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GENERAL TOPOLOGY

Nonhomogeneity of Remainders, II

by

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Summary. We present an example of a separable metrizable topological group G having the property that no remainder of it is (topologically) homogeneous.

1. Introduction. *All topological spaces under discussion are Tychonoff.*

A space X is *homogeneous* if for any two points $x, y \in X$ there is a homeomorphism h from X onto itself such that $h(x) = y$. If bX is a compactification of a space X, then $bX \setminus X$ is called its *remainder*.

In 1956, Walter Rudin [13] proved that the Čech–Stone remainder $\beta\omega \setminus \omega$, where ω is the discrete space of non-negative integers, is not homogeneous under CH. This result was later generalized considerably by Frolík [9] who showed in ZFC that $\beta X \setminus X$ is not homogeneous, for any nonpseudocompact space X. For other results in the same spirit, see e.g. [6], [7], [10].

Hence the study of (non)homogeneity of Čech–Stone remainders has a long history. In this note we continue our study begun in [4] concerning the (non)homogeneity of arbitrary remainders of topological spaces. Special attention is given to remainders of non-locally compact topological groups. For some recent facts on such remainders, see Arhangel'skii [1] and [2]. One of them, established in [1], is: every remainder of a topological group is either Lindelöf or pseudocompact.

The aim of this note is to present an example of a separable metrizable topological group G no remainder of which is homogeneous. The first examples of topological groups that share this property can be found in [4]; these examples have various interesting properties but are not metrizable.

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2. The example. For a space X, we let $\mathscr{H}(X)$ denote its group of homeomorphisms. We will make good use of the Alexandroff–Hausdorff Theorem that every uncountable Borel subset of a Polish space contains a Cantor set [11, p. 447].

A group G is called *Boolean* if each of its elements has order at most 2. Clearly, every Boolean group is Abelian. We use additive notation for Abelian groups.

Our example is the example from van Mill [12] of a separable metrizable topological group G having no homeomorphisms other than translations. Such a group is easily seen to be Boolean. We will state the properties of G that we will need in the verification that it has no homogeneous remainder.

(P1) G is a subgroup of a Boolean topological group H which is homeomorphic to Hilbert space ℓ^2.
(P2) Every homeomorphism of G is a translation.
(P3) G intersects every Cantor set in H.
(P4) G is locally connected.
(P5) G has index $\mathfrak{c}$, i.e., $|H/G| = \mathfrak{c}$.

Properties (P1)–(P4) are stated explicitly in [12]. It is not clear whether Property (P5) follows from the construction there. However, the variations of G in Arhangel'skii and van Mill [3] all have index $\mathfrak{c}$. This follows from the definition of G_κ and the proof of Lemma 6.8, both on page 922 of [3].

It is clear that G is not locally compact, being a dense subgroup of H.

Lemma 2.1. *If $K \subseteq H$ is a Cantor set, then $K \setminus G \neq \emptyset$.*

Proof. Indeed, pick an arbitrary $x \in H \setminus G$. Such a point exists by (P5). Hence $(x + K) \cap G \neq \emptyset$ by (P3), or, equivalently, $K \cap (x + G) \neq \emptyset$. ■

Lemma 2.2. *Let U be a nonempty open and connected subset of G. If $A \subseteq H$ is countable, then $U \setminus A$ is connected.*

Proof. Striving for a contradiction, assume that there exist disjoint and relatively open subsets E and F of $U \setminus A$ such that $E \cup F = U \setminus A$. Pick disjoint open subsets E' and F' of U such that $E' \cap (U \setminus A) = E$ and $F' \cap (U \setminus A) = F$ [8, 2.1.7]. Let U' be an open subset of H such that $U' \cap G = U$. By (P3), G is dense in H, and hence there are disjoint open subsets E'' and F'' of U' such that $E'' \cap G = E'$ and $F'' \cap G = F'$. Consequently, the set $S = U' \setminus (E'' \cup F'')$ separates the connected open subset U' of H. Hence S is uncountable, H being homeomorphic to ℓ^2. Then S contains a Cantor set K. By (P3), $G \cap K$ has size $\mathfrak{c}$. But $G \cap K$ is contained in the countable set A, which is a contradiction. ■

Now assume that aG is an arbitrary compactification of G. We will show that $aG \setminus G$ is not homogeneous.

Let bG be a metrizable compactification of G such that $bG \leq aG$ in the usual order of compactifications [8, 3.5.F]. Let $f\colon aG \to bG$ be a continuous function which restricts to the identity on G. Since both bG and H are Polish, by the Lavrentieff Theorem [8, 4.3.21] there are G_δ-subsets S of bG and T of H both containing G such that the identity function $G \to G$ can be extended to a homeomorphism $h\colon S \to T$. We claim that $H \setminus T$ is countable. It is an F_σ-subset of H and hence if it were uncountable, it would contain a Cantor set which would intersect G by (P3), and this is absurd.

Since $|H/G| = \mathfrak{c}$ by (P5), there exist $p, q \in H$ such that

$$(\dagger) \qquad (p+G) \cap (q+G) = \emptyset, \quad (p+G) \cup (q+G) \subseteq T \setminus G.$$

By abuse of notation, we will identify S and T so that we can think of the cosets $p+G$ and $q+G$ as subsets of the remainder $bG \setminus G$. Let $A \subseteq G$ be a discrete sequence converging to p in bG, and take a limit point a of A in aG. Moreover, take $b \in bG \setminus G$ such that $f(b) \notin p+G$. We will show that no homeomorphism of $aG \setminus G$ takes a to b. Striving for a contradiction, assume that $\xi \in \mathscr{H}(aG \setminus G)$ is such that $\xi(a) = b$.

Lemma 2.3. *If U is a nonempty connected open subset of G, and V is an open subset of aG such that $V \cap G = U$, then $V \setminus G$ is connected* (*and nonempty*)*.*

Proof. That $V \setminus G$ is nonempty is clear.

Assume that E and F are disjoint nonempty open subsets of $aG \setminus G$ such that $E \cup F = V \setminus G$. Since $aG \setminus G$ is dense in aG, there are disjoint open subsets E' and F' of V such that $E' \cap (aG \setminus G) = E$ and $F' \cap (aG \setminus G) = F$. Observe that $K = V \setminus (E' \cup F')$ separates V and hence U. Clearly, S is locally compact, being closed in the locally compact open subset V of aG. But S is also contained in G, hence it is σ-compact (being separable and metrizable). Hence from Lemma 2.1, we conclude that K is countable. But this contradicts Lemma 2.2. ■

Lemma 2.4. *ξ can be extended to a homeomorphism $\bar{\xi}\colon aG \to aG$.*

Proof. Here we apply an idea of Curtis and van Mill [5, 4.1]. Fix $x \in G$. By (P4), G is locally connected at x. Hence we may fix a decreasing neighborhood base $(U_n)_n$ at x consisting of connected open subsets of G. For every n, let V_n in aG be open such that $V_n \cap G = U_n$. By Lemma 2.3, $V_n \setminus G$ is connected and nonempty, hence $\xi(V_n \setminus G)$ is connected, from which it follows that

$$T_x = \bigcap_{n<\omega} \overline{\xi(V_n \setminus G)},$$

being the intersection of a decreasing sequence of nonempty continua, is a nonempty continuum in aG.

We first claim that T_x is contained in G. Indeed, if $p \in aG \setminus G$, then there exists $n < \omega$ such that $\xi^{-1}(p) \notin \overline{U}_n$ (here the closure is taken in aG). This implies that $p \notin \overline{\xi(V_n \setminus G)}$ (simply observe that $V_n \subseteq \overline{U}_n$).

We next claim that T_x is a degenerate continuum. Indeed, if T_x were nondegenerate, it would contain a Cantor set, which would violate Lemma 2.1. So we conclude that T_x is a single point, say $\{g_x\}$.

Now define $\bar{\xi}\colon aG \to aG$ by

$$\bar{\xi}(x) = \begin{cases} \xi(x) & (x \in aG \setminus G), \\ g_x & (x \in G). \end{cases}$$

It is easy to see that $\bar{\xi}$ is continuous and has a continuous inverse, hence is a homeomorphism. ∎

By (P2), $\eta = \bar{\xi}\restriction G$ is a translation. Hence there exists $g \in G$ such that $\eta(x) = x + g$ for every $x \in G$. Since a is a limit point of the discrete set A, $\bar{\xi}(a)$ is a limit point of $g + A$. But $g + A$ converges in bG to $g + p$, hence

$$f(\bar{\xi}(a)) = f(\xi(a)) = g + p \in p + G.$$

As a consequence, $\xi(a) \neq b$, since $f(b) \notin p + G$.

It is clear that G, being a Bernstein set, is very bad from the descriptive point of view.

QUESTION 2.5. Let G be a Polish (Borel, analytic) separable metrizable topological group. Is there a compactification bG of G such that $bG \setminus G$ is homogeneous?

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