Optimisation of Railway Polynomial Transition Curves with Different Number of Terms

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This paper presents the results of research of influence of degree and number of polynomial terms of railway transition curves (TCs) on their dynamical properties. The research performed for needs of this work are planned numeric tests. The search for the proper shape means the evaluation of the curve properties based on chosen dynamical quantity and generation of such shape with the use of mathematically understood optimisation methods. For the needs of this work advanced vehicle model, dynamical track-vehicle and vehicle-passenger interactions as well as optimisation methods were exploited.
The subject of the analysis are odd degree polynomial transition curves of the 5th, 7th, 9th and 11th degree. For each degree of the polynomial a range of acceptable number of polynomial terms was determined. One dynamical quantity - minimum of integral of vehicle body lateral acceleration - being the results of simulation of railway vehicle advanced model is exploited in the determination of quality function (QF)..
Keywords: polynomial railway transition curves, computer simulation, optimisation.

1. INTRODUCTION

Issues taken in this article in general relate to the search of a proper shape of railway transition curves (TCs) including advanced vehicle model, the dynamical interactions: vehicle-track and vehicle-passenger and optimisation methods. By searching of a proper shape authors of this article mean to generate such a shape which is a result of the use of mathematical methods of optimisation and evaluation of properties of different shapes on the basis of selected dynamical quantity. In particular, it relates to the examination of the impact of the degree and number of terms of polynomial TCs on their dynamical properties. In the study presented below authors applied only one optimisation criterion. Quality function (QF) is an integral of absolute value of lateral acceleration of vehicle body mass centre along the route. Full description of the criterion applied is presented in detail in work [4].

2. THE FORM OF TRANSITION CURVES TAKEN FOR ANALYSIS

The authors of the article studied the polynomial transition curves (TCs) of odd degree from 5^{th} to 11^{th}. The parameters of the transition curve are presented in the form of the following equations:

$$y = \frac{1}{R}\left(\frac{A_n l^n}{l_0^{n-2}} + \frac{A_{n-1} l^{n-1}}{l_0^{n-3}} + \frac{A_{n-2} l^{n-2}}{l_0^{n-4}} + \frac{A_{n-3} l^{n-3}}{l_0^{n-5}} + ... + \frac{A_4 l^4}{l_0^2} + \frac{A_3 l^3}{l_0^1}\right), \tag{2.1}$$

$$k = \frac{d^2 y}{dl^2} = \frac{1}{R}\left[n(n-1)\frac{A_n l^{n-2}}{l_0^{n-2}} + (n-1)(n-2)\frac{A_{n-1} l^{n-3}}{l_0^{n-3}} + ... + 3 \cdot 2\frac{A_3 l^1}{l_0^1}\right], \tag{2.2}$$

$$h = H\left[n(n-1)\frac{A_n l^{n-2}}{l_0^{n-2}} + (n-1)(n-2)\frac{A_{n-1} l^{n-3}}{l_0^{n-3}} + ... + 4 \cdot 3\frac{A_4 l^2}{l_0^2} + 3 \cdot 2\frac{A_3 l^1}{l_0^1}\right], \tag{2.3}$$

$$\begin{aligned} i = \frac{dh}{dl} = H\Bigg[& n(n-1)(n-2)\frac{A_n l^{n-3}}{l_0^{n-2}} + (n-1)(n-2)(n-3)\frac{A_{n-1} l^{n-4}}{l_0^{n-3}} \\ & + ... + 5 \cdot 4 \cdot 3\frac{A_5 l^2}{l_0^3} + 4 \cdot 3 \cdot 2\frac{A_4 l^1}{l_0^2} + 3 \cdot 2 \cdot 1\frac{A_3 l^0}{l_0^1}\Bigg], \end{aligned} \tag{2.4}$$

where y, k, h, and i define curve lateral co-ordinate, curvature, superelevation, and inclination of superelevation ramp, respectively. The R, H, l_0, and l define curve minimum radius (at its end), maximum susperelevation (at the curve end), total curve length, and curve current length, respectively. The A_i are polynomial coefficients ($i = n$, n-1,…., 4, 3) where n is polynomial degree. Number of the polynomial terms (terms in Eqs. (2.1)-(2.4)) must not be smaller than 2. On the other hand the smallest degree n_{min} of the last term in Eq. (2.1) must be $n_{min} \geq 3$. Such definition of the curves gives possibility of proper values k and h at TC's terminal points. They should be equal to 0 at the initial points and 1/R and H at the end points. Note, that values for both ones always are equal to 0 for l=0. In order to ensure values 1/R and H for l=L, normalisation of the coefficients is necessary ([8]). In this work, condition on tangence of k, h and i was also imposed: i.e. $k'(0)$=0, $k'(l_0)$=0, $i'(0)$=0 and $i'(l_0)$=0. So each time if authors say about a tangence of k, h and i, they mean tangence of these quantities to the courses of these functions in contiguos straight track (ST) and circular arc (CC).

For each polynomial curve (also odd degree) geometrical demands were imposed that one wants or does not want to take into account. Possible combinations of coefficients for those demands are shown in Tab. 2.1. Generally, in this paper authors focused only on polynomial curves with maximum number of terms.

Table 1. Possible polynomial configurations for different geometrical demands

Type of demand ⇒ / Polynomial degree (terms number in the initial polyn.) ⇓	**Demand IDZ=1** (proper values of r and h in TCs' terminal points) **Number of terms**	**Demand IDZ=2** (tangence of r and h functions at TCs' terminal points) **Number of terms**	**Demand IDZ=3** (tangence of h slope, i.e. of i, at TCs' terminal points) **Number of terms**
5th degree (IWI=2)	IW=2; IW=3	IW=2(single curve)	Not to be satisfied
7th degree (IWI=3)	IW=2; …, IW=5	IW=2(single curve); IW=3; …; IW=4	IW=3(single curve)
9th degree (IWI=4)	IW=2; …, IW=7	IW=2(single curve); IW=3; …; IW=6	IW=3(single curve) IW=3; …; IW=5
11th degree (IWI=5)	IW=2; …; IW=9	IW=2(single curve); IW=3; …; IW=8	IW=3(single curve) IW=3; …; IW=7

3. NOMINAL VEHICLE MODEL

In the study one model of wagon was used. The model represents 2-axle HSFV1 freight car of the average values of parameters. The same model of the system was used in the earlier studies by present authors [4] – [10]. Its structure is shown in Fig. 3c. It is supplemented with discrete models of vertically and laterally flexible track shown in Fig 3a and 3b, respectively. Linearity of the vehicle suspension was assumed. So, linear stiffness and damping elements in vehicle suspension were applied. The same concerns the track models. Here, also linear stiffness and damping elements were applied. One can find all parameters of the model used in detail in [5].

Vehicle model is equipped with a pair of wheel/rail profiles that correspond to the real ones. That is a pair of the nominal (i.e. unworn) S1002/UIC60 profiles that are used all over the Europe. Non-linear geometry of this pair is introduced into the model in a form of table with the contact parameters. In order to calculate non-linear tangential contact forces between wheel and rail well known FASTSIM program by J.J. Kalker was applied. Normal forces in the contact are not constant but influenced by both the geometry and the dynamical effects that make value of a wheelset vertical load variable.

solution is reached earlier, i.e. for $i<i_{lim}$, then the optimisation process stops automatically and the corresponding results are recorded.

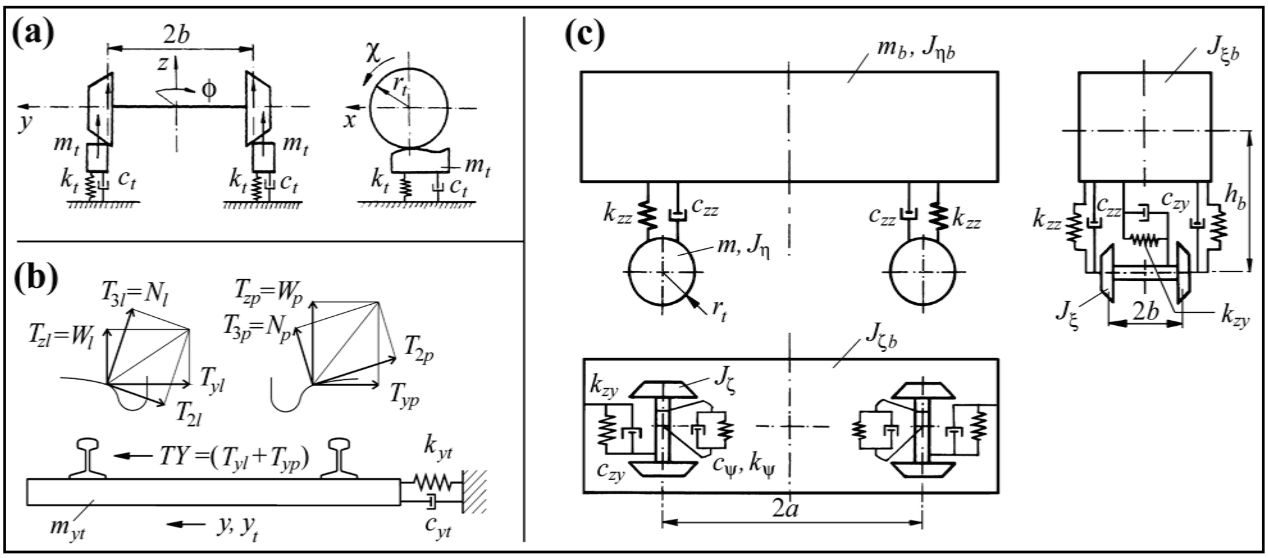


Fig. 3.1. System's nominal model: (a) track vertically, (b) track laterally, (c) vehicle.

4. SOFTWARE USED IN THE RESEARCH

Scheme of the software ([7], [8]) used in optimisation TCs shape is shown in Fig. 4.1. The major objects within this scheme are two iteration loops visible there. The first is the integration loop. This loop is stopped when distance l_{lim}, being the length of route (usually compound route straight track ST, transition curve TC and circular arc CC), is reached the model. The second is the optimisation process loop. It is stopped when number of iterations reaches limit value i_{lim}. This value means that i_{lim} simulations of vehicle motion have to be performed in order to stop optimisation process. In the calculations done so far i_{lim}=100 was used as standard value. If the optimum

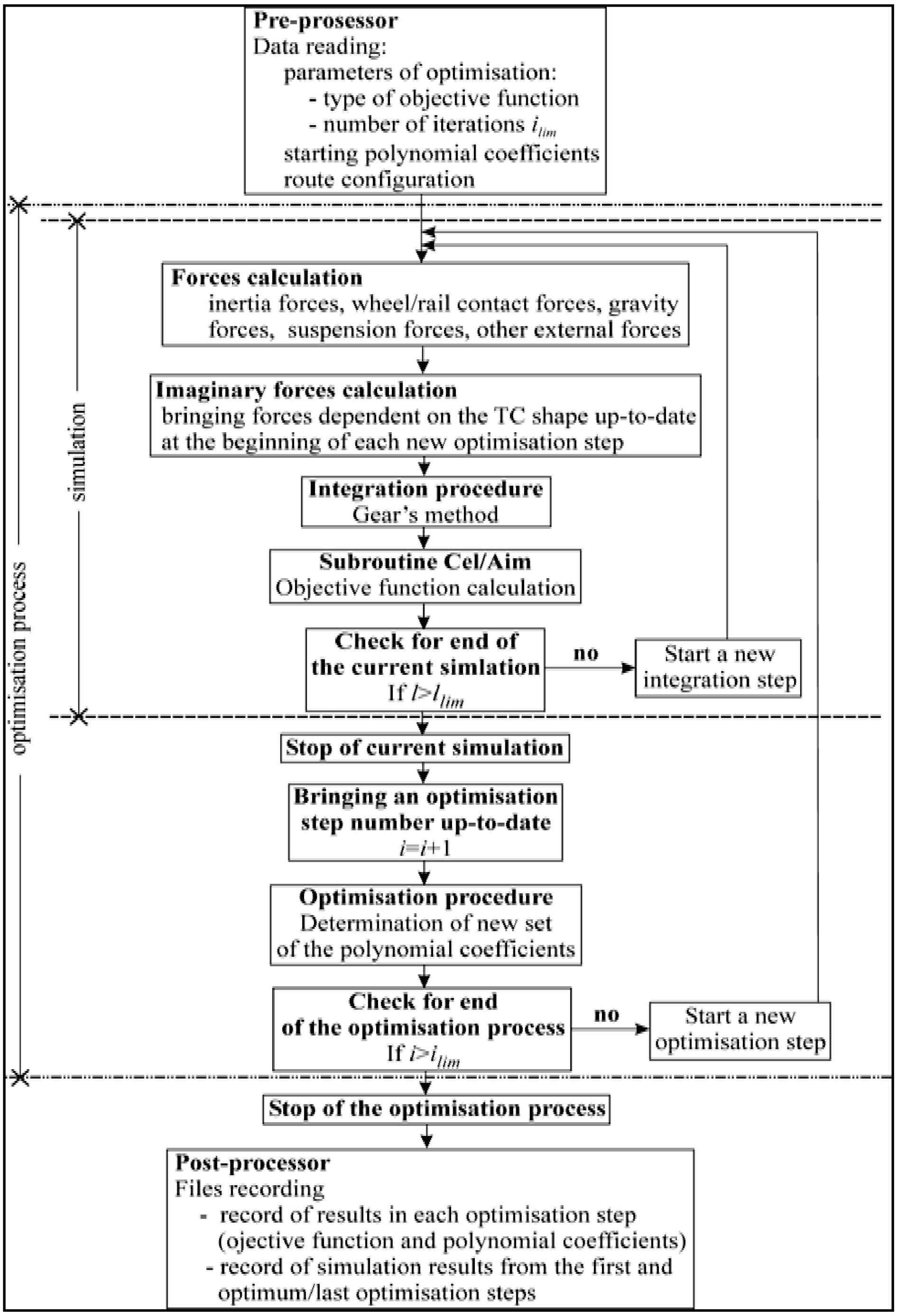


Fig. 4.1. General scheme of the software to optimise transition curves' shape.

5. RESULTS OF THE STUDIES

The results of the optimisation in general consist of: optimum polynomial TCs' coefficients, values of the objective (quality) function, graphical representations of the curve and its curvature, displacements and accelerations of the vehicle body mass centre. In the studies authors assumed values of: arc radius R equal to 600 m, cant H=0.15 m and maximum velocity of wheel vertical rise along the superelevation ramp f_{lim}=56 mm/s. As vehicle velocity v authors adopted two values. First velocity - 24.26 m/s - guarantees ideal balance between transversal components of gravity and centrifugal force and second velocity - 30.79 m/s - guarantees the maximum admissible vehicle velocity in curved track a_{lim} - 0.6 m/s^2. Length l_0 of the TCs for a given degree of polynomial authors calculated according to the algorithm given in [1]. In Figs 5.1-5.2 courses for the standard TCs and 3rd degree parabola are marked by a solid line and the dashed line is used for courses for optimum TCs.

5.1. RAILWAY POLYNOMIAL TCS OF 5^{th} AND 7^{th} DEGREE WITH MAXIMUM NUMBER OF TERMS

As a standard curve (initial in optimisation process) for the 5^{th} degree authors took Bloss curve ([1], [2]). The curvature of this TC has a tangence in extreme points 0 and l_0.

In the case of the optimisation process shown below the following conditions of optimisation were assumed: l_0=97.47 m, R=600 m, a_{lim}=0 m/s^2, v=24.26 m/s, f_{lim} = 56 mm/s, IW = 3.

Optimum curve found by the program is represented by the following formula:

$$y = \frac{1}{600}\left(-0,00085025 \cdot \left(\frac{1}{97.47}\right)^3 + 0,0048605 \cdot \left(\frac{1}{97.47}\right)^2 + 0,15978 \cdot \left(\frac{1}{97.47}\right)\right) \tag{5.1}$$

Figure 5.1a shows the comparison of the lateral co-ordinates of Bloss curve (standard), the curve (5.1), and 3^{rd} degree parabola, figure 5.1b compares curvatures corresponding to these transition curves.

The ratio of value of the quality function for the transition curve (5.1) to the value of quality function for the standard curve of 5^{th} degree (Bloss curve) was 0.11=(0.005999 [m/s^2]/0.050122 [m/s^2]), and to the value for the 3^{rd} degree parabola - 1.03=(0.005999 [m/s^2]/ 0.0057982 [m/s^2]) ([7]).

For polynomial curves of 7^{th} degree similar optimisation processes were performed as for the 5^{th} degree. As simulation conditions authors assumed: l_0=154.68 m, a_{lim}=0.6 m/s^2, v=30.79 m/s, f_{lim}=56 mm/s (IW=5, in accordance with Tab. 2.1). Standard transition curve of 7^{th} degree is presented in [2].

The results for the polynomials of 7^{th} degree – lateral co-ordinates, curvatures, displacement and acceleration of vehicle body – are qualitatively very similar to the results for the 5^{th} degree.

The ratio value of the quality function for optimum curve / value of the quality function for a standard curve of 7^{th} degree was 0.62=(0.0054504 [m/s^2]/0.008741 [m/s^2]), and the ratio of value of quality function for optimum curve to value for 3^{rd} degree parabola was 0.98=(0.0054504 [m/s^2]/0.00552392 [m/s^2]). Last ratio – 0.98 – is very close to 1.00 ([7]).

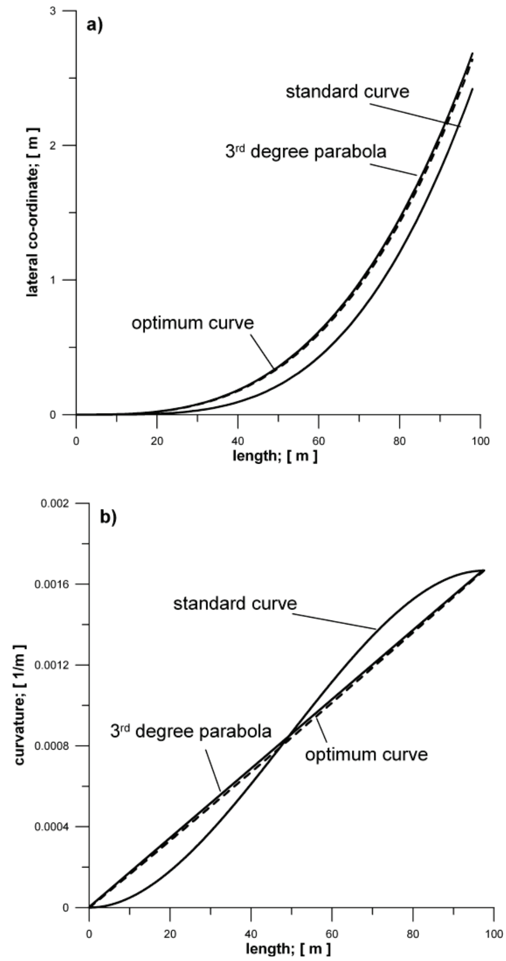
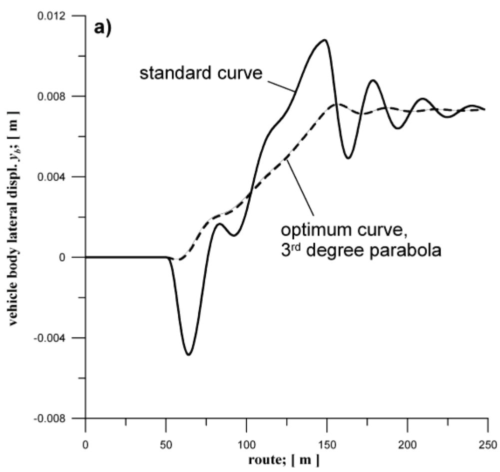


Fig. 5.1 Comparison: a) lateral co-ordinates, b) curvatures of transition curves – standard curve of 5^{th} degree, the curve (5.1), and the 3^{rd} degree parabola



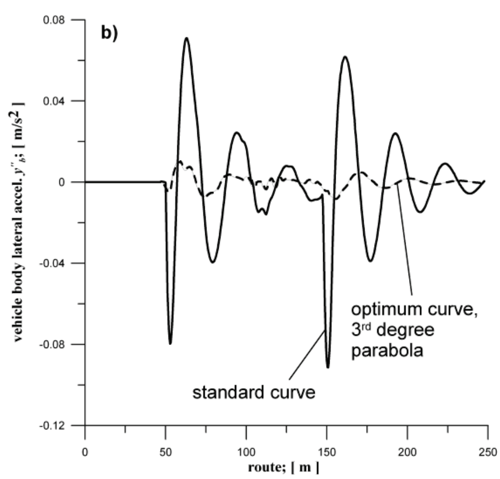


Fig. 5.2 Comparison lateral: a) displacements, b) accelerations of vehicle body

The main conclusion from the analysis of these results is that the optimum polynomial transition curves of the 5th and 7th (values of quality function – 1.03 and 0.98) approach to linear curvature of the 3rd degree parabola ([7], [8]). Optimisation procedure can't find a solution significantly better than mentioned 3rd degree parabola. In works [7] and [8] their authors showed this fact for both degrees of polynomial – 5th and 7th – and both vehicle velocities – 24.26 m/s and 30.79 m/s.

5.2. RAILWAY POLYNOMIAL TCS OF 9th AND 11th DEGREE WITH DIFFERENT NUMBER OF TERMS

For needs of this section authors performed the research of polynomial transition curves of 9th degree with the number of terms - 7, …, 3, and the 11th degree with the number of terms - 9, …, 3 (see Tab. 2.1). Optimisations which were made by the authors aimed to analyze the influence of the number of terms of a polynomial transition curves on the value of the quality function adopted.

Conditions of optimisation of the first part of the calculations done for needs of this chapter are as follows: l_0=142.15 m, a_{lim}=0 m/s^2, v=24.26 m/s, f_{lim}=56 mm/s, number of polynomial terms=3,…7.

Figure 5.3a shows a comparison of curvatures of transition curves of 9th degree with a different number of terms and 3rd degree parabola, figure 5.3b compares superelevation ramp slopes corresponding to these curves (without 3rd degree parabola, which is a straight line). Marking 9d6, for example, says that this is a polynomial of 9th degree with the number of terms equal to 6. Courses for curves with maximum number of terms are marked by dashed line. Figure 5.4a shows a comparison of lateral displacements of vehicle body for standard TC of 9th degree, optimum TCs with number of terms 3, …, 7, and 3rd degree parabola. Figure 5.4b compares lateral accelerations of vehicle body corresponding to these curves.

Values of quality function for optimum transition curves of 9th degree with the number of terms from 3 to 7 and the ratio of the these values to the values of quality function for the standard (initial) transition curve of 9th degree ([2]) and 3rd degree parabola are shown in Tab. 5.1 ([7]). From this table we see that the smallest value of quality function has transition curve with maximum number of terms – 7. This optimum transition curve has also the smallest value of ratio of value of quality function to values of quality function for standard curve of 9th degree and 3rd degree parabola.

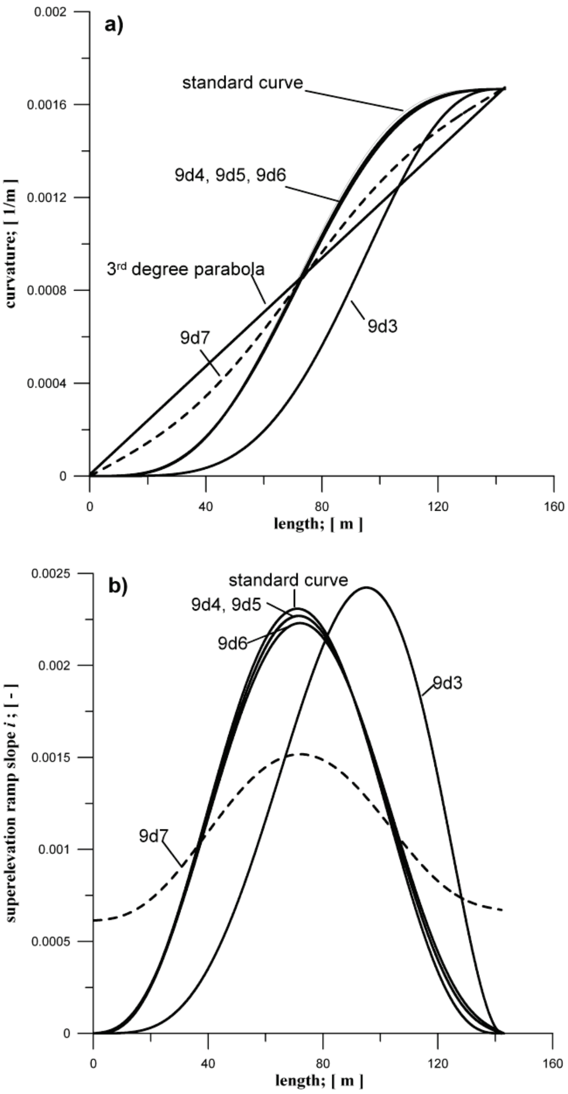


Fig. 5.3 Comparison of: a) curvatures, b) superelevation ramp slopes of transition curves of 9th degree with a different number of terms - 3, …, 7, and 3rd degree parabola

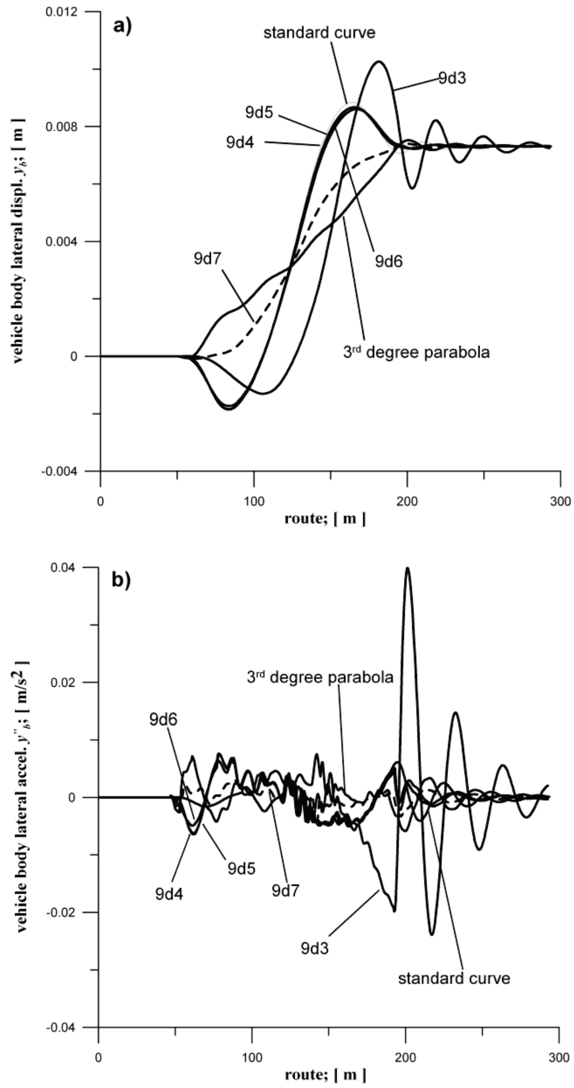


Fig. 5.4 Comparison of lateral: a) displacements, b) accelerations of the vehicle body for curves from Fig. 5.3

Table 5.1 Values of the quality function (QF) for optimum curves of 9th degree

Curve	*Value of QF [m/s^2]*	*Ratio of values QF to value of standard curve of 9th degree*	*Ratio values of QF to value of 3rd degree parabola*
9d7	0.0018622	0.35	0.59
9d6	0.0041683	0.64	1.33
9d5	0.0053467	0.82	1.71
9d4	0.0042870	0.66	1.37
9d3	0.0125410	1.93	4.02

For the second analised degree of polynomial – the 11th one – the following conditions of optimisation were assumed: l_0 = 202.94 m, a_{lim} = 0,6 m/s^2, v = 30.79 m/s, f_{lim} = 56 mm/s, number of polynomial terms=3,…9.

Here, like in 9th degree of polynomial, also curve with maximum number of terms – 9 – has both the smallest value of quality function and ratio of value of quality function to values of quality function for standard (initial) curve of 11th degree ([2]) and 3rd degree parabola (see Tab. 5.2) ([7]). The results for the polynomials of 11th degree – lateral co-ordinates, curvatures, displacement and acceleration of vehicle body – are qualitatively very similar to the results for the 9th degree.

Table 5.2 Values of the quality function (QF) for optimum curves of 11th degree

Curve	*Values of QF [m/s²]*	*Ratio of values QF to value of standard curve of 11th degree*	*Ratio values of QF to value of 3rd degree parabola*
11d9	0.0019648	0.86	0.45
11d8	0.0022324	0.97	0.46
11d7	0.0022443	0.98	0.46
11d6	0.0028045	0.98	0.56
11d5	0.0022350	0.98	0.56
11d4	0.0110750	3.88	2.27
11d3	0.0158260	5.54	3.25

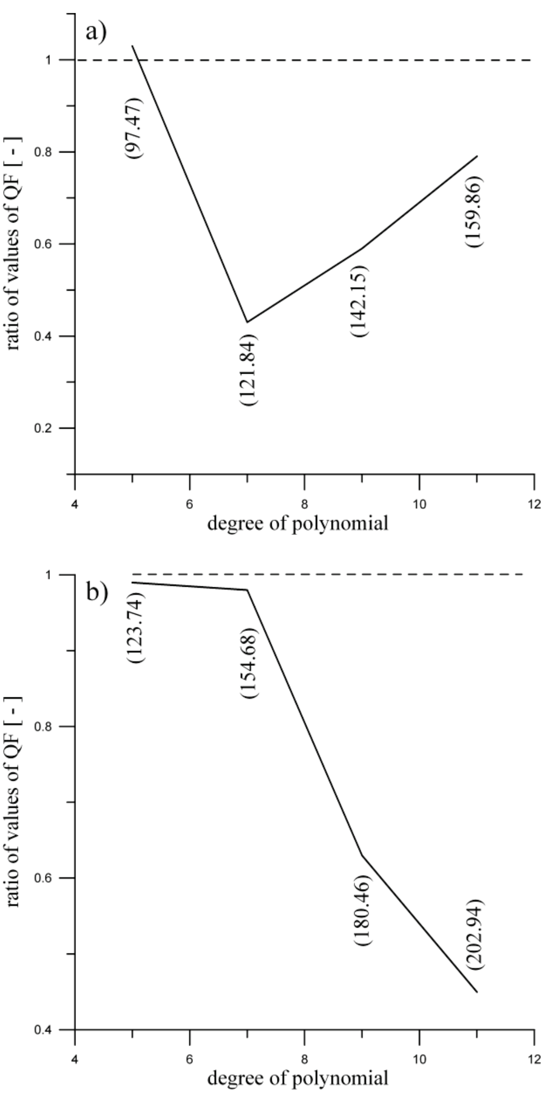


Figure 5.5 Comparison of ratio of values of QF for optimum TCs with maximum number of terms to value for 3rd degree parabola: a) v=24.26 m/s, b) v=30.79 m/s. Values in brackets are lengths of TCs

Figure 5.5 shows a comparison of ratio of values of quality function for optimum transition curves of 5th, 7th, 9th and 11th degree with maximum number of terms to value for 3rd degree parabola for two adopted velocities of vehicle v – 24.26 m/s and 30.79 m/s. In general, optimum transition curves obtained are better than traditionally used 3rd degree parabola in the vast majority of cases analyzed.

6. CONCLUSIONS

The basic conclusion that arises from an analysis performed in this study is that the rise of a degree of polynomial of transition curve causes a decrease in the value of the quality function assumed. Polynomial curve is more suitable for the transition curve when it has higher degree. This fact has been confirmed by studies of dynamic properties of transition curves made for degrees of polynomial from 5 to 11 for two different velocities of the vehicle and the different lengths of transition curves – [7], [8]. Corresponding values of quality function are for velocity v=24.26 m/s: 0.005999 [m/s^2] (5th), 0.002024 [m/s^2] (7th), 0.0018622 [m/s^2] (9th), 0.00188 [m/s^2] (11th), and for velocity v=30.79 m/s: 0.009894 [m/s^2] (5th), 0.0054504 [m/s^2] (7th), 0.0030869 [m/s^2] (9th) and 0.0019648 [m/s^2] (11th).

Second important conclusion of the study is that the increase of the number of terms in a given polynomial also causes a decrease of the value of the quality function. Transition curves of 9th and 11th degree have the smallest values of the quality function with number of terms - 7 and 9, respectively, although the functions of the curvature of these curves do not have a tangence in the extreme (initial and final) points ([7], [8]). Analysis of the values of quality function from Tabs. 5.1 and 5.2 shows that such curves have also the smallest the ratio of the values of the optimum quality function to values of quality function for standard curves of a given degree and 3rd degree parabola.

In this work authors showed also that using optimum transition curves of 5th and 7th degree with maximum number of terms has no sense because these curves are very closed for course for 3rd degree parabola.

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