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# THE TEMPERATURE OF A BRAKE DISC DURING FRICTIONAL HEATING WITH LINEAR DISTRIBUTION OF THE CONTACT PRESSURE

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**Abstract:** In this paper the analytical solution of the boundary–value heat conduction problem for a brake rotor was developed. A solid brake disc is heated by frictional heat flux during braking with constant deceleration. Intensity of the heat flux affecting friction surface of the disc is proportional to the specific power of friction. It was assumed that contact pressure between the pad and the disc increases linearly, from zero in the initial moment of the braking process to the maximum value in standstill. Calculations were carried out on variables and parameters in the dimensionless form. The obtained results were compared with adequate resultsduring braking with constant deceleration, with an assumption of pressure constant in time.

**Keywords:** contact pressure, frictional heating, heat conduction equation, temperature, friction, brake disc.

## Introduction

**Nomenclature**:
$a$ – effective depth of the heat penetra-tion [m];
erf($x$) – Gauss error function;
erfc($x$)=1 – erf($x$) – complementary error function;
$\mathrm{ierfc}(x) = \pi^{-1/2}\exp(-x^2) - x\,\mathrm{erfc}(x)$ – integral of the complementary error function;
$f$ – friction coefficient;
$K$ – thermal conductivity [W K$^{-1}$ m$^{-1}$];
$k$ – thermal diffusivity [m$^2$ s$^{-1}$];
$p$ – contact pressure [Pa];
$q$ – intensity of the frictional heat flux[W m$^{-2}$];
$t$ – time [s];
$t_s$ – braking time[s];
$T$ – temperature [K];
$T^*$ – dimensionless temperature;
$T_0$ – temperature scaling factor [K];
$T_a$ – ambient temperature [K];
$V$ – velocity sliding [m s$^{-1}$ ];
$\tau$ – Fourier number;
$\zeta$ – dimensionless spatial coordinate.

In the mechanical brakes large amount of heat is generated due to friction. High temperature on the contact surface causes changes of thermophysical properties and deterioration of frictional couple, which results in reduction of the braking efficiency [8]. The temperature is a main parameter for the assessment of abrasive wear and thermal stability of friction systems. Therefore, knowledge of the temperature field is a priority in the brake system design process. One of the effective methods used to predict the maximal temperature on the pad/disc interface is formulation and solving of friction heat problems. These are boundary–value heat conduction problems with defined heat flux on the friction surface. The intensity of frictional heat flux is proportional to the specific power of friction forces. It is equal to product of friction coefficient, contact pressure and sliding speed [3]. In this study a mathematical model of transient temperature field in a brake disc was developed using this method. It was assumed that, the braking process proceeds with linear distribution of pressure and constant deceleration.

## Statement to the problem

Frictional heating of the contact surface of a brake disc is considered. The assumptions were applied:
- the material of the disc is homogeneous and isotropic;
- spatial distribution of the contact pressure on pad/disc interface increases linearly:

$$p(t) = p_0 p^*(t),\ p^*(t) = t/t_s,\ 0 \le t \le t_s\,; \qquad (1)$$

- braking proceeds with constant deceleration i.e. velocity of the vehicle decreases linearly from the maximal value $V_0$ in the initial time $t = 0$ , to zero in the retention time $t = t_s$:

$$V(t) = V_0 V^*(t),\ V^*(t) = 1 - t/t_s,\ 0 \le t \le t_s\,; \qquad (2)$$

- the sum of heat fluxes intensities directed perpendicularly from the friction surface to the inside of the brake disc, is equal to the specific power of friction forces:

$$q(t) = q_0 q^*(t),\ 0 \le t \le t_s, \qquad (3)$$

where:

$$q_0 = f p_0 V_0, \qquad (4)$$

$$q^*(t) = p^*(t)V^*(t),\ 0 \le t \le t_s\,; \tag{5}$$

- gradient of temperature in radial and circumferential directions are negligible;
- the free surface of the brake rotor is adiabatic;
- in the initial time of the braking process the temperature of the disc is constant, equal to $T_a$.

According to the above assumptions the transient temperature field of the disc is one–dimensional. To find the distribution of the temperature, we had the following parabolic heat conduction problem boundary–value for semi–space in the Cartesian coordinate system *Oxyz* (Fig. 1).

$$\frac{\partial^2 T(z,t)}{\partial z^2} = \frac{1}{k}\frac{\partial T(z,t)}{\partial t},\ 0 \le z < \infty,\ 0 \le t \le t_s\,, \tag{6}$$

$$K\left.\frac{\partial T(z,t)}{\partial z}\right|_{z=0} = -q(t),\ 0 \le t \le t_s\,, \tag{7}$$

$$T(z,t) \to 0,\ z \to \infty,\ 0 \le t \le t_s\,, \tag{8}$$

$$T(z,0) = T_a\,,\ 0 \le z < \infty,\,, \tag{9}$$

where intensity of the heat flux *q(t)* has form (3–5).

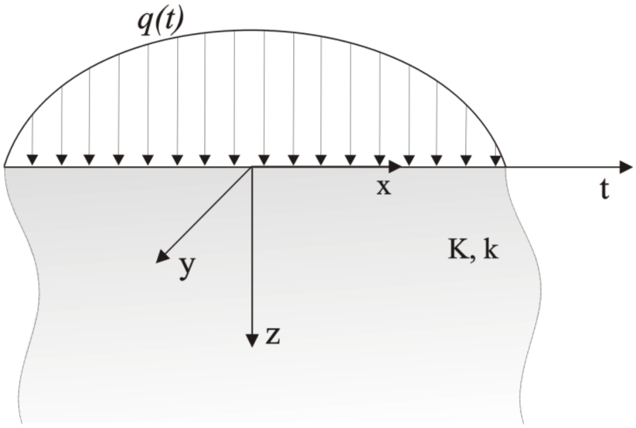


Fig. 1. Scheme of the problem.

Applying the following dimensionless variables and parameters:

$$\zeta = \frac{z}{a}\,,\ \tau = \frac{kt}{a^2}\,,\ \tau_s = \frac{kt_s}{a^2}\,,\ T_0 = \frac{q_0 a}{K},\ T^* = \frac{T - T_a}{T_0}, \tag{10}$$

($a = \sqrt{3kt_s}$ – effective depth of the heat penetration inside the brake disc [4, 5]), considered boundary–value problem (6–9) was written in dimensionless form:

$$\frac{\partial^2 T^*(\zeta,\tau)}{\partial \zeta^2} = \frac{1}{k^2}\frac{\partial T^*(\zeta,\tau)}{\partial \tau},\ 0 \le \zeta\ , 0 \le \tau \le \tau_s\,, \tag{11}$$

$$\left.\frac{\partial T^*(\zeta,\tau)}{\partial \zeta}\right|_{\zeta=0} = -q^*(\tau),\ 0 \le \tau \le \tau_s\,, \tag{12}$$

$$T^*(\zeta,\tau) \to 0,\ \zeta \to \infty,\ 0 \le \tau \le \tau_s\,, \tag{13}$$

$$T^*(\zeta,0) = 0,\ 0 \le \zeta\ , \tag{14}$$

where

$$q^*(\tau) = \frac{\tau}{\tau_s}\left(1 - \frac{\tau}{\tau_s}\right), 0 \le \tau \le \tau_s\,. \tag{15}$$

## Solution and verification to the problem

Solution to the formulated boundary-value problem of heat conduction (11–15) was found based on Duhamel's theorem [6], which for the considered case can be written in the form:

$$T^*(\zeta,\tau) = \int_0^\tau q^*(s)\frac{\partial}{\partial \tau}T^{*(0)}(\zeta,\tau - s)ds\,, \zeta \ge 0,\ 0 \le \tau \le \tau_s\,, \tag{16}$$

where [2]

$$T^{*(0)}(\zeta,\tau) = 2\sqrt{\tau}\,\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right),\ \zeta \ge 0\,,\ 0 \le \tau \le \tau_s\,, \tag{17}$$

is solution to the problem (11)–(15) with constant intensity of heat flux $q^*(\tau) = 1$ in boundary condition (12).

Substituting function $q^*(\tau)$ (15) and solution (17) in equation (16), we have:

$$T^*(\zeta,\tau) = \int_0^\tau \frac{s}{\tau_s}\left(1 - \frac{s}{\tau_s}\right)\frac{\partial}{\partial \tau}\left[2\sqrt{\tau - s}\ \mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau - s}}\right)\right]ds\,,\quad 0 \le \tau \le \tau_s\,. \tag{18}$$

Taking into account the value of the following derivative [1]:

$$\frac{d}{dx}\mathrm{erfc}(x) = -\frac{2x}{\sqrt{\pi}}e^{-x^2}\,, \tag{19}$$

derivative of integral of the complementary error function was counted:

$$\frac{d}{dx}[\mathrm{ierfc}(x)] = -\frac{2x}{\sqrt{\pi}}e^{-x^2} - \mathrm{erfc}(x) - x\frac{d}{dx}\mathrm{erfc}(x) = -\mathrm{erfc}(x)\,\cdot \tag{20}$$

Using relation (20), we achieved:

$$\frac{\partial}{\partial \tau}\left[2\sqrt{\tau - s}\ \mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau - s}}\right)\right] = \frac{1}{\sqrt{\tau - s}}\left[\frac{1}{\sqrt{\pi}}e^{-\left(\frac{\zeta}{2\sqrt{\tau - s}}\right)^2} + \frac{\zeta}{2\sqrt{\tau - s}}\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau - s}}\right)\right] + \frac{\zeta}{2(\tau - s)}\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau - s}}\right) = \frac{e^{-\left(\frac{\zeta}{2\sqrt{\tau - s}}\right)^2}}{\sqrt{\pi(\tau - s)}}\,. \tag{21}$$

Substituting the partial derivative (21) to the formula (18), we have:

$$T^*(\zeta,\tau)=\frac{1}{\tau_s}I_1(\zeta,\tau)-\frac{1}{\tau_s^2}I_2(\zeta,\tau), \tag{22}$$

where

$$I_1(\zeta,\tau)=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{s}{\sqrt{\tau-s}}e^{-\left(\frac{\zeta}{2\sqrt{\tau-s}}\right)^2}ds,$$

$$I_2(\zeta,\tau)=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{s^2}{\sqrt{\tau-s}}e^{-\left(\frac{\zeta}{2\sqrt{\tau-s}}\right)^2}ds. \tag{23}$$

Subsequently, function $I_1(\zeta,\tau)$ was presented in the following form:

$$I_1(\zeta,\tau)=J_1(\zeta,\tau)-J_2(\zeta,\tau), \tag{24}$$

where

$$J_1(\zeta,\tau)=\frac{\tau}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-s}}e^{-\left(\frac{\zeta}{2\sqrt{\tau-s}}\right)^2}ds,$$

$$J_2(\zeta,\tau)=\frac{1}{\sqrt{\pi}}\int_0^\tau \sqrt{\tau-s}\,e^{-\left(\frac{\zeta}{2\sqrt{\tau-s}}\right)^2}ds. \tag{25}$$

Using the method of substitution the formula (25) was received:

$$J_1(\zeta,\tau)=\left\{\begin{array}{l} x=\frac{1}{\sqrt{\tau-s}} \\ ds=\frac{2}{x^3}dx \end{array}\right\}=\frac{2\tau}{\sqrt{\pi}}L_2(\zeta,\tau), \tag{26}$$

where

$$L_2(\zeta,\tau)=\int_{\frac{1}{\sqrt{\tau}}}^{\infty} e^{-\left(\frac{\zeta}{2}\right)^2x^2}\frac{dx}{x^2}. \tag{27}$$

Counting the integral (27) [7]:

$$L_2(\zeta,\tau)=\left[-\frac{1}{x}e^{-(\frac{\zeta}{2}x)^2}-\sqrt{\pi}\frac{\zeta}{2}\operatorname{erf}\left(\frac{\zeta}{2}x\right)\right]\Bigg|_{\frac{1}{\sqrt{\tau}}}^{\infty},\ \zeta>0, \tag{28}$$

from equation (26) it was found:

$$J_1(\zeta,\tau)=\frac{2\tau}{\sqrt{\pi}}\left[-\frac{1}{x}e^{-(\frac{\zeta}{2}x)^2}-\sqrt{\pi}\frac{\zeta}{2}\operatorname{erf}\left(\frac{\zeta}{2}x\right)\right]\Bigg|_{\frac{1}{\sqrt{\tau}}}^{\infty}=\frac{2\tau\sqrt{\tau}}{\sqrt{\pi}}e^{-\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2}-2\tau\frac{\zeta}{2}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)=2\tau\sqrt{\tau}\operatorname{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right). \tag{29}$$

Function $J_2(\zeta,\tau)$ (25) was found in the similar way:

$$J_2(\zeta,\tau)=\left\{\begin{array}{l} x=\frac{1}{\sqrt{\tau-s}} \\ ds=\frac{2}{x^3}dx \end{array}\right\}=\frac{2}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{\tau}}}^{\infty} e^{-\left(\frac{\zeta}{2}\right)^2x^2}\frac{dx}{x^4}=\frac{2}{\sqrt{\pi}}L_4(\zeta,\tau). \tag{30}$$

Exploiting the following recurrence relation [7]:

$$L_p(\zeta,\tau)=\int\frac{e^{-(ax)^2}}{x^n}dx=-\frac{e^{-(ax)^2}}{(n-1)x^{n-1}}-\frac{2a^2}{n-1}\int\frac{e^{-(ax)^2}}{x^{n-2}}dx,$$

$$a>0,\ n=2,3,..., \tag{31}$$

and relation (28), we achieved:

$$L_4(\zeta,\tau)=\left\{-\frac{e^{-\left(\frac{\zeta}{2}\right)^2x^2}}{3x^3}-\frac{2}{3}\left(\frac{\zeta}{2}\right)^2\left[-\frac{1}{x}e^{-\left(\frac{\zeta}{2}\right)^2x^2}-\sqrt{\pi}\left(\frac{\zeta}{2}\right)\operatorname{erf}\left(\frac{\zeta}{2}x\right)\right]\right\}\Bigg|_{\frac{1}{\sqrt{\tau}}}^{\infty}=\left\{\frac{2}{3}\sqrt{\pi}\left(\frac{\zeta}{2}\right)^2\left[\frac{\zeta}{2}+x^{-1}\operatorname{ierfc}\left(\frac{\zeta}{2}x\right)\right]-\frac{e^{-\left(\frac{\zeta}{2}\right)^2x^2}}{3x^3}\right\}\Bigg|_{\frac{1}{\sqrt{\tau}}}^{\infty}. \tag{32}$$

Taking into account function $L_4(\zeta,\tau)$ (32) in formula (30), we obtained:

$$J_2(\zeta,\tau)=\frac{2}{\sqrt{\pi}}\left\{\frac{2}{3}\sqrt{\pi}\left(\frac{\zeta}{2}\right)^2\left[\frac{\zeta}{2}+\frac{1}{x}\operatorname{ierfc}\left(\frac{\zeta}{2}x\right)\right]-\frac{e^{-\left(\frac{\zeta}{2}x\right)^2}}{3x^3}\right\}\Bigg|_{\frac{1}{\sqrt{\tau}}}^{\infty}=\frac{2}{3}\tau\sqrt{\tau}\left\{\left[1-2\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\operatorname{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)+\frac{\zeta}{2\sqrt{\tau}}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\}. \tag{33}$$

Substituting function $J_1(\zeta,\tau)$ (29) and $J_2(\zeta,\tau)$ (33) into the right side of equation (24), we received:

$$I_1(\zeta,\tau)=\frac{2\tau\sqrt{\tau}}{3}\left\{2\left[1+\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\operatorname{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)-\left(\frac{\zeta}{2\sqrt{\tau}}\right)\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\}. \tag{34}$$

Function $I_2(\zeta,\tau)$ (23) was written as a difference of the integrals:

$$I_2(\zeta,\tau) = J_3(\zeta,\tau) - J_4(\zeta,\tau), \tag{35}$$

where

$$J_3(\zeta,\tau) = \frac{\tau^2}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-s}}\, e^{-\left(\frac{\zeta}{2\sqrt{\tau-s}}\right)^2} ds,$$

$$J_4(\zeta,\tau) = \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{(\tau-s)(\tau+s)}{\sqrt{\tau-s}}\, e^{-\left(\frac{\zeta}{2\sqrt{\tau-s}}\right)^2} ds. \tag{36}$$

Based on relation (29), we can write:

$$J_3(\zeta,\tau) = 2\tau^2\sqrt{\tau}\,\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right). \tag{37}$$

To find integral in equation (36), the method of substitution was used. As a result we obtained:

$$J_4(\zeta,\tau) = \left\{\begin{matrix} x = \dfrac{1}{\sqrt{\tau-s}} \\ ds = \dfrac{2}{x^3}dx \end{matrix}\right\} = \frac{2}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{\tau}}}^{\infty}\left(2\tau - x^{-2}\right) e^{-\left(\frac{\zeta}{2}\right)^2 x^2}\frac{dx}{x^4}. \tag{38}$$

Function $J_4(\zeta,\tau)$ (38) was written in the form:

$$J_4(\zeta,\tau) = \frac{4\tau}{\sqrt{\pi}}L_4(\zeta,\tau) - \frac{2}{\sqrt{\pi}}L_6(\zeta,\tau). \tag{39}$$

Based on equations (31) and (32) it was found:

$$L_6(\zeta,\tau) = \int_{\frac{1}{\sqrt{\tau}}}^{\infty}\frac{e^{-\left(\frac{\zeta}{2}\right)^2 x^2}}{x^6}dx = -\frac{e^{-\left(\frac{\zeta}{2}\right)^2 x^2}}{5x^5}\Bigg|_{\frac{1}{\sqrt{\tau}}}^{\infty} - \frac{2}{5}\left(\frac{\zeta}{2}\right)^2 L_4(\zeta,\tau) =$$

$$= \frac{1}{5}\tau^2\sqrt{\tau}\exp\left[-\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right] - \frac{2}{15}\left(\frac{\zeta}{2}\right)^2\tau\sqrt{\tau\pi}\left\{\left[1-2\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) + \frac{\zeta}{2\sqrt{\tau}}\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\}. \tag{40}$$

Taking into account relations (32) and (40) in formula (38), it was counted:

$$J_4(\zeta,\tau) = \frac{4\tau^2\sqrt{\tau}}{3}\left[1+\frac{1}{5}\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\left\{\left[1-2\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) + \frac{\zeta}{2\sqrt{\tau}}\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\} - \frac{2}{5}\frac{\tau^2\sqrt{\tau}}{\sqrt{\pi}}\exp\left[-\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]. \tag{41}$$

Substituting functions $J_3(\zeta,\tau)$ (37) and $J_4(\zeta,\tau)$ (41) to the right side of equation (35), we received:

$$I_2(\zeta,\tau) == \frac{2\tau^2\sqrt{\tau}}{15}\left\{\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\left[8+18\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2+4\left(\frac{\zeta}{2\sqrt{\tau}}\right)^4\right] - \left(\frac{\zeta}{2\sqrt{\tau}}\right)\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\left[7+2\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\right\}. \tag{42}$$

In regard to functions $I_1(\zeta,\tau)$ (34) , $I_2(\zeta,\tau)$ (42) and relation (22), we found the dimensionless temperature field:

$$T^*(\zeta,\tau) = \frac{2}{3}\frac{\tau\sqrt{\tau}}{\tau_s}\left\{2\left[1+\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \left(\frac{\zeta}{2\sqrt{\tau}}\right)\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\} -$$

$$-\frac{2\tau^2\sqrt{\tau}}{15\tau_s^2}\left\{\left[8+18\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2+4\left(\frac{\zeta}{2\sqrt{\tau}}\right)^4\right]\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \left(\frac{\zeta}{2\sqrt{\tau}}\right)\left[7+2\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\}, \quad \zeta \ge 0,\ 0 \le \tau \le \tau_s. \tag{43}$$

To verify the correctness of the obtained solutions (43) the accordance of the boundary (12), (13) and initial (14) conditions with achieved temperature fields were examined. First, the accordance of boundary condition (12) was checked, which defines thermal load on the outer surface of the strip $\zeta = 0$. For this purpose, the derivative of complementary error function (19) and the derivative of its integral (20) were taken into account, we achieved:

$$\frac{\partial}{\partial\zeta}\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) = -\frac{1}{2\sqrt{\tau}}\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right),$$

$$\frac{d}{d\zeta}\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) = -\frac{\zeta}{\sqrt{\tau\pi}}e^{-\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2}. \tag{44}$$

Differentiating solution (43) with respect $\zeta$, with regard to the derivatives (44), was found:

$$\frac{\partial T^*(\zeta,\tau)}{\partial\zeta} = \frac{2}{3}\frac{\tau\sqrt{\tau}}{\tau_s}\left\{\frac{2}{\sqrt{\tau}}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\mathrm{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \frac{1}{\sqrt{\tau}}\left[\frac{3}{2}+\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) + \frac{\zeta^2}{2\tau\sqrt{\pi}}e^{-\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2}\right\} -$$

$$-\frac{2\tau^2\sqrt{\tau}}{15\tau_s^2}\left\{-\frac{1}{2\sqrt{\tau}}\left[15+18\left(\frac{\zeta}{2\tau}\right)+\left(\frac{8}{\sqrt{\tau}}+6\right)\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\mathrm{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) + \frac{\zeta^2}{2\tau\sqrt{\pi}}\left[7+2\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]e^{-\left(\frac{\zeta}{2\sqrt{\tau}}\right)^2}\right\}. \tag{45}$$

Substituting in equation (45) $\zeta = 0$, we received:

$$\left.\frac{\partial T^*(\zeta,\tau)}{\partial \zeta}\right|_{\zeta=0} = -\frac{\tau}{\tau_s} + \frac{\tau^2}{\tau_s^2} = -\frac{\tau}{\tau_s}\left(1 - \frac{\tau}{\tau_s}\right), \quad (46)$$

which provides the fulfillment of the boundary condition (12), in regard to the form of function $q^*(\tau)$ (15).
Taking into account, that [1]:

$$\lim_{x\to\infty} \operatorname{erfc}(x) = 0\,, \quad (47)$$

we obtained:

$$\lim_{x\to\infty} \operatorname{ierfc}(x) = \lim_{x\to\infty}\left[\frac{e^{-x^2}}{\sqrt{\pi}} - x\operatorname{erfc}(x)\right] = 0. \quad (48)$$

The limit of the solution (43) as $\zeta$ approaches infinity ($\zeta \to \infty$) in regard to value of the limits (47)–(48), was calculated:

$$\lim_{x\to\infty} T^*(\zeta,\tau) = 0\,, \quad (49)$$

which shows the fulfillment of the condition (13).
Substituting $\tau = 0$ in the solution (43) we can easy compute:

$$T^*(\zeta,0) = 0\,, \quad (50)$$

in this way, conformity of the result (43) with the initial condition (14) has been proved.

## Numerical analysis

Numerical analysis was carried out based on the devised solution (43), which describes transient temperature field in a brake disc heated by frictional heat flux with linearly increasing contact pressure. The results were compared with the following, adequate results, which were received in the article [9], with an assumption of constant pressure:

$$T^*(\zeta,\tau) = 2\sqrt{\tau}\operatorname{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \frac{2\tau\sqrt{\tau}}{\tau_s}\left\{\frac{2}{3}\left[1 + \left(\frac{\zeta}{2\sqrt{\tau}}\right)^2\right]\operatorname{ierfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \frac{1}{3}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right)\right\}, \quad \zeta \ge 0\,, 0 \le \tau \le \tau_s\,. \quad (51)$$

During braking with linear distribution of contact pressure (1) and linearly decreasing sliding velocity (2), dimensionless total amount of thermal energy directed to a brake disc is:

$$Q^* = \int_0^{\tau_s} q^*(\tau)d\tau = \int_0^{\tau_s} \frac{\tau}{\tau_s}\left(1 - \frac{\tau}{\tau_s}\right)d\tau = \left(\frac{\tau^2}{2\tau_s} - \frac{\tau^3}{3\tau_s^2}\right)\Bigg|_0^{\tau_s} = \frac{1}{6}\tau_s\,. \quad (52)$$

Whereas, during braking with constant pressure $p^*(\tau) = p^* > 0$ and constant deceleration this value is equal:

$$Q^* = \int_0^{\tau_s} q^*(\tau)d\tau = p^*\int_0^{\tau_s}\left(1 - \frac{\tau}{\tau_s}\right)d\tau = p^*\left(\tau - \frac{\tau^2}{2\tau_s}\right)\Bigg|_0^{\tau_s} = \frac{1}{2}p^*\tau_s\,. \quad (53)$$

Taking into account (52) and (53), to maintain the same total amount of heat, during calculations of temperature field in the case with constant pressure, the value $p^*=1/3$ was adopted.
The input dimensionless conditions used to numerical analysis were: distance from friction surface $\zeta$ and Fourier numbers (dimensionless time) $\tau$ and $\tau_s$. It was assumed that, dimensionless braking time is $\tau_s=1$.
Evolution of the dimensionless temperature $T^*$ in time, on few depth $\zeta$ was presented in Fig. 2. At the beginning of the process the temperature increases, attains the maximum value and decreases until the moment of standstill. The maximum dimensionless temperatures $T^*=0{,}425$ (Fig. 2a) and $T^*=0{,}177$ (Fig. 2b) are reached on the contact surface $\zeta = 0$, exactly in the half of braking time ($\tau = 0{,}5$). The maximum temperature achieved in the braking with linearly changing pressure is 140% higher than in the braking with constant pressure.

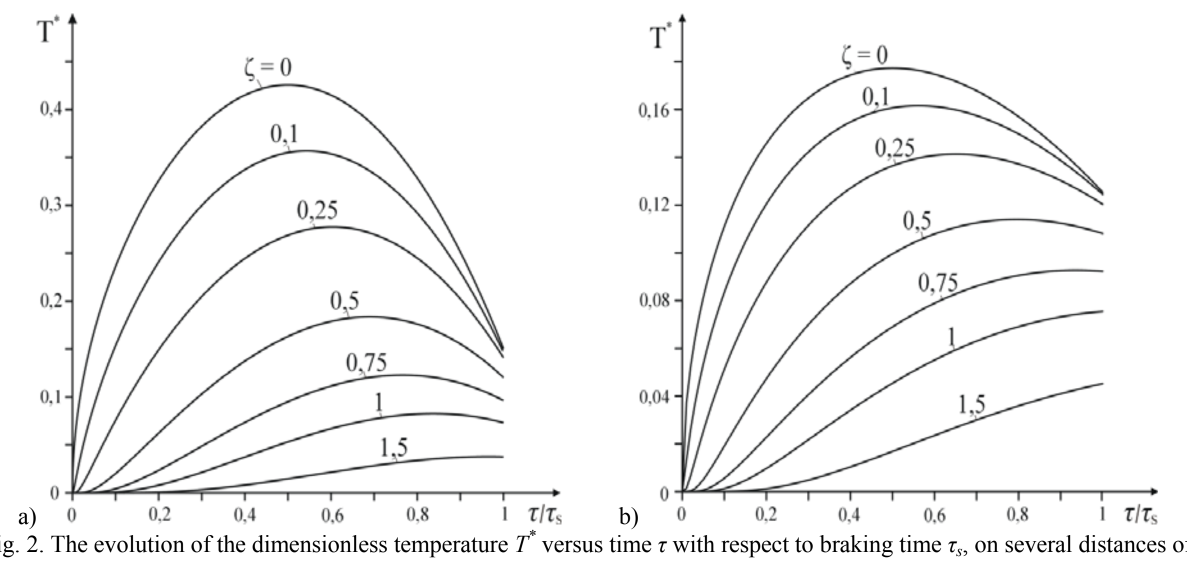


Fig. 2. The evolution of the dimensionless temperature $T^*$ versus time $\tau$ with respect to braking time $\tau_s$, on several distances of friction surface $\zeta$: a) linearly increasing contact pressure; b) constant pressure.

Cooling of the outer surface of the disc after maximum temperature has been reached, in the case with linear distribution of pressure is more intense than with constant pressure. The temperature is lower and the time to reach maximum temperature value increases with increasing distance from the friction surface to the center of the disc. In the braking with constant pressure maximum temperatures on particular distances $\zeta$ are achieved with higher time offset in the standstill direction. Monotonically temperature increases during the entire braking process: with linear distribution of the pressure it takes place under the distance $\zeta = 1,5$, while with constant pressure – under the distance $\zeta = 1$.

Distribution of the dimensionless temperature $T^*$ versus dimensionless depth $\zeta$ in several values of Fourier number $\tau$ were shown in Fig. 3. The temperature monotonically decreases with increasing of the distance from the heated surface. The largest gradient between temperature on the friction surface $\zeta = 0$ and at the distance $\zeta = 1,5$, occurs in the half time of braking process $\tau = 0,5$ (Fig. 3). The mentioned gradient reaches minimum value in the stop moment $\tau = \tau_s = 1$. Effective depth of the heat penetration, i.e. distance from friction surface, on which the temperature achieved 5% of the maximum value on heated surface. During braking with linearly increasing pressure, effective depth of the heat penetration is equal $\zeta_{eff} = 1$ (Fig. 3a) and the temperature decreases more rapidly (Fig. 3a). Whereas, during braking with constant pressure the mentioned depth has a higher value $\zeta_{eff} > 1,5$ (Fig. 3b).

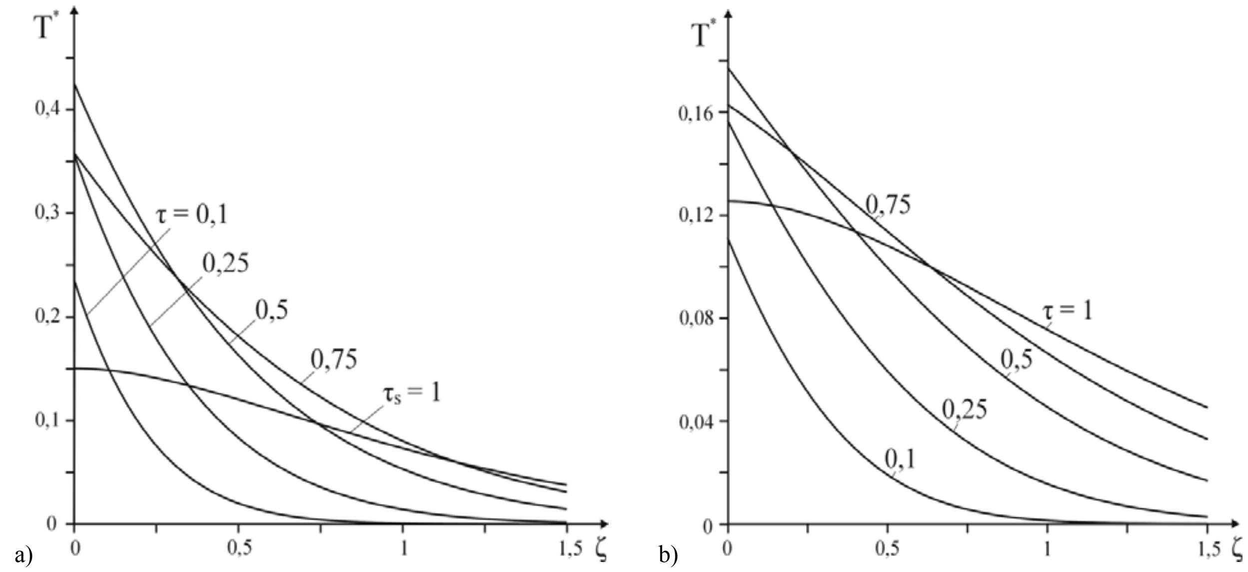


Fig. 3. Distribution of the dimensionless temperature $T^*$ inside disc at few different dimensionless time moments $\tau$: a) linearly increasing contact pressure; b) constant pressure.

## Conclusions

Conducted analysis demonstrates, that:

-with equal value of total amount of heat, during braking with constant pressure reached temperatures near the working surface are significantly lower. Adjacent the distance $\zeta = 1$, obtained temperatures in both cases are equal. Below this depth, the temperatures achieved in case of constant pressure are higher;

- cooling of the heated surface during braking with linear distribution of the pressure is more intensive;
- effective depth of the heat penetration is higher during braking with constant pressure;
- the time to reach maximum temperature value increases with increasing distance from the friction surface to the center of the disc. In the braking with constant pressure the maximum temperatures on particular distances $\zeta$ are achieved with higher time offset in the standstill direction;
- the temperature increases monotonically during the entire braking process with linear distribution of the pressure which takes place on the distance $\zeta \geq 1,5$, while with constant pressure – on the distance $\zeta \geq 1$.

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