VPS VIBRATIONS IN PHYSICAL SYSTEMS

DOI: 10.21008/j.0860-6897.2021.2.12
Vibrations in Physical Systems, 32(2):2021212, 2021

Influence of the Variability of the Shape of Short Axisymmetric Rods on the Value of Basic Natural Frequency of Longitudinal Vibrations

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Abstract Rods with a relatively short length in relation to their transverse dimensions with a variable cross-section along the axis, subjected to longitudinal vibrations, have the character of stocky rods, for which the classic geometric principles used for rods are not fully met. For small engineering elements, the postulated design frequencies of natural vibrations in the very high cycle fatigue analyses reach the values of a dozen or so, small, several dozen kHz. In the article, the sensitivity of the value of the basic natural frequency to the change in the shape of the element is analysed and discussed. The analysis is conducted using finite element method.

Keywords: rods, longitudinal vibrations, natural frequencies

1. Introduction

Deformation of elastic systems with a continuous mass distribution is described by set of the differential equations of the theory of elasticity. With small deformations and the use of material that is described by the constitutive Hook's law, these equations are linear. Based on these equations, it is possible to describe the propagation of various types of elastic waves in a structure (in particular for infinite structure), or to describe various types of structure vibrations (for elements with defined dimension) [1,2].

In the mechanics of continuous systems, in particular with regard to engineering applications (shape of structural elements, mathematical model, material), structures are commonly clasified, primarily due to their shape, and in particular due the dominance of two dimensions over the third (so-called two-dimensional structures), or the dominance of one dimension above the rest two (so-called one-dimensional structures), they are called continuous technical systems. Their dynamics, and vibrations in particular, can be described by simplified equations of the mechanics of continuous media. Another important feature is the curvature or the flatness of the element (two-dimensional element), and the straightness of the axis of the element (one-dimensional element). In the case of one-dimensional systems, depending on the stiffness of the structure and the type of deformation (vibration type), the following specific technical continuous systems are specified and called: string (bending vibrations, lack of natural bending stiffness; necessity to pre-tension), beam (bending vibrations), shaft (vibrations torsional) and rod (longitudinal vibration). The equations commonly used to describe vibrations are built with specific assumptions regarding deformation, and in particular with the assumption of the form of internal displacements in the cross-sections of elements. The fundamental assumption that allows for the formulation of appropriate principles (assumptions, hypotheses) is the assumption that the parameter (parameters) describing the shape of the cross-section geometry is small in relation to the parameter describing the length/width of the element. This allows for the assumption of the flatness of the cross-sections during deformation (bars, beams, shafts) or the rules of normal lines (membranes, plates, shells) [3]. This behaviour takes place at a certain distance from the supporting points of the elements, which is called the de Saint-Venant principle. The concept of a small value, or a sufficiently large distance, and thus the limit of the theory's applicability, is of a conventional nature, and is associated with an increasing error of description when departing from the assumption. In this case, it is necessary to apply the complete theory of the three-dimensional deformable body, i.e. the so-called the equations of the theory of elasticity (if sufficient for the linear theory of elasticity). It is worth paying attention to the article by Krawczuk, Grabowska and Palacz for rods [4].

Due to practical applications, in particular in design of samples for gigacycle testing the fatigue strength of materials, which are to be carried out in a relatively short time, the problem of dynamics (vibrations) analysis of samples with transverse dimensions expressed in millimeters and length expressed in centimeters, usually with axisymmetric geometry and for which the resonant frequency is of the order of 20 kHz (generated by piezoelectric actuators), will arise. Such a frequency allows Very High Cycle Fatigue (VHCF) tests to be carried out in a short time, measured in seconds. However, due to the dimensions of samples for typical construction materials (material properties), their geometry must have the nature of an element with a variable cross-section due to two opposite basic reasons: for the generation of a longitudinal wave (holder shape and sufficient longitudinal and transverse stiffness of the element) requiring a larger cross-section and separation of the area of stress concentration of uniaxial tension/compression nature requiring a smaller cross-section. Therefore, commonly used in such cases are elements with a variable cross-section which, due to their shape, can be called stocky rods, for which the condition of small transverse dimensions (variable along the length) in relation to the length of the element is certainly not fully met, and fulfilling de Saint Venant's principle assumptions. On the other hand, however, due to the symmetry of the shape, with appropriately designed handles and excitations, a longitudinal wave, generally of a plane wave character, is generated in the structure [5].

In the literature a group of works on longitudinal vibrations of stocky rods, in which the analyses are based on analytical solutions can be found. Generally, it should be stated that analytical solutions to the natural vibration problem can be obtained for selected shapes of cross-section variability described by several characteristic analytical functions. In the book by Bathias and Ribeiro [5] was discussed and several of these solutions were used. First of all, they relate to the articles of Kozlov and Selitser [6], Kong, Saanouni and Bathias (exponential form) [7] and Wu with Bathias (catenoide) [5]. The article by Yardimoglu and Aydin [8] considers the shape of the powers of the sine function. Hull [9] considered examples about linear, triangular, Bessel function and Gamma function variations. Freitas et.al. [10] considered three types of cross-section: tapered, exponential and hyperbolic. In the article by Gorbachev [11] an integral formula is used to formulate the general solution of the differential equation with variable coefficients. In the book Solecki and Szymkiewicz [12] consider cases of a rod shaped like exponential and powered linear functions types.

If it is necessary to analyse elements with any defined variability of cross-section along the length, approximate methods should be used. D. Xu, J. Du, Z. Liu [13] implemented theoretical formulation in MATLAB computing environment for solving non-typical cases of geometry variation. At present, the use of FEM offers great possibilities for analysing rods of any shape. This modelling enables an analysis for a specific case of shape and does not lead directly to general analytical dependencies relating to the relationship between the shape of the cross-section and resonance frequencies. Examples of this type of analysis are included in the book by C. Bathias and P.C. Ribeiro [5]. Also the author [14] used this method in preliminary analyses of nayral longitudinal vibrations of stocky rod elements. The considered examples relate essentially to the family of samples that can be used in the *ItalSigma* testing machine, and their results show a high sensitivity of the element shape to the basic natural frequency of longitudinal vibrations. For general shape of cross section determination of basic natural frequency can be approximately estimated using Galerkin or Rayleigh-Ritz methods too [3,12,15].

The above-discussed aspects of modelling were the justification for undertaking analyses showing the influence of the bar shape in a qualitative way (without an analytical formula), using finite element method (FEM). The research also took into account the influence of the diameter in the area where the sample may be attached to the gigacycle fatigue testing machine (handles at both ends with a constant cross-section) and the length of this area (handle length) in relation to the length of the area with a variable cross-section. The results of the analyses are presented and discussed in the article.

2. Analytical description

Let us consider longitudinal vibrations of the isotropic, homogeneous, straight rod with different cross-section along its axis x. For the linear case, the equation of longitudinal vibration has the form (1):

$$\rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]=q(x,t)\,, \qquad (1)$$

after differentiation (2):

$$\rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2} - E\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x} - EA(x)\frac{\partial^2 u(x,t)}{\partial x^2} = q(x,t), \tag{2}$$

where: $u(x,t)$ - longitudinal (along x axis) displacement of the cross-section [m], $A(x)$ – area of cross-section varying with x [m^2], ρ – material density [kg/m^3], E – Young modulus [Pa], $q(x,t)$ – external distributed force [N/m]. The fundamental assumption of the theory is that during deformation, the cross-sections are perpendicular to the x axis and are planar [2]. The longitudinal force in all cross-sections are due to the mentioned above assumptions equal to product of the normal stress (constant over the whole cross-section) and actual area of the cross-section (3):

$$N(x,t) = A(x)\sigma_x(x,t) = A(x)E\varepsilon_x(x,t) = A(x)E\frac{\partial u(x,t)}{\partial x}. \tag{3}$$

For constant cross-section along axis of rod, equation (2) takes the simply form (4):

$$\frac{\rho}{E}\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{EA}q(x,t). \tag{4}$$

For analysis of natural vibrations leading to the determination of set of natural vibrations of rod with finite length it should be assumed that there is no external excitations, hence $q(x,t) = 0$ and the vibration equation of the rod takes the form (5):

$$\rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2} - E\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x} - EA(x)\frac{\partial^2 u(x,t)}{\partial x^2} = 0. \tag{5}$$

The analytical solution can be found by the application of the Fourier method, when the desired function describing the longitudinal vibrations $u(x,t)$ is proposed in the form of the product of the independent time functions T(t) and the spatial variable $X(x)$. After some of standard transformations used in this type of analysis [3, 12, 15], a solution is obtained in the form of a generally formulated sum (6):

$$u(x,t) = \sum_{n=1}^{+\infty} X_n(x)\, T_n(t), \tag{6}$$

written in a more detailed form (7),

$$u(x,t) = \sum_{n=1}^{+\infty} C_n X_n(x)\sin\left(\omega_n t + \varphi_n\right), \tag{7}$$

where C_n is a series of integration constants that are determined in details after formulating and application of boundary conditions.

The series of functions $X_n(x)$ is determined in general after solving an ordinary differential equation with variable coefficients (8):

$$\frac{d^2 X_n(x)}{dx^2} - \frac{\frac{dA(x)}{dx}}{A(x)}\frac{dX_n(x)}{dx} + \frac{\rho}{E}\omega_n^2 X_n(x) = 0 \tag{8}$$

The problem formulated in this way has analytical solutions only for selected functions $A(x)$ (selected types of cross-section variability). Some of the characteristic cross-sections and the obtained forms of solutions known in the literature can be found in the articles reviewed in the Introduction.

In the case of vibrations of a rod with a constant cross-section, the vibration equation has the form (9):

$$\frac{\rho}{E}\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0. \tag{9}$$

For the combination of typical boundary conditions - free end or fixed type, analytical formulas for determining the natural frequency ω_n and their form $X_n(x)$ can be given. They have been collected in Tab. 1.

Tab. 1. Formulas for natural frequencies and mode shape functions for longitudinal vibrations of rod

Boundary condition	ω_n	$X_n(y)$
fixed - free	$\frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}}$	$\sin\left(\frac{(2n-1)\pi}{2L}x\right)$
fixed - fixed	$\frac{n\pi}{L}\sqrt{\frac{E}{\rho}}$	$\sin\left(\frac{n\pi}{L}x\right)$
free - free	$\frac{n\pi}{L}\sqrt{\frac{E}{\rho}}$	$\cos\left(\frac{n\pi}{L}x\right)$

3. Analyses of geometry influences

3.1. General remarks

The analysis of the influence of geometry on the basic natural frequencies of longitudinal stocky rods with a variable cross-section was performed for an axially symmetrical cylindrical element with the total length L. The bar also has a symmetry plane perpendicular to its axis, which includes the cross-section passing through the centre of the bar's length. The cylindrical outer sections of the rod (along the rod length) have a diameter D and a length 0,5c each. For one group of analyses, the central cylindrical section with diameter d and length a is taken into account. For the second group of analyses, this section is reduced to a circular section with zero length (a = 0). Between the middle section there are symmetrically along the length two sections of variable diameter. In the cross-section of a rod, the boundary line is a fragment of a circle with a radius of R. This line tangentially connects to the central element, if present, or is a whole fragment of a circle with a radius of R (if there is no cylindrical segment with a diameter d). In general, the condition $L = a + b + c$ is satisfied. The drawing showing the cross-section of the analysed element is shown in Fig. 1. In the analyses, the element is made of steel (ρ = 7680 kg/m^3, E = 210 GPa).

The sensitivity analysis was focused on the value of the basic natural frequency (the lowest natural frequency) of the longitudinal vibrations of the element with free-free boundary conditions.

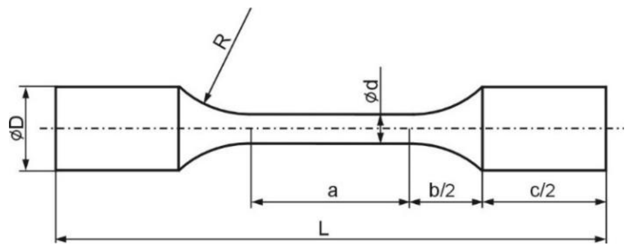

Fig. 1. Geometry of the analysed elements (cross-section)

Simulations were done by application of the *Ansys* finite element package using *plane42* elements with axisymmetric option. The meshed model in the form of half cross-section is shown in Fig. 2 (left), where x axis is the axis of symmetry. For verification of the spatial model was used for test of correctness of applied in the analyses modelling by the plane axisymmetric elements. For spatial modelling *solid272* elements were applied, and the meshed model is shown in Fig. 2 (right). The difference between obtained values of basic frequencies for both models for the reference element (a=0, b=80 mm, c=19.6 mm, d=3 mm, D=10 mm) seems to be enough small (f_1=20724 Hz for *plane42* model, f_1=20371 Hz for solid272 model), therefore for the analyses the plane axisymmetric model was used.

It is important to emphasize that the FEM simulations show the expected situation that despite the change in the cross-section of the element, due to its axial symmetry – the flatness of the longitudinal wave is maintained. It is shown by the distribution of displacements along the bar axis visualized for the reference model in Fig . 3.

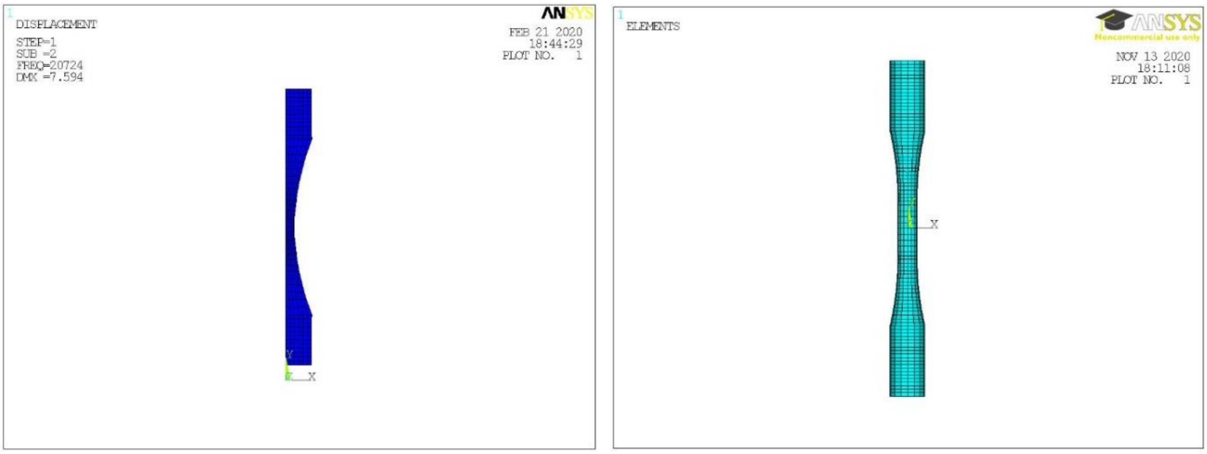

Fig. 2. FEM mesh of axisymmetric model in *Ansys*: plane (left), spatial (right)

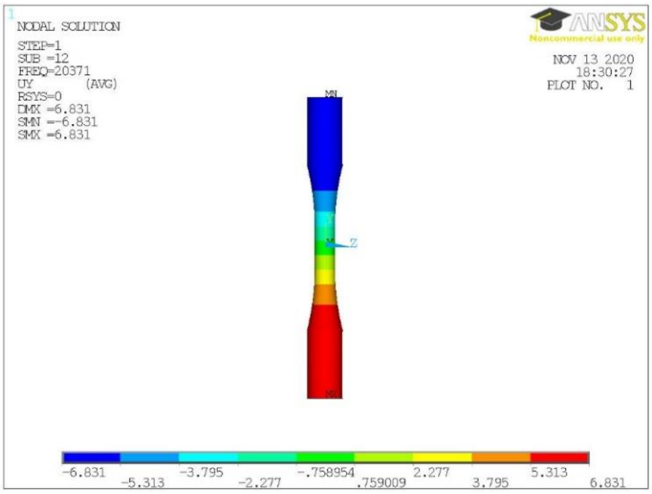

Fig. 3. Axial (u_x) displacements for the basic frequency mode

3.2. Analysis of elements without a cylindrical middle part

The first group of simulations concerned an element that did not contain a cylindrical segment in its central part intentionally with a constant diameter (a = 0). The aim of the analyses was to show the sensitivity of the basic natural frequency of longitudinal vibrations to a change in geometric parameters: the length of the part with a constant diameter (c), the length of the part with a variable cross-section (b), the diameter of the part with a constant cross-section (D), the diameter of the cross-section with the smallest diameter (d). The tested element had a constant total length L = 59.6 mm. For reference, the value of the natural basic frequency was given for a cylindrical element with a diameter of D = 10 mm. These values were determined analytically (based on the formula given in the third row in Tab. 1) and using the FEM model (for verification of the FEM model). The tested shapes of elements refer to the shape of the ItalSigma gigacycle fatigue test [14].

The results of FEM simulations for some characteristic shapes were given in Tab. 2 [14]. For comparison the values of basic natural frequencies of the cylinders with handle diameter and minimal probe diameter (analytical solutions) are shown too.

Tab. 2. Basic natural frequencies f_1 for different probe shapes (a=0)

L [mm]	0.5c [mm]	0.5b [mm]	D [mm]	d [mm]	f_1 [Hz]
		WHOLE CYLLINDER (ANALYTICAL)			
59.6	9.8	40	10	10	43 868
		WHOLE CYLLINDER (FEM)			
59.6	9.8	40	10	10	43 809
		ANALYSIS (FEM) - REFERENCE ELEMENT			
		plane42			
59.6	9.8	40	10	3	20 724
		solid272			
59.6	9.8	40	10	3	20 371
		ANALYSIS (FEM) – variable c			
56.0	8	40	10	3	22 144
70.0	15	40	10	3	17 726
80.0	20	40	10	3	15 753
		ANALYSIS (FEM) – variable b			
49.6	9.8	30	10	3	24 749
69.6	9.8	50	10	3	17 872
79.6	9.8	60	10	3	15 741
		ANALYSIS (FEM) – variable D			
59.6	9.8	40	8	3	24 091
59.6	9.8	40	12	3	18 190
59.6	9.8	40	15	3	15 557
		ANALYSIS (FEM) – variable d			
59.6	8	40	10	2	15 716
59.6	8	40	10	4	25 050

3.3. Analysis of elements with a cylindrical middle part

The second group of analyses concerns elements that in the central part contain a cylindrical part with a constant diameter d=5 mm and length a=12 mm. The length of the variable diameter part is also constant (0.5b=19 mm). The variable parameters were the diameter of the outer part with a constant diameter (D) and its length (0.5c). Consequently, the elements have a different overall length (L). The analysed shapes correspond to the shape of the ItalSigma sample, with the general assumption that the basic natural frequency should have a value of about f_1≈20 kHz. The obtained values are summarized in Tab 3.

Tab. 3. Basic natural frequencies f_1 for different probe shapes (a=0)

L [mm]	a [mm]	0.5c [mm]	0.5b [mm]	D [mm]	d [mm]	f_1 [Hz]
100	12	25	19	10	5	16 647
90	12	20	19	10	5	18 335
90	12	20	19	9	5	19 503
88	12	19	19	9	5	19 927
86	12	18	19	10	5	19 144
86	12	18	19	9	5	20 375

 DOI: 10.21008/j.0860-6897.2021.2.12

4. Conclusions

Generally the obtained results shows, that the values of basic natural frequencies are very sensitive on the geometry of the element and are far from the values obtained for the case of the element with constant cross-section (cylinder with constant value of diameter D).

The form of basic longitudinal type frequency mode is exactly satisfying the condition the standing wave is plane in the entire volume of the element. It is important from the application in design of the probe for gigacyle fatigue tests.

For limited types of the probe shape the analytical formulas for natural frequencies are known in literature.

However the basic natural frequency of the rod with variable cross-section can be approximately estimated by using Galerkin or Rayleigh-Ritz method, currently the finite element method. Application of finite element method make possible to estimate the value and distribution of stresses in the probe.

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