l_1-dx(x_1))\cdot(l_2-dy(x_2))\cdot(l_1dy(x_2)+l_2dx(x_1)-dx(x_1)dy(x_2))}$

$\eta=\dfrac{1}{l_1\cdot l_2}\sqrt{\dfrac{2(B_r-B_m)}{(B_r+B_m)}(l_1-dx(x_1))\cdot(l_2-dy(x_2))\cdot(l_1dy(x_2)+l_2dx(x_1)-dx(x_1)dy(x_2))}$ We have analyzed free vibrations of a simple supported square plate with the constant width of the ribs. Taking into the consideration *tolerance model*, we obtain from (8) differential equation describing dynamic behavior of the considered plate

$$F^{1111}\frac{\partial^4 V_3}{\partial x^4}+2(F^{1122}+2F^{1212})\frac{\partial^4 V_3}{\partial x^2 \partial y^2}+F^{2222}\frac{\partial^4 V_3}{\partial y^4}+<\widetilde{\rho}>\ddot{V}_3=0 \tag{11}$$

where for square plate and $d_x = d_y = d$, $l_1 = l_2 = l$ we have

$$\begin{aligned} F^{1111} &= F^{2222} = \widetilde{B}^{1111} - (1+\nu^2)(\widetilde{B}^{1I11})^2 K^{1I1I} - 2\nu(\widetilde{B}^{1I11})^2 K^{1I2II} , \\ F^{1122} &= \nu B^{1111} - (1+\nu^2)(B^{1I11})^2 K^{1I2II} - 2\nu(B^{1I11})^2 K^{1I1I} , \\ F^{1212} &= F^{1221} = \frac{1-\nu}{2} B^{1111} - 2(\frac{1-\nu}{2})^2 (B^{1I11})^2 (K^{1II1II} + K^{1II2I}) , \end{aligned} \tag{12}$$

Exemplified modulus:

$$K^{1I1I} = \frac{\widetilde{B}^{2II2II}}{\widetilde{B}^{1I1I}\widetilde{B}^{2II2II} - (\widetilde{B}^{1I2II})^2} , \quad K^{1I2II} = \frac{-\widetilde{B}^{1I2II}}{\widetilde{B}^{1I1I}\widetilde{B}^{2II2II} - (\widetilde{B}^{1I2II})^2} ,$$

$$\widetilde{B}^{1111} = B_r[(1-nd)(nd+\alpha(1-nd))+nd] , \quad <\widetilde{\rho}>= \widetilde{\rho}_r[(1-nd)(nd+\beta(1-nd))+nd] ,$$

$$\alpha = B_m / B_r , \qquad \beta = \widetilde{\rho}_m / \widetilde{\rho}_r , \qquad nd = d / l .$$

In the above formulae we have assume: Poisson's ratio $\nu = \nu_m = \nu_r$, B_m, B_r stiffness of the matrix and rib respectively, $\widetilde{\rho}_m, \widetilde{\rho}_r$ - mass density of the matrix and rib related to the plate midplane.

The equation (11) is in the form analogous to equation of motion of homogeneous orthotropic plate. This equation will be solved similar to known method for simply supported rectangular plates. Restricting our considerations to harmonic vibrations $V(x^1, x^2, t) = V(x^1, x^2)e^{i\omega t}$ we derive equation

$$\frac{\partial^4 V}{\partial x^4} + 2\eta\varepsilon^2 \frac{\partial^4 V}{\partial x^2 \partial y^2} + \varepsilon^4 \frac{\partial^4 V}{\partial y^4} - \frac{<\widetilde{\rho}>}{F^{1111}}\omega^2 V = 0 \tag{13}$$

where $\varepsilon^4 = \frac{F^{2222}}{F^{1111}}$, $\eta = \frac{F^{1122} + 2F^{1212}}{\sqrt{F^{2222}/F^{1111}}}$

Substituting $V(x,y) = V_{mn} \sin(\frac{m\pi}{L_x}x)\sin(\frac{n\pi}{L_y}y)$ into equation (13) we derive formula for free vibration frequencies

$$\omega_{mn} = \frac{\pi^2}{{L_x}^2}\sqrt{\frac{F^{1111}}{<\widetilde{\rho}>}}\sqrt{m^4 + 2\eta\varepsilon^2(\frac{mnL_x}{L_y})^2 + \varepsilon^4(\frac{nL_x}{L_y})^4} \tag{14}$$

The results obtained above were compared to finite element method calculated by Abaqus program [11]. It was considered two-dimensional shell element with a thickness equal to 0,10m. The way of modeling of the plate in Abaqus program was described in the paper [2]. Ribs are represented by the slave and matrix by master surface. The boundary conditions were established as simply supported along the circumference of the plate. Calculations were provided for the linear perturbation (frequency). As mesh element we assume S4R element as a 4-node doubly curved thin (or thick) shell which provides reduced integration, hourglass control and finite membrane strains. The mesh was added to the matrix and ribs separately bearing in mind that for the slave surface the mesh needs to be denser. To verify model equations and Abaqus program there will be compared values of the first four vibration frequencies.

4. Results

Free vibrations frequencies for the plate with constant width of the ribs and geometric and material parameters shown below for different shape functions are in Table 1.

Geometric data: $h = 0{,}1m$, size of the plate: $L_1 = L_2 = 4{,}0m$, width of the ribs: $d = 0{,}05m$, size of the cell: $l_1 = l_2 = 0{,}20m$. Material data: $E_r = 210GPa,\ \nu_r = \nu_m = 0{,}3,\ \rho_r = 7800kg/m^3,\ E_m = 20GPa,\ \rho_m = 2400kg/m^3$

Table 1. First four free vibrations frequencies for different shape functions

	1st mode	2nd mode	3rd mode	4th mode	Versions 1-4/ Abaqus
	[Hz]	[Hz]	[Hz]	[Hz]	
Version 1	141,783	361,147	361,147	567,133	1,62%
Version 2	161,497	405,614	405,614	645,987	13,63%
Version 3	170,305	426,001	426,001	681,22	18,09%
Version 4	182,561	455,748	455,748	730,245	23,59%
Abaqus	139,490	357,32	357,32	555,060	

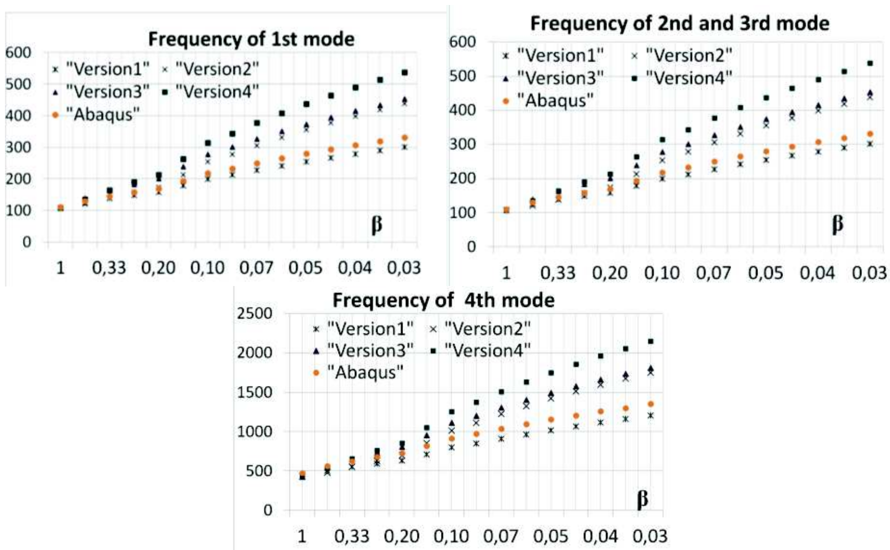

Figure 4. First four free vibrations frequencies depending on parameter β

In the Figure 4 there are shown free frequencies of the first four modes. On the horizontal axis is presented parameter $\beta = E_m / E_r$. The calculations are made for the constant density equal to $2400 kg / m^3$ and respectively for different amplitudes (10b).

5. Conclusion

It can be observed that free vibrations for different *versions* vary from 2% till 24%. Only the 2nd and 4th versions depend on Young's modulus, We can recognize that the results shown in the Figure 4 are convergent for homogenous plate $(\beta = 1)$. The higher the β parameter is, the higher is the difference between parameters**.** The most consistent with Abaqus' outcome is the 1st *version*. Further research, in which influence of different Young's modulus on the free vibrations will be investigated.

References

1. C. Woźniak, B. Michalak, J. Jędrysiak, (eds.), *Thermomechanics of Heterogeneous Solids and Structures*, Lodz University of Technology, Lodz 2008.
2. B. Michalak, M. Rabenda, *Tolerance modelling of dynamic behavior of thin plates with dense system of ribs in two direction*, paper under review in Archives of Civil and Mechanical Engineering.
3. M. Rabenda, B. Michalak, *Natural vibrations of prestressed thin functionally graded plates with dense system of ribs in two directions*, Composite Structures, **133** (2015) 1016 – 1023.
4. E. Baron , *On modelling of periodic plates with inhomogeneity period of an order of the plate thickness*, J.Theor. Appl. Mech., **44** (2006) 3 – 18.
5. S. J. Matysiak, W. Nagórko, *On the wave propagation in periodically laminated composites*, Bulletin of the Polish Academy of Science - Technical Sciences, **43** (1995) 1 – 12.
6. K. Jeleniewicz, W. Nagórko, *Free vibrations in periodically ribbed elastic plate*, Acta Scientiarum Polonorum, s. Architectura, **9**(2) (2010) 11 – 21.
7. B. Michalak, *Vibrations of plates with initial geometrical periodical imperfections interacting with a periodic elastic foundation*, Archive of Applied Mechanics, **70** (2000) 508 – 518.
8. J. Marczak, J. Jędrysiak, *Tolerance modelling of vibrations of periodic three-layered plates with inert core*, **134** (2015) 854 – 861.
9. B. Michalak, A. Wirowski, *Dynamic modeling of thin plate made of certain functionally graded materials*, Meccanica, **47** (2012) 1487 – 1498.
10. C. Woźniak, M. Wagrowska, O. Szlachetka, *Asymptotic modelling and design of some microlayered functionally graded heat conductors*, ZAMM, **92** (2012) 841 – 848.
11. https://portal.plgrid.pl/web/guest/home