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Operation and reliability optimization of complex technical systems

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Abstract

Modelling and prediction of the operation and reliability of technical multistate ageing systems related to their operation processes called complex systems are briefly presented and applied to prediction of the operation processes and reliability characteristics of an exemplary complex non-homogeneous system composed of a series-parallel and a series-"*m* out of *l*" subsystems linked in series, changing its reliability structure and its components reliability parameters at variable operation conditions. Further, the linear programming is proposed to the operation and reliability optimization of complex technical systems operating at variable operation conditions. The method consists in determining the optimal values of limit transient probabilities at the system operation states that maximize the system lifetimes in the reliability state subsets. The proposed method is practically applied to the operation and reliability optimization of the considered exemplary complex system.

1. Introduction

The complexity of the systems' operation processes and their influence on changing in time the systems' structures and their components' reliability parameters are very often met in real practice. Thus, the practical importance of an approach linking the system reliability models and the system operation processes models into an integrated general model in reliability assessment of real technical systems is evident. The convenient tools for analyzing these problems are semi-Markov modelling the systems' operation processes [1]-[4], [12], [16]-[17], [20], [26] and multistate approach to the systems' reliability evaluation [6], [10], [21], [29]-[32]. The common usage of those two tools in order to construct the joint general system reliability model related to its operation process [7]-[9], [11], [13]-[16], [23]-[25] and to apply it to the reliability analysis of complex technical systems is briefly presented and applied to the reliability evaluation of an exemplary complex system. The complex technical systems reliability improvement and decreasing the risk of exceeding a critical reliability state are of great value in the industrial practice [16], [18]-[19], [28]. There are needed the tools allowing for changing their operation processes after comparing the values of these characteristics with their values before their operation processes optimization in order to improve their reliability [5], [13]-[14], [16], [21], [27].

2. Complex system operation modeling

We assume that the system during its operation process is taking $v, v \in N,$ different operation states $z_1, z_2, \ldots, z_v$. Further, we define the system operation process $Z(t)$, $t \in <0,+\infty)$, with discrete operation states from the set $\{z_1, z_2, \ldots, z_v\}$. Moreover, we assume that the system operation process $Z(t)$ is a semi-Markov process [3], [12]-[13], [26] with the conditional sojourn times θ_{bl} at the operation states z_b when its next operation state is z_l, $b, l = 1,2,...,v,$ $b \neq l.$ Under these assumptions, the system operation process may be described by:

- the vector $[p_b(0)]_{1 \times v}$ of the initial probabilities $p_b(0) = P(Z(0) = z_b),$ $b = 1,2,...,v,$ of the system operation process $Z(t)$ staying at particular operation states at the moment $t = 0$;
- the matrix $[p_{bl}]_{v \times v}$ of probabilities p_{bl}, $b, l = 1,2,...,v,$ $b \neq l,$ of the system operation process $Z(t)$ transitions between the operation states z_b and z_l;

- the matrix $[H_{bl}(t)]_{v\times v}$ of conditional distribution functions $H_{bl}(t) = P(\theta_{bl} < t)$, $t \geq 0$, $b, l = 1,2,...,v$, $b \neq l$, of the system operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states.

As the mean values $E[\theta_{bl}]$ of the conditional sojourn times θ_{bl} are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t), \ b,l = 1,2,...,v, \ b \neq l, \quad (1)$$

then from the formula for total probability, it follows that the unconditional distribution functions of the sojourn times θ_b, $b = 1,2,...,v$, of the system operation process $Z(t)$ at the operation states z_b, $b = 1,2,...,v$, are given by [3], [16], [21], [26]

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \ t \geq 0, \ b = 1,2,...,v.$$

Hence, the mean values $E[\theta_b]$ of the system operation process $Z(t)$ unconditional sojourn times θ_b, $b = 1,2,...,v$, at the operation states are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl}, \ b = 1,2,...,v, \quad (2)$$

where M_{bl} are defined by the formula (1).

The limit values of the system operation process $Z(t)$ transient probabilities at the particular operation states $p_b(t) = P(Z(t) = z_b)$, $t \in < 0,+\infty)$, $b = 1,2,...,v$, are given by [3], [15], [21], [26]

$$p_b = \lim_{t \to \infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \ b = 1,2,...,v, \quad (3)$$

where M_b, $b = 1,2,...,v$, are given by (2), while the steady probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases} \quad (4)$$

Other interesting characteristics of the system operation process $Z(t)$ possible to obtain are its total sojourn times $\hat{\theta}_b$ at the particular operation states z_b, $b = 1,2,...,v$, during the fixed system opetation time. It is well known [3], [16], [21], [26] that the system operation process total sojourn times $\hat{\theta}_b$ at the particular operation states z_b, for sufficiently large operation time θ, have approximately normal distributions with the expected value given by

$$\hat{M}_b = E[\hat{\theta}_b] = p_b \theta, \ b = 1,2,...,v, \quad (5)$$

where p_b are given by (3).

Example

We consider a series system S composed of the subsystems S_1 and S_2, with the scheme showed in *Figure 1*.

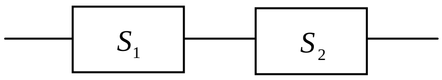

Figure 1. The scheme of the exemplary system S reliability structure (6)

We assume that the subsystem S_1 is a series-parallel system with the scheme given in *Figure 2* and the subsystem S_2 illustrated in *Figure 3* is either a series-parallel system or a series-“2 out of 4” system.

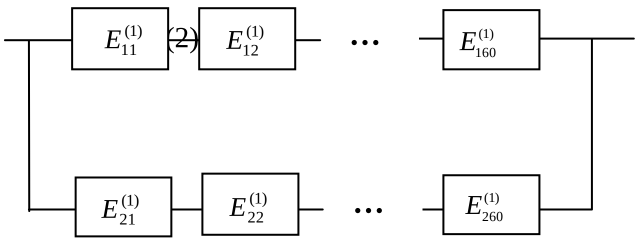

Figure 2. The scheme of the subsystem S_1 reliability structure

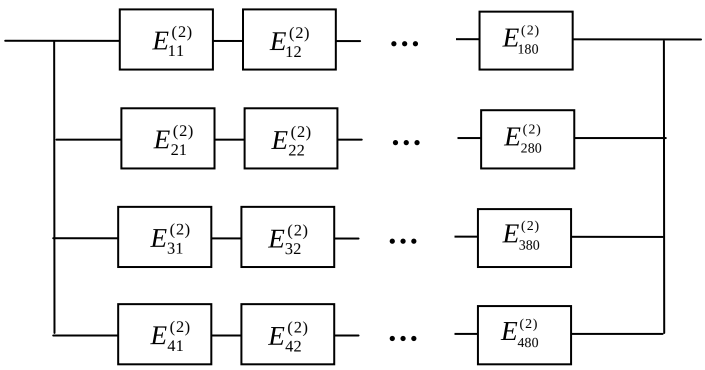

Figure 3. The scheme of the subsystem S_2 reliability structure

The subsystems S_1 and S_2 are forming a general series reliability structure of the system presented in *Figure 1*. However, this system reliability structure and its subsystems and components reliability depend on its changing in time operation states [13], [16], [26].

Under the assumption that the system operation conditions are changing in time, we arbitrarily fix the number of the system operation process states $\nu = 4$ and we distinguish the following as its operation states:

- an operation state z_1 – the system is composed of the subsystem S_1 with the scheme showed in *Figure 2* that is a series-parallel system,
- an operation state z_2 – the system is composed of the subsystem S_2 with the scheme showed in *Figure 3* that is a series-parallel system,
- an operation state z_3 – the system is a series system with the scheme showed in *Figure 1* composed of the subsystems S_1 and S_2 that are series-parallel systems with the schemes respectively given in *Figure 2* and *Figure 3*,
- an operation state z_4 – the system is a series system with the scheme showed in *Figure 1* composed of the subsystem S_1 and S_2, while the subsystem S_1 is a series-parallel system with the scheme given in *Figure 2* and the subsystem S_2 is a series-“2 out of 4” system with the scheme given in *Figure 3*.

The influence of the above system operation states changing on the changes of the exemplary system reliability structure is indicated in these operation states above definitions and illustrated in *Figures 1-3*. Its influence on the system components reliability will be defined in this example continuation in Section 3.

We arbitrarily assume that the probabilities p_{bl} of the exemplary system operation process transitions from operation state z_b into the operation state z_l are given in the matrix below

$$[p_{bl}] = \begin{bmatrix} 0 & 0.25 & 0.30 & 0.45 \\ 0.20 & 0 & 0.25 & 0.55 \\ 0.15 & 0.20 & 0 & 0.65 \\ 0.40 & 0.25 & 0.35 & 0 \end{bmatrix}. \quad (6)$$

We also arbitrarily fix the conditional mean values $M_{bl} = E[\theta_{bl}],\ b,l = 1,2,3,4,$ of the exemplary system sojourn times at the particular operation states as follows:

$$M_{12} = 190,\ M_{13} = 480,\ M_{14} = 200,$$
$$M_{21} = 100,\ M_{23} = 80,\ M_{24} = 60,$$
$$M_{31} = 870,\ M_{32} = 480,\ M_{34} = 300,$$

$$M_{41} = 320,\ M_{42} = 510,\ M_{43} = 440. \quad (7)$$

This way, the exemplary system operation process is defined and we may find its main characteristics. Namely, applying (2), (6) and (7), the unconditional mean sojourn times at the particular operation states are:

$$M_1 = 281.5,\ \ M_2 = 73.0,$$
$$M_3 = 421.5,\ \ M_4 = 409.5. \quad (8)$$

Further, according to (4), after considering (6), we find the steady probabilities

$$\pi_1 \cong 0.216,\ \ \pi_2 \cong 0.191,$$
$$\pi_3 \cong 0.237,\ \ \pi_4 \cong 0.356. \quad (9)$$

After considering the result (8) and (9), according to (3), the limit values of the exemplary system operation process transient probabilities $p_b(t)$ at the operation states z_b are:

$$p_1 \cong 0.190,\ \ p_2 \cong 0.043,$$
$$p_3 \cong 0.312,\ \ p_4 \cong 0.455. \quad (10)$$

Hence, the expected values of the total sojourn times $\hat{\theta}_b,\ b = 1,2,3,4,$ of the exemplary system operation process at the particular operation states z_b, $b = 1,2,3,4,$ during the fixed operation time $\theta = 1$ year = 365 days, after applying (5), amount (in days):

$$\hat{M}_1 \cong 69.3,\ \ \hat{M}_2 \cong 15.7,$$
$$\hat{M}_3 \cong 113.9,\ \ \hat{M}_4 \cong 166.1. \quad (11)$$

3. Complex system reliability modeling

We assume that the changes of the operation states of the system operation process $Z(t)$ have an influence on the system multistate components E_i, $i = 1,2,...,n,$ reliability and the system reliability structure as well. Consequently, we denote the system multistate component E_i, $i = 1,2,...,n,$ conditional lifetime in the reliability state subset $\{u, u+1,...,z\}$ while the system is at the operation state z_b, $b = 1,2,...,v,$ by $T_i^{(b)}(u)$ and its conditional reliability function by the vector

$$[R_i(t,\cdot)]^{(b)} = [1,\ [R_i(t,1)]^{(b)},\ ...,\ [R_i(t,z)]^{(b)}\ ],$$

with the coordinates defined by

$$[R_i(t,u)]^{(b)} = P(T_i^{(b)}(u) > t \,|\, Z(t) = z_b)$$

for $t \in <0, \infty)$, $u = 1,2,\ldots,z$, $b = 1,2,\ldots,\nu$.

The reliability function $[R_i(t,u)]^{(b)}$ is the conditional probability that the component E_i lifetime $T_i^{(b)}(u)$ in the reliability state subset $\{u, u+1,\ldots,z\}$ is greater than *t*, while the system operation process *Z*(*t*) is at the operation state z_b.

Similarly, we denote the system conditional lifetime in the reliability state subset $\{u, u+1,\ldots,z\}$ while the system is at the operation state z_b, $b = 1,2,\ldots,\nu$, by $T^{(b)}(u)$ and the conditional reliability function of the system by the vector

$$[\boldsymbol{R}(t,\cdot)]^{(b)} = [1,\ [\boldsymbol{R}(t,1)]^{(b)}, \ldots,\ [\boldsymbol{R}(t,z)]^{(b)}],$$

with the coordinates defined by

$$[\boldsymbol{R}(t,u)]^{(b)} = P(T^{(b)}(u) > t \,|\, Z(t) = z_b) \qquad (12)$$

for $t \in <0, \infty)$, $u = 1,2,\ldots,z$, $b = 1,2,\ldots,\nu$.

The reliability function $[\boldsymbol{R}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,\ldots,z\}$ is greater than *t*, while the system operation process *Z*(*t*) is at the operation state z_b.

Further, we denote the system unconditional lifetime in the reliability state subset $\{u, u+1,\ldots,z\}$ by $T(u)$ and the unconditional reliability function of the system by the vector

$$\boldsymbol{R}(t,\cdot) = [1,\ \boldsymbol{R}(t,1), \ldots,\ \boldsymbol{R}(t,z)\ ],$$

with the coordinates defined by

$$\boldsymbol{R}(t,u) = P(T(u) > t) \qquad (13)$$

for $t \in <0, \infty)$, $u = 1,2,\ldots,z$.

In the case when the system operation time θ is large enough, the coordinates of the unconditional reliability function of the system defined by (13) are given by

$$\boldsymbol{R}(t,u) \cong \sum_{b=1}^{\nu} p_b [\boldsymbol{R}(t,u)]^{(b)} \text{ for } t \geq 0,\ u = 1,2,\ldots,z, \qquad (14)$$

where $[\boldsymbol{R}(t,u)]^{(b)}$, $u = 1,2,\ldots,z$, $b = 1,2,\ldots,\nu$, are the coordinates of the system conditional reliability functions defined by (12) and p_b, $b = 1,2,\ldots,\nu$, are the system operation process limit transient probabilities given by (3).

Thus, the mean value $\mu(u) = E[T(u)]$ of the system unconditional lifetime $T(u)$ in the reliability state subset $\{u, u+1,\ldots,z\}$ is given by [16], [26]

$$\mu(u) \cong \sum_{b=1}^{\nu} p_b \mu_b(u),\ u = 1,2,\ldots,z, \qquad (15)$$

where $M_b(u) = E[T^{(b)}(u)]$ are the mean values of the system conditional lifetimes $T^{(b)}(u)$ in the reliability state subset $\{u, u+1,\ldots,z\}$ at the operation state z_b, $b = 1,2,\ldots,\nu$, given by

$$\mu_b(u) = \int_0^{\infty} [\boldsymbol{R}(t,u)]^{(b)} dt,\ u = 1,2,\ldots \text{(22)} \qquad (16)$$

$[\boldsymbol{R}(t,u)]^{(b)}$, $u = 1,2,\ldots,z$, $b = 1,2,\ldots,\nu$, are defined by (12) and p_b are given by (3). Since the relationships between the system unconditional lifetimes $\bar{T}(u)$ in the particular reliability states and the system unconditional lifetimes $T(u)$ in the reliability state subsets can be expressed by

$$\bar{T}(u) = T(u) - T(u+1),\ u = 0,1,\ldots,z-1,$$
$$\bar{T}(z) = T(z),$$

then we get the following formulae for the mean values of the unconditional lifetimes of the system in particular reliability states

$$\bar{\mu}(u) = \mu(u) - \mu(u+1),\ u = 0,1,\ldots,z-1,$$
$$\bar{\mu}(z) = \mu(z), \qquad (17)$$

where $\mu(u)$, $u = 0,1,\ldots,z$, are given by (15).

Moreover, if *s*(*t*) is the system reliability state at he moment *t*, $t \in <0, \infty)$, and *r*, $r \in \{1,2,\ldots,z\}$, is the system critical reliability state, then the system risk function

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \leq t),\ t \in <0, \infty),$$

defined as the probability that the system is in the subset of states worse than the critical state *r*, *r* ∈ {1,...,*z*} while it was in the state *z* at the (25) moment *t* = 0 is given by [16]

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}(t,r),\ t \in <0, \infty), \qquad (18)$$

where $\boldsymbol{R}(t,r)$ is the coordinate of the system unconditional reliability function given by (14) for

$u = r$ and if τ is the moment when the system risk function exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \quad (19)$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the risk function $\boldsymbol{r}(t)$ given by (18).

Example (continuation)
In Section 2, it is fixed that the exemplary system reliability structure and its subsystems and components reliability depend on its changing in time operation states. Considering the assumptions and agreements of these sections, we assume that its subsystems S_υ, $\upsilon = 1,2,$ are composed of four-state, i.e. z = 3, components $E_{ij}^{(\upsilon)}$, $\upsilon = 1,2,$ having the conditional reliability functions given by the vector

$$[R_{ij}^{(\upsilon)}(t,\cdot)]^{(b)} = [1,\ [R_{ij}^{(\upsilon)}(t,1)]^{(b)},\ [R_{ij}^{(\upsilon)}(t,2)]^{(b)},\ [R_{ij}^{(\upsilon)}(t,3)]^{(b)}],\quad b = 1,2,3,4,$$

with the exponential co-ordinates

$$[R_{ij}^{(\upsilon)}(t,1)]^{(b)} = \exp[-[\lambda_{ij}^{(\upsilon)}(1)]^{(b)}],$$
$$[R_{ij}^{(\upsilon)}(t,2)]^{(b)} = \exp[-[\lambda_{ij}^{(\upsilon)}(2)]^{(b)}],$$
$$[R_{ij}^{(\upsilon)}(t,3)]^{(b)} = \exp[-[\lambda_{ij}^{(\upsilon)}(3)]^{(b)}],$$

different at various operation states z_b, $b = 1,2,3,4,$ and with the intensities of departure from the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, respectively

$$[\lambda_{ij}^{(\upsilon)}(1)]^{(b)},\ [\lambda_{ij}^{(\upsilon)}(2)]^{(b)},\ [\lambda_{ij}^{(\upsilon)}(3)]^{(b)},\ b = 1,2,3,4.$$

The influence of the system operation states changing on the changes of the system reliability structure and its components reliability functions is as follows.
At the system operation state z_1, the system is composed of the series-parallel subsystem S_1 with the structure showed in *Figure 2*, containing two identical series subsystems ($k^{(1)} = 2$), each composed of sixty components ($l_1^{(1)} = 60$, $l_2^{(1)} = 60$) with the exponential reliability functions. In both series subsystems of the subsystem S_1 there are respectively:
- the components $E_{ij}^{(1)}$, $i = 1,2,$ $j = 1,2,...,40,$ with the conditional reliability function coordinates

$$[R_{ij}^{(1)}(t,1)]^{(1)} = \exp[-0.0008t],$$
$$[R_{ij}^{(1)}(t,2)]^{(1)} = \exp[-0.0009t],$$
$$[R_{ij}^{(1)}(t,3)]^{(1)} = \exp[-0.0010t],\ i = 1,2,\ j = 1,2,...,40;$$

- the components $E_{ij}^{(1)}$, $i = 1,2,$ $j = 41,42,...,60,$ with the conditional reliability function coordinates

$$[R_{ij}^{(1)}(t,1)]^{(1)} = \exp[-0.0011t],$$
$$[R_{ij}^{(1)}(t,2)]^{(1)} = \exp[-0.0012t],$$
$$[R_{ij}^{(1)}(t,3)]^{(1)} = \exp[-0.0013t],$$
$$i = 1,2,\ j = 41,42,...,60.$$

Thus, at the operational state z_1, the system is identical with the subsystem S_1 that is a four-state series-parallel system with its structure shape parameters , $l_1^{(1)} = 60$, $l_2^{(1)} = 60$, and according to Ptoposition 1 given in [17], its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(1)} = [1, [\boldsymbol{R}(t,1)]^{(1)}, [\boldsymbol{R}(t,2)]^{(1)}, [\boldsymbol{R}(t,3)]^{(1)}] \quad (20)$$

for $t \geq 0$, where

$$[\boldsymbol{R}(t,1)]^{(1)} = 2\exp[-0.054t] - \exp[-0.108t],$$
$$[\boldsymbol{R}(t,2)]^{(1)} = 2\exp[-0.060t] - \exp[-0.120t],$$
$$[\boldsymbol{R}(t,3)]^{(1)} = 2\exp[-0.066t] - \exp[-0.132t]. \quad (21)$$

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$ at the operation state z_1, calculated from the results given by (21), according to (16), respectively are:

$$\mu_1(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(1)}\,dt \cong 27.78,$$
$$\mu_1(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(1)}dt = 25.00,$$
$$\mu_1(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(1)}dt \cong 22.73. \quad (22)$$

At the system operation state z_2, the system is composed of the series-parallel subsystem S_2 with the structure showed in *Figure 3*, containing four identical series subsystems ($k^{(2)} = 4$), each composed of eighty components ($l_1^{(2)} = 80$, $l_2^{(2)} = 80$, $l_3^{(2)} = 80$, $l_4^{(2)} = 80$) with the exponential reliability functions. In all series subsystems of the subsystem S_2 there are respectively:
- the components $E_{ij}^{(2)}$, $i = 1,2,3,4,$ $j = 1,2,...,40,$ with the conditional reliability function co-ordinates

$[R_{ij}^{(2)}(t,1)]^{(2)} = \exp[-0.0014t]$,
$[R_{ij}^{(2)}(t,2)]^{(2)} = \exp[-0.0015t]$,
$[R_{ij}^{(2)}(t,3)]^{(2)} = \exp[-0.0016t]$, $i = 1,2,3,4$,
$j = 1,2,...,40$;

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 21,22,...,40$, with the conditional reliability function co-ordinates

$[R_{ij}^{(2)}(t,1)]^{(2)} = \exp[-0.0018t]$,
$[R_{ij}^{(2)}(t,2)]^{(2)} = \exp[-0.0020t]$,
$[R_{ij}^{(2)}(t,3)]^{(2)} = \exp[-0.0022t]$, $i = 1,2,3,4$,
$j = 41,42,...,80$.

Thus, at the operation state z_2, the system is identical with the subsystem S_2 that is a four-state series-parallel system with its structure shape parameters $k^{(2)} = 4$, $l_1^{(2)} = 80$, $l_2^{(2)} = 80$, $l_3^{(2)} = 80$, $l_4^{(2)} = 80$, and according to Proposition 1 given in [17], its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(2)} = [1, [\boldsymbol{R}(t,1)]^{(2)}, [\boldsymbol{R}(t,2)]^{(2)}, [\boldsymbol{R}(t,3)]^{(2)}] \quad (23)$$

for $t \geq 0$, where

$$[\boldsymbol{R}(t,1)]^{(2)} = 4\exp[-0.128t] - 6\exp[-0.256t] + 4\exp[-0.384t] - \exp[-0.512t],$$
$$[\boldsymbol{R}(t,2)]^{(2)} = 4\exp[-0.140t] - 6\exp[-0.280t] + 4\exp[-0.420t] - \exp[-0.560t],$$
$$[\boldsymbol{R}(t,3)]^{(2)} = 4\exp[-0.152t] - 6\exp[-0.304t] + 4\exp[-0.456t] - \exp[-0.608t]. \quad (24)$$

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$ at the operation state z_1, calculated from the results given by (24), according to (16), respectively are:

$$\mu_2(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(2)} dt \cong 16.27,$$
$$\mu_2(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(2)} dt \cong 14.88,$$
$$\mu_2(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(2)} dt \cong 13.71. \quad (25)$$

At the system operation state z_3, the system is a series system with the structure showed in *Figure 1*, composed of two series-parallel subsystems S_1 and S_2 illustrated respectively in *Figure 2* and *Figure 3*. The subsystem S_1 with the structure showed in *Figure 2*, consists of two identical series subsystems ($k^{(3)} = 2$), each composed of sixty components ($l_1^{(3)} = 60$, $l_2^{(3)} = 60$) with the exponential reliability functions. In both series subsystems of the subsystem S_1 there are respectively:

- the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 1,2,...,40$, with the conditional reliability function co-ordinates

$[R_{ij}^{(1)}(t,1)]^{(3)} = \exp[-0.0009t]$,
$[R_{ij}^{(1)}(t,2)]^{(3)} = \exp[-0.0010t]$,
$[R_{ij}^{(1)}(t,3)]^{(3)} = \exp[-0.0011t]$, $i = 1,2$, $j = 1,2,...,40$;

- the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 41,42,...,60$, with the conditional reliability function co-ordinates

$[R_{ij}^{(1)}(t,1)]^{(3)} = \exp[-0.0012t]$,
$[R_{ij}^{(1)}(t,2)]^{(3)} = \exp[-0.0014t]$,
$[R_{ij}^{(1)}(t,3)]^{(3)} = \exp[-0.0016t]$, $i = 1,2$,
$j = 41,42,...,60$.

Thus, at the operation state z_3, the subsystem S_1 is a four-state series-parallel system with its structure shape parameters $k^{(3)} = 2$, $l_1^{(3)} = 60$, $l_2^{(3)} = 60$, and according to Proposition 1 given in [17], its conditional reliability function is given by

$$[\boldsymbol{R}^{(1)}(t,\cdot)]^{(3)} = [1, [\boldsymbol{R}^{(1)}(t,1)]^{(3)}, [\boldsymbol{R}^{(1)}(t,2)]^{(3)}, [\boldsymbol{R}^{(1)}(t,3)]^{(3)}] \quad (26)$$

for $t \geq 0$, where

$$[\boldsymbol{R}^{(1)}(t,1)]^{(3)} = 2\exp[-0.060t] - \exp[-0.120t],$$
$$[\boldsymbol{R}^{(1)}(t,2)]^{(3)} = 2\exp[-0.068t] - \exp[-0.136t],$$
$$[\boldsymbol{R}(t,3)]^{(1)} = 2\exp[-0.076t] - \exp[-0.152t]. \quad (27)$$

The subsystem S_2 with the structure showed in Figure 3, consists of four identical series subsystems ($k^{(3)} = 4$), each composed of eighty components ($l_1^{(3)} = 80$, $l_2^{(3)} = 80$, $l_3^{(3)} = 80$, $l_4^{(3)} = 80$) with the exponential reliability functions given below. In all series subsystems of the subsystem S_2 there are respectively:

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 1,2,...,40$, with the conditional reliability function co-ordinates

$[R_{ij}^{(2)}(t,1)]^{(3)} = \exp[-0.0010t]$,
$[R_{ij}^{(2)}(t,2)]^{(3)} = \exp[-0.0011t]$,
$[R_{ij}^{(2)}(t,3)]^{(3)} = \exp[-0.0012t]$, $i = 1,2,3,4$,
$j = 1,2,...,40$;

- the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 41,42,...,80$, with the conditional reliability function co-ordinates

$$[R_{ij}^{(2)}(t,1)]^{(3)} = \exp[-0.0014t],$$
$$[R_{ij}^{(2)}(t,2)]^{(3)} = \exp[-0.0016t],$$
$$[R_{ij}^{(2)}(t,3)]^{(3)} = \exp[-0.0018t], \; i = 1,2,3,4, \; j = 41,42,...,80.$$

Thus, at the operation state z_3, the subsystem S_2 is a four-state series-parallel system with its structure shape parameters $k^{(3)} = 4$, $l_1^{(3)} = 80$, $l_2^{(3)} = 80$, $l_3^{(3)} = 80$, $l_4^{(3)} = 80$, and according to Proposition 1 given in [17], its conditional reliability function is given by

$$[\boldsymbol{R}^{(2)}(t,\cdot)]^{(3)} = [1, \; [\boldsymbol{R}^{(2)}(t,1)]^{(3)}, [\boldsymbol{R}^{(2)}(t,2)]^{(3)}, [\boldsymbol{R}^{(2)}(t,3)]^{(3)}], \; t \geq 0, \tag{28}$$

where

$$[\boldsymbol{R}^{(2)}(t,1)]^{(3)} = 4\exp[-0.096t] - 6\exp[-0.192t] + 4\exp[-0.288t] - \exp[-0.384t],$$
$$[\boldsymbol{R}^{(2)}(t,2)]^{(3)} = 4\exp[-0.108t] - 6\exp[-0.216t] + 4\exp[-0.324t] - \exp[-0.432t],$$
$$[\boldsymbol{R}^{(2)}(t,3)]^{(3)} = 4\exp[-0.120t] - 6\exp[-0.240t] + 4\exp[-0.360t] - \exp[-0.480t]. \tag{29}$$

Considering that the system at the operation state z_3 is a four-state series system composed of subsystems S_1 and S_2, after applying the formulae appearing after Definition 3.4 in [16] and (27) and (29), its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(3)} = [1, [\boldsymbol{R}(t,1)]^{(3)}, [\boldsymbol{R}(t,2)]^{(3)}, [\boldsymbol{R}(t,3)]^{(3)}] \tag{30}$$

for $t \geq 0$, where

$$[\boldsymbol{R}(t,1)]^{(3)} = 8\exp[-0.156t] - 12\exp[-0.252t] + 8\exp[-0.348t] - 2\exp[-0.424t] - 4\exp[-0.216t] + 6\exp[-0.312t] - 4\exp[-0.408t] + \exp[-0.504t],$$
$$[\boldsymbol{R}(t,2)]^{(3)} = 8\exp[-0.176t] - 12\exp[-0.284t] + 8\exp[-0.392t] - 2\exp[-0.500t] - 4\exp[-0.236t] + 6\exp[-0.344t] - 4\exp[-0.452t] + \exp[-0.560t],$$
$$[\boldsymbol{R}(t,3)]^{(3)} = 8\exp[-0.196t] - 12\exp[-0.316t] + 8\exp[-0.436t] - 2\exp[-0.556t] - 4\exp[-0.256t] + 6\exp[-0.376t] - 4\exp[-0.496t] + \exp[-0.616t]. \tag{31}$$

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$ at the operation state z_3, calculated from the results given by (31), according to (16), respectively are:

$$\mu_3(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(3)} dt \cong 14.82,$$
$$\mu_3(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(3)} dt \cong 13.04,$$
$$\mu_3(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(3)} dt \cong 11.48. \tag{32}$$

At the system operation state z_4, the system is a series system with the scheme showed in *Figure 1*, composed of the subsystem S_1 and S_2 illustrated respectively in *Figure 2* and *Figure 3*, whereas the subsystem S_1 is a series-parallel system and the subsystem S_2 is a series-"2 out of 4" system.

The subsystem S_1 consists of two identical series subsystems ($k^{(4)} = 2$), each composed of sixty components ($l_1^{(4)} = 60$, $l_2^{(4)} = 60$) with the exponential reliability functions the same as at the operation state z_1. Thus, according to (21), the subsystem S_1 conditional reliability function at the operation state z_4, is given by

$$[\boldsymbol{R}^{(1)}(t,\cdot)]^{(4)} = [1, [\boldsymbol{R}^{(1)}(t,1)]^{(4)}, [\boldsymbol{R}^{(1)}(t,2)]^{(4)}, [\boldsymbol{R}^{(1)}(t,3)]^{(4)}] \tag{33}$$

for $t \geq 0$, where

$$[\boldsymbol{R}^{(1)}(t,1)]^{(4)} = 2\exp[-0.054t] - \exp[-0.108t],$$
$$[\boldsymbol{R}^{(1)}(t,2)]^{(4)} = 2\exp[-0.060t] - \exp[-0.120t],$$
$$[\boldsymbol{R}(t,3)]^{(4)} = 2\exp[-0.066t] - \exp[-0.132t]. \tag{34}$$

The subsystem S_2 consists of four identical series subsystems ($k^{(4)} = 4$), each composed of eighty components ($l_1^{(4)} = 80$, $l_2^{(4)} = 80$, $l_3^{(4)} = 80$, $l_4^{(4)} = 80$) with the exponential reliability functions the same as at the operation state z_2 and is a series-"2 out of 4" system ($m = 2$). Thus, at the operation state z_4, the subsystem S_2 is a four-state series-"2 out of 4" system, with its structure shape parameters $k^{(4)} = 4$, $l_1^{(4)} = 80$, $l_2^{(4)} = 80$, $l_3^{(4)} = 80$, $l_4^{(4)} = 80$, and according to Proposition 1 given in [17], its conditional reliability function is given by

$$[\boldsymbol{R}^{(2)}(t,\cdot)]^{(4)} = [1, \; [\boldsymbol{R}^{(2)}(t,1)]^{(4)}, [\boldsymbol{R}^{(2)}(t,2)]^{(4)}, [\boldsymbol{R}^{(2)}(t,3)]^{(4)}] \tag{35}$$

for $t \geq 0$, where

$$[\boldsymbol{R}^{(2)}(t,1)]^{(4)} = 6\exp[-0.256t] - 8\exp[-0.384t] + 3\exp[-0.512t],$$
$$[\boldsymbol{R}^{(2)}(t,2)]^{(4)} = 6\exp[-0.280t] - 8\exp[-0.420t] + 3\exp[-0.560t],$$
$$[\boldsymbol{R}^{(2)}(t,3)]^{(4)} = 6\exp[-0.304t] - 8\exp[-0.456t] + 3\exp -0.608t]. \quad (36)$$

Considering that the system at the operation state z_4 is a four-state series system composed of subsystems S_1 and S_2, after applying the formulae appearing after Definition 3.4 in [16] and (34) and (36), its conditional reliability function is given by

$$[\boldsymbol{R}(t,\cdot)]^{(4)} = [1, [\boldsymbol{R}(t,1)]^{(4)}, [\boldsymbol{R}(t,2)]^{(4)}, [\boldsymbol{R}(t,3)]^{(4)}] \quad (37)$$

for $t \geq 0$, where

$$[\boldsymbol{R}(t,1)]^{(4)} = 12\exp[-0.310t] - 6\exp[-0.364t] - 16\exp[-0.438t] + 8\exp[-0.492t] + 6\exp[-0.566t] - 3\exp[-0.620t],$$
$$[\boldsymbol{R}(t,2)]^{(4)} = 12\exp[-0.340t] - 6\exp[-0.400t] - 16\exp[-0.480t] + 8\exp[-0.540t] + 6\exp[-0.620t] - 3\exp[-0.680t],$$
$$[\boldsymbol{R}(t,3)]^{(4)} = 12\exp[-0.370t] - 6\exp[-0.436t] - 16\exp[-0.522t] + 8\exp[-0.588t] + 6\exp[-0.674t] - 3\exp[-0.740t]. \quad (38)$$

The mean values of the system sojourn times $T(u)$ in the reliability state subsets after applying the formula (38) and (16), are:

$$\mu_4(1) = \int_0^\infty [\boldsymbol{R}(t,1)]^{(4)}\, dt \cong 7.72,$$
$$\mu_4(2) = \int_0^\infty [\boldsymbol{R}(t,2)]^{(4)} dt \cong 7.04,$$
$$\mu_4(3) = \int_0^\infty [\boldsymbol{R}(t,3)]^{(4)} dt \cong 6.47. \quad (39)$$

In the case when the system operation time is large enough its unconditional four-state reliability function is given by the vector

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), \boldsymbol{R}(t,2), \boldsymbol{R}(t,3)] \text{ for } t \geq 0, \quad (40)$$

where according to (14) and considering the exemplary system operation process transient probabilities at the operation states determined by (10), the vector co-ordinates are given respectively by

$$\boldsymbol{R}(t,1) = p_1[\boldsymbol{R}(t,1)]^{(1)} + p_2[\boldsymbol{R}(t,1)]^{(2)} + p_3[\boldsymbol{R}(t,1)]^{(3)} + p_4[\boldsymbol{R}(t,1)]^{(4)}$$
$$= 0.190 \cdot [\boldsymbol{R}(t,1)]^{(1)} + 0.043 \cdot [\boldsymbol{R}(t,1)]^{(2)} + 0.312 \cdot [\boldsymbol{R}(t,1)]^{(3)} + 0.455 \cdot [\boldsymbol{R}(t,1)]^{(4)},$$
$$\boldsymbol{R}(t,2) = p_1[\boldsymbol{R}(t,2)]^{(1)} + p_2[\boldsymbol{R}(t,2)]^{(2)} + p_3[\boldsymbol{R}(t,2)]^{(3)} + p_4[\boldsymbol{R}(t,2)]^{(4)}$$
$$= 0.190 \cdot [\boldsymbol{R}(t,2)]^{(1)} + 0.043 \cdot [\boldsymbol{R}(t,2)]^{(2)} + 0.312 \cdot [\boldsymbol{R}(t,2)]^{(3)} + 0.455 \cdot [\boldsymbol{R}(t,2)]^{(4)},$$
$$\boldsymbol{R}(t,3) = p_1[\boldsymbol{R}(t,3)]^{(1)} + p_2[\boldsymbol{R}(t,3)]^{(2)} + p_3[\boldsymbol{R}(t,3)]^{(3)} + p_4[\boldsymbol{R}(t,3)]^{(4)}$$
$$= 0.190 \cdot [\boldsymbol{R}(t,3)]^{(1)} + 0.043 \cdot [\boldsymbol{R}(t,3)]^{(2)} + 0.312 \cdot [\boldsymbol{R}(t,3)]^{(3)} + 0.455 \cdot [\boldsymbol{R}(t,3)]^{(4)}, \quad (41)$$

where the coordinates $[\boldsymbol{R}(t,1)]^{(1)}$, $[\boldsymbol{R}(t,1)]^{(2)}$, $[\boldsymbol{R}(t,1)]^{(3)}$, $[\boldsymbol{R}(t,1)]^{(4)}$, $[\boldsymbol{R}(t,2)]^{(1)}$, $[\boldsymbol{R}(t,2)]^{(2)}$, $[\boldsymbol{R}(t,2)]^{(3)}$, $[\boldsymbol{R}(t,2)]^{(4)}$, $[\boldsymbol{R}(t,3)]^{(1)}$, $[\boldsymbol{R}(t,3)]^{(2)}$, $[\boldsymbol{R}(t,3)]^{(3)}$, $[\boldsymbol{R}(t,3)]^{(4)}$ are given by (21), (24), (31), (38).
The graph of the four-state exemplary system reliability function is illustrated in *Figure 4*.

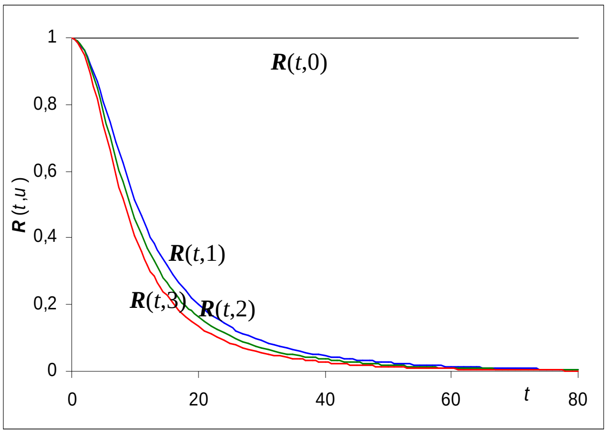

Figure 4. The graph of the exemplary system reliability function $\boldsymbol{R}(t,\cdot)$ coordinates

The expected values of the system unconditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, calculated from the results given by (41) according to (16) and considering (15) and (22), (25), (32), (39), respectively are:

$$\mu(1) = p_1\mu_1(1) + p_2\mu_2(1) + p_3\mu_3(1) + p_4\mu_4(1)$$
$$= 0.190 \cdot 27.78 + 0.043 \cdot 16.27 + 0.312 \cdot 14.82 + 0.455 \cdot 7.72 \cong 14.11,$$
$$\mu(2) = p_1\mu_1(2) + p_2\mu_2(2) + p_3\mu_3(2) + p_4\mu_4(2)$$
$$= 0.190 \cdot 25.00 + 0.043 \cdot 14.88 + 0.312 \cdot 13.04$$

$$+0.455\cdot 7.04 \cong 12.66,$$

$$\mu(3) = p_1\mu_1(3) + p_2\mu_2(3) + p_3M_3(3) + p_4\mu_4(3)$$
$$= 0.190\cdot 22.73 + 0.043\cdot 13.71 + 0.312\cdot 11.48$$
$$+0.455\cdot 6.47 \cong 11.43. \quad (42)$$

Farther, considering (17) and (42), the mean values of the system unconditional lifetimes in the particular reliability states 1, 2, 3, respectively are:

$$\bar{\mu}(1) = \mu(1) - \mu(2) = 1.45,$$
$$\bar{\mu}(2) = \mu(2) - \mu(3) = 1.23,$$
$$\bar{\mu}(3) = \mu(3) = 11.43. \quad (43)$$

Since the critical reliability state is r = 2, then the system risk function, according to (18), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}(t,2) \text{ for } t \geq 0, \quad (44)$$

where $\boldsymbol{R}(t,2)$ is given by (41).

Hence, by (19), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 2.25. \quad (45)$$

The graph of the risk function $\boldsymbol{r}(t)$ of the exemplary four-state system operating at the variable conditions is given in *Figure 5*.

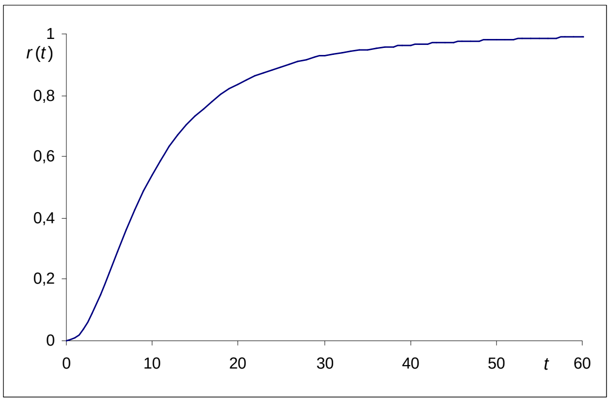

Figure 5. The graph of the exemplary system risk function $\boldsymbol{r}(t)$

4. Complex system reliability and operation optimization

Considering the equation (14), it is natural to assume that the system operation process has a significant influence on the system reliability. This influence is also clearly expressed in the equation (15) for the mean values of the system unconditional lifetimes in the reliability state subsets.

From the linear equation (15), we can see that the mean value of the system unconditional lifetime $\mu(u)$, $u = 1,2,..., z,$ is determined by the limit values of transient probabilities p_b, $b = 1,2,...,\nu$, of the system operation process at the operation states given by (3) and the mean values $\mu_b(u)$, $b = 1,2,...,\nu$, $u = 1,2,..., z,$ of the system conditional lifetimes in the reliability state subsets $\{u, u+1,..., z\}$, $u = 1,2,..., z,$ given by (16). Therefore, the system lifetime optimization approach based on the linear programming [13], [15], [17], [22]. can be proposed. Namely, we may look for the corresponding optimal values $\dot{p}_b$, $b = 1,2,...,\nu$, of the transient probabilities p_b, $b = 1,2,...,\nu$, of the system operation process at the operation states to maximize the mean value $\mu(u)$ of the unconditional system lifetimes in the reliability state subsets $\{u, u+1,..., z\}$, $u = 1,2,..., z,$ under the assumption that the mean values $\mu_b(u)$, $b = 1,2,...,\nu$, $u = 1,2,..., z,$ of the system conditional lifetimes in the reliability state subsets are fixed. As a special and practically important case of the above formulated system lifetime optimization problem, if r, $r = 1,2,..., z,$ is a system critical reliability state, we may look for the optimal values $\dot{p}_b$, $b = 1,2,...,\nu$, of the transient probabilities p_b, $b = 1,2,...,\nu$, of the system operation process at the system operation states to maximize the mean value $\mu(r)$ of the unconditional system lifetime in the reliability state subset $\{r, r+1, ,..., z\}$, $r = 1,2,..., z,$ under the assumption that the mean values $\mu_b(r)$, $b = 1,2,...,\nu$, $r = 1,2,..., z,$ of the system conditional lifetimes in this reliability state subset are fixed. More exactly, we may formulate the optimization problem as a linear programming model with the objective function of the following form

$$\mu(r) = \sum_{b=1}^{\nu} p_b \mu_b(r) \quad (46)$$

for a fixed $r \in \{1,2,..., z\}$ and with the following bound constraints

$$\breve{p}_b \leq p_b \leq \hat{p}_b, \; b = 1,2,...,\nu, \; \sum_{b=1}^{\nu} p_b = 1, \quad (47)$$

where $\mu_b(r)$, $\mu_b(r) \geq 0$, $b = 1,2,...,\nu$, are fixed mean values of the system conditional lifetimes in the reliability state subset $\{r, r+1,..., z\}$ and

$$\breve{p}_b, \; 0 \leq \breve{p}_b \leq 1 \text{ and } \hat{p}_b, \; 0 \leq \hat{p}_b \leq 1, \; \breve{p}_b \leq \hat{p}_b, \quad (48)$$

$b = 1,2,...,\nu,$

are lower and upper bounds of the unknown transient probabilities p_b, $b = 1,2,...,\nu,$ respectively.

Now, we can obtain the optimal solution of the formulated by (46)-(48) the linear programming problem, i.e. we can find the optimal values $\dot{p}_b$ of the transient probabilities p_b, $b = 1,2,...,\nu,$ that maximize the objective function given by (1).

First, we arrange the system conditional lifetime mean values $\mu_b(r)$, $b = 1,2,...,\nu,$ in non-increasing order $\mu_{b_1}(r) \geq \mu_{b_2}(r) \geq . \quad . \quad . \quad \geq \mu_{b_\nu}(r),$ where $b_i \in \{1,2,...,\nu\}$ for $i = 1,2,...,\nu.$

Next, we substitute

$$x_i = p_{b_i},\ \breve{x}_i = \breve{p}_{b_i},\ \hat{x}_i = \hat{p}_{b_i} \ \text{ for } \ i = 1,2,...,\nu \tag{49}$$

and we maximize with respect to x_i, $i = 1,2,...,\nu,$ the linear form (46) that after this transformation takes the form

$$\mu(r) = \sum_{i=1}^{\nu} x_i \mu_{b_i}(r) \tag{50}$$

for a fixed $r \in \{1,2,..., z\}$ with the following bound constraints

$$\breve{x}_i \leq x_i \leq \hat{x}_i,\ i = 1,2,...,\nu,\ \sum_{i=1}^{\nu} x_i = 1, \tag{51}$$

where $\mu_{b_i}(r)$, $\mu_{b_i}(r) \geq 0$, $i = 1,2,...,\nu,$ are fixed mean values of the system conditional lifetimes in the reliability state subset $\{r, r+1,..., z\}$ arranged in non-increasing order and

$$\breve{x}_i,\ 0 \leq \breve{x}_i \leq 1 \ \text{ and } \ \hat{x}_i,\ 0 \leq \hat{x}_i \leq 1,\ \breve{x}_i \leq \hat{x}_i, \quad i = 1,2,...,\nu, \tag{52}$$

are lower and upper bounds of the unknown probabilities x_i, $i = 1,2,...,\nu,$ respectively.

To find the optimal values of x_i, $i = 1,2,...,\nu,$ we define

$$\breve{x} = \sum_{i=1}^{\nu} \breve{x}_i,\ \hat{y} = 1 - \breve{x} \tag{53}$$

and

$$\breve{x}^0 = 0,\ \hat{x}^0 = 0 \ \text{ and } \ \breve{x}^I = \sum_{i=1}^{I} \breve{x}_i,\ \hat{x}^I = \sum_{i=1}^{I} \hat{x}_i \quad \text{for } \ I = 1,2,...,\nu. \tag{54}$$

Next, we find the largest value $I \in \{0,1,...,\nu\}$ such that

$$\hat{x}^I - \breve{x}^I < \hat{y} \tag{55}$$

and we fix the optimal solution that maximize (50) in the following way:

i) if $I = 0$, the optimal solution is

$$\dot{x}_1 = \hat{y} + \breve{x}_1 \ \text{ and } \ \dot{x}_i = \breve{x}_i \ \text{ for } \ i = 2,3,...,\nu; \tag{56}$$

ii) if $0 < I < \nu$, the optimal solution is

$$\dot{x}_i = \hat{x}_i \ \text{ for } \ i = 1,2,..., I,\ \dot{x}_{I+1} = \hat{y} - \hat{x}^I + \breve{x}^I + \breve{x}_{I+1} \quad \text{and } \ \dot{x}_i = \breve{x}_i \ \text{ for } \ i = I+2, I+3,...,\nu; \tag{57}$$

iii) if $I = \nu$, the optimal solution is

$$\dot{x}_i = \hat{x}_i \ \text{ for } \ i = 1,2,...,\nu. \tag{58}$$

Finally, after making the inverse to (49) substitution, we get the optimal limit transient probabilities

$$\dot{p}_{b_i} = \dot{x}_i \ \text{ for } \ i = 1,2,...,\nu, \tag{59}$$

that maximize the system mean lifetime in the reliability state subset $\{r, r+1,..., z\}$, defined by the linear form (46), giving its maximum value in the following form

$$\dot{\mu}(r) = \sum_{b=1}^{\nu} \dot{p}_b \mu_b(r) \tag{60}$$

for a fixed $r \in \{1,2,..., z\}$.

From the expression (60) for the maximum mean value $\dot{\mu}(r)$ of the system unconditional lifetime in the reliability state subset $\{r, r+1,..., z\}$, replacing in it the critical reliability state r by the reliability state u, $u = 1,2,..., z,$ we obtain the corresponding optimal solutions for the mean values of the system unconditional lifetimes in the reliability state subsets $\{u, u+1,..., z\}$ of the form

$$\dot{\mu}(u) = \sum_{b=1}^{\nu} \dot{p}_b \mu_b(u) \ \text{ for } \ u = 1,2,..., z. \tag{61}$$

Further, according to (13)-(14), the corresponding optimal unconditional multistate reliability function of the system is the vector

$$\dot{\boldsymbol{R}}(t,\cdot) = [1,\ \dot{\boldsymbol{R}}(t,1),\ ...,\ \dot{\boldsymbol{R}}(t,z)\ ], \tag{62}$$

with the coordinates given by

$$\dot{\boldsymbol{R}}(t,u) \cong \sum_{b=1}^{v} \dot{p}_b [\boldsymbol{R}(t,u)]^{(b)} \text{ for } t \ge 0, \quad u = 1,2,...,z. \tag{63}$$

And, by (17), the optimal solutions for the mean values of the system unconditional lifetimes in the particular reliability states are

$$\dot{\bar{\mu}}(u) = \dot{\mu}(u) - \dot{\mu}(u+1), \ u = 1,...,z-1, \quad \dot{\bar{\mu}}(z) = \dot{\mu}(z). \tag{64}$$

Moreover, considering (18) and (19), the corresponding optimal system risk function and the optimal moment when the risk exceeds a permitted level δ, respectively are given by

$$\dot{\boldsymbol{r}}(t) = 1 - \dot{\boldsymbol{R}}(t,r), \ t \ge 0, \tag{65}$$

and

$$\dot{\tau} = \dot{\boldsymbol{r}}^{-1}(\delta), \tag{66}$$

where $\dot{\boldsymbol{R}}(t,r)$ is given by (63) for $u = r$ and $\dot{\boldsymbol{r}}^{-1}(t)$, if it exists, is the inverse function of the optimal risk function $\dot{\boldsymbol{r}}(t)$.

Replacing in (3) the limit transient probabilities p_b of the system operation process at the operation states by their optimal values $\dot{p}_b$, maximizing the mean value $\mu(r)$ of the system lifetime in the reliability states subset $\{r, r+1,..., z\}$ defined by (46) and the mean values M_b of the unconditional sojourn times at the operation states by their corresponding unknown optimal values $\dot{M}_b$, we get the system of equations

$$\dot{p}_b = \frac{\pi_b \dot{M}_b}{\sum_{l=1}^{v} \pi_l \dot{M}_l}, \ b = 1,2,...,v. \tag{67}$$

After simple transformations the above system takes the form

$$\begin{aligned} &(\dot{p}_1 - 1)\pi_1 \dot{M}_1 + \dot{p}_1 \pi_2 \dot{M}_2 + ... + \dot{p}_1 \pi_v \dot{M}_v = 0 \\ &\dot{p}_2 \pi_1 \dot{M}_1 + (\dot{p}_2 - 1)\pi_2 \dot{M}_2 + ... + \dot{p}_2 \pi_v \dot{M}_v = 0 \\ &... \\ &\dot{p}_v \pi_1 \dot{M}_1 + \dot{p}_v \pi_2 \dot{M}_2 + ... + (\dot{p}_v - 1)\pi_v \dot{M}_v = 0, \end{aligned} \tag{68}$$

where $\dot{M}_b$ are unknown variables we want to find, $\dot{p}_b$ are optimal transient probabilities determined by (59) and π_b are steady probabilities determined by (4) .

Since the system of equations (68) is homogeneous and it can be proved that the determinant of its main matrix is equal to zero, then it has nonzero solutions and moreover, these solutions are ambiguous. Thus, if we fix some of the optimal values $\dot{M}_b$ of the mean values M_b of the unconditional sojourn times at the operation states, for instance by arbitrary fixing one or a few of them, we may find the values of the remaining once and this way to get the solution of this equation.

Having this solution, it is also possible to look for the optimal values $\dot{M}_{bl}$ of the mean values M_{bl} of the conditional sojourn times at the operation states using the following system of equations

$$\sum_{l=1}^{v} p_{bl} \dot{M}_{bl} = \dot{M}_b, \ b = 1,2,...,v, \tag{69}$$

obtained from (2) by replacing M_b by $\dot{M}_b$ and M_{bl} by $\dot{M}_{bl}$, were p_{bl} are known probabilities of the system operation process transitions between the operation states z_b i z_l, $b,l = 1,2,...,v.$

Another very useful and much easier to be applied in practice tool that can help in planning the operation processes of the complex technical systems are the system operation process optimal mean values of the total system operation process sojourn times $\hat{\theta}_b$ at the particular operation states z_b, $b = 1,2,...,v,$ during the fixed system operation time θ, that can be obtain by the replacing in the formula (5) the transient probabilities p_b at the operation states z_b by their optimal values $\dot{p}_b$ and resulting in the following expession

$$\dot{\hat{M}}_b = \dot{E}[\hat{\theta}_b] = \dot{p}_b \theta, \ b = 1,2,...,v. \tag{70}$$

The knowledge of the optimal values $\dot{M}_b$ of the mean values of the unconditional sojourn times and the optimal values $\dot{M}_{bl}$ of the mean values of the conditional sojourn times at the operation states and the optimal mean values $\dot{\hat{M}}_b$ of the total sojourn times at the particular operation states during the fixed system operation time may by the basis for changing the complex technical systems operation processes in order to ensure these systems operation more reliable.

Example (continuation)

We consider a series system S composed of the subsystems S_1 and S_2, with the scheme showed in

Figures 1-3. This system reliability structure and its components reliability parameters depend on its changing in time operation states with arbitrarily fixed the number of the system operation process states $\nu = 4$ and their influence on the system reliability indicated in Sections 2-3 where its main reliability characteristics are predicted.

To find the optimal values of those system reliability characteristics, we conclude that the objective function defined by (46), in this case, as the exemplary system critical state is $r = 2$, according to (42), takes the form

$$\mu(2) = p_1 \cdot 25.00 + p_2 \cdot 14.88 + p_3 \cdot 13.04 + p_4 \cdot 7.04. \quad (71)$$

Arbitrarily assumed, the lower $\breve{p}_b$ and upper $\hat{p}_b$ bounds of the unknown optimal values of transient probabilities p_b, $b = 1,2,3,4,$ respectively are:

$$\breve{p}_1 = 0.201,\ \breve{p}_2 = 0.03,\ \breve{p}_3 = 0.245,\ \breve{p}_4 = 0.309,\ \hat{p}_1 = 0.351,\ \hat{p}_2 = 0.105,\ \hat{p}_3 = 0.395,\ \hat{p}_4 = 0.459. \quad (72)$$

Therefore, according to (47), we assume the following bound constraints

$$0.201 \le p_1 \le 0.351,\ 0.030 \le p_2 \le 0.105,\ 0.245 \le p_3 \le 0.395,\ 0.309 \le p_4 \le 0.459.\ \sum_{b=1}^{4} p_b = 1, \quad (73)$$

Now, before we find optimal values $\dot{p}_b$ of the transient probabilities p_b, $b = 1,2,3,4,$ that maximize the objective function (71), w arrange the system conditional lifetime mean values $\mu_b(2)$, $b = 1,2,3,4,$ in non-increasing order

$$\mu_1(2) \ge \mu_2(2) \ge \mu_3(2) \ge \mu_4(2).$$

Further, according to (49), we substitute

$$x_1 = p_1,\ x_2 = p_2,\ x_3 = p_3,\ x_4 = p_4, \quad (74)$$

and

$$\breve{x}_1 = \breve{p}_1 = 0.201,\ \breve{x}_2 = \breve{p}_2 = 0.030,\ \breve{x}_3 = \breve{p}_3 = 0.245,\ \breve{x}_4 = \breve{p}_4 = 0.309; \quad (75)$$

$$\hat{x}_1 = \hat{p}_1 = 0.351,\ \hat{x}_2 = \hat{p}_2 = 0.105,\ \hat{x}_3 = \hat{p}_3 = 0.395,\ \hat{x}_4 = \hat{p}_4 = 0.459, \quad (76)$$

and we maximize with respect to x_i, $i = 1,2,3,4,$ the linear form (71) that according to (50)-(51) takes the form

$$\mu(2) = x_1 \cdot 25.00 + x_2 \cdot 14.88 + x_3 \cdot 13.04 + x_4 \cdot 7.04, \quad (77)$$

with the following bound constraints

$$0.201 \le x_1 \le 0.351,\ 0.030 \le x_2 \le 0.105,\ 0.245 \le x_3 \le 0.395,\ 0.309 \le x_4 \le 0.459.\ \sum_{i=1}^{4} x_i = 1. \quad (78)$$

According to (53), we calculate

$$\breve{x} = \sum_{i=1}^{4} \breve{x}_i = 0.785,\ \hat{y} = 1 - \breve{x} = 1 - 0.785 = 0.215 \quad (79)$$

and according to (54), we determine

$$\breve{x}^0 = 0,\ \hat{x}^0 = 0,\ \hat{x}^0 - \breve{x}^0 = 0,$$
$$\breve{x}^1 = 0.201,\ \hat{x}^1 = 0.351,\ \hat{x}^1 - \breve{x}^1 = 0.150,$$
$$\breve{x}^2 = 0.231,\ \hat{x}^2 = 0.456,\ \hat{x}^2 - \breve{x}^2 = 0.225,$$
$$\breve{x}^3 = 0.476.\ \hat{x}^3 = 0.851,\ \hat{x}^3 - \breve{x}^3 = 0.375,$$
$$\breve{x}^4 = 0.785\ \ \hat{x}^4 = 1.31\ \ \hat{x}^4 - \breve{x}^4 = 0.525. \quad (80)$$

From the above, as according to (79), the inequality (55) takes the form (30)

$$\hat{x}^I - \breve{x}^I < 0.215, \quad (81)$$

it follows that the largest value $I \in \{0,1,2,3,4\}$ such that this inequality holds is $I = 1$.

Therefore, we fix the optimal solution that maximize linear function (77) according to the rule (57). Namely, we get

$$\dot{x}_1 = \hat{x}_1 = 0.351,$$
$$\dot{x}_2 = \hat{y} - \hat{x}^1 + \breve{x}^1 + \breve{x}_2 = 0.215 - 0.351 + 0.201 + 0.030 = 0.095,$$
$$\dot{x}_3 = \breve{x}_3 = 0.245,\ \dot{x}_4 = \breve{x}_4 = 0.309.$$

Finally, after making the inverse to (74) substitution, we get the optimal transient probabilities

$$\dot{p}_1 = \dot{x}_1 = 0.351,\ \dot{p}_2 = \dot{x}_2 = 0.095,\ \dot{p}_3 = \dot{x}_3 = 0.245,\ \dot{p}_4 = \dot{x}_4 = 0.309, \quad (82)$$

that maximize the exemplary system mean lifetime $\mu(2)$ in the reliability state subset $\{2,3\}$ expressed by the linear form (71) giving, according to (60) and (82), its optimal value

$$\dot{\mu}(2) = \dot{p}_1 \cdot 25.00 + \dot{p}_2 \cdot 14.88 + \dot{p}_3 \cdot 13.04 + \dot{p}_4 \cdot 7.04 = 0.351 \cdot 25.00 + 0.095 \cdot 14.88 + 0.245 \cdot 13.04 + 0.309 \cdot 7.07 \cong 15.56. \quad (83)$$

Substituting the optimal solution (82) into the formula (61), we obtain the optimal solution for the mean values of the exemplary system unconditional lifetimes in the reliability state subsets {1,2,3} and {3}, that are as follows

$$\dot{\mu}(1) = \dot{p}_1 \cdot 27.78 + \dot{p}_2 \cdot 16.27 + \dot{p}_3 \cdot 14.82 + \dot{p}_4 \cdot 7.72 = 0.351 \cdot 27.78 + 0.095 \cdot 16.27 + 0.245 \cdot 14.82 + 0.309 \cdot 7.72 \cong 17.31, \quad (84)$$

$$\dot{\mu}(3) = \dot{p}_1 \cdot 22.73 + \dot{p}_2 \cdot 13.71 + \dot{p}_3 \cdot 11.48 + \dot{p}_4 \cdot 6.47 = 0.351 \cdot 22.73 + 0.095 \cdot 13.71 + 0.245 \cdot 11.48 + 0.309 \cdot 6.47 \cong 14.09 \quad (85)$$

and according to (64), the optimal values of the mean values of the system unconditional lifetimes in the particular reliability states 1, 2 and 3, respectively are

$$\dot{\bar{\mu}}(1) = \dot{\mu}(1) - \dot{\mu}(2) = 1.75,$$
$$\dot{\bar{\mu}}(2) = \dot{\mu}(2) - \dot{\mu}(3) = 1.47,$$
$$\dot{\bar{\mu}}(3) = \dot{\mu}(3) = 14.09. \quad (86)$$

Moreover, according to (62)-(63), the corresponding optimal unconditional multistate reliability function of the system is of the form

$$\dot{\boldsymbol{R}}(t,\cdot) = [1, \dot{\boldsymbol{R}}(t,1), \dot{\boldsymbol{R}}(t,2), \dot{\boldsymbol{R}}(t,3)] \quad (87)$$

for $t \geq 0$, with the coordinates given by

$$\dot{\boldsymbol{R}}(t,1) = 0.351 \cdot [\boldsymbol{R}(t,1)]^{(1)} + 0.095 \cdot [\boldsymbol{R}(t,1)]^{(2)} + 0.245 \cdot [\boldsymbol{R}(t,1)]^{(3)} + 0.309 \cdot [\boldsymbol{R}(t,1)]^{(4)},$$
$$\dot{\boldsymbol{R}}(t,2) = 0.351 \cdot [\boldsymbol{R}(t,2)]^{(1)} + 0.095 \cdot [\boldsymbol{R}(t,2)]^{(2)} + 0.245 \cdot [\boldsymbol{R}(t,2)]^{(3)} + 0.309 \cdot [\boldsymbol{R}(t,2)]^{(4)},$$
$$\dot{\boldsymbol{R}}(t,3) = 0.351 \cdot [\boldsymbol{R}(t,3)]^{(1)} + 0.0095 \cdot [\boldsymbol{R}(t,3)]^{(2)} + 0.245 \cdot [\boldsymbol{R}(t,3)]^{(3)} + 0.309 \cdot [\boldsymbol{R}(t,3)]^{(4)}, \quad (88)$$

where $[\boldsymbol{R}(t,1)]^{(b)}$, $[\boldsymbol{R}(t,2)]^{(b)}$, $[\boldsymbol{R}(t,3)]^{(b)}$, $b = 1,2,3,4,$ are fixed in Section 3.

The graph of the exemplary system optimal reliability function $\dot{\boldsymbol{R}}(t,\cdot)$ given by (87)-(88) is presented in *Figure 6*.

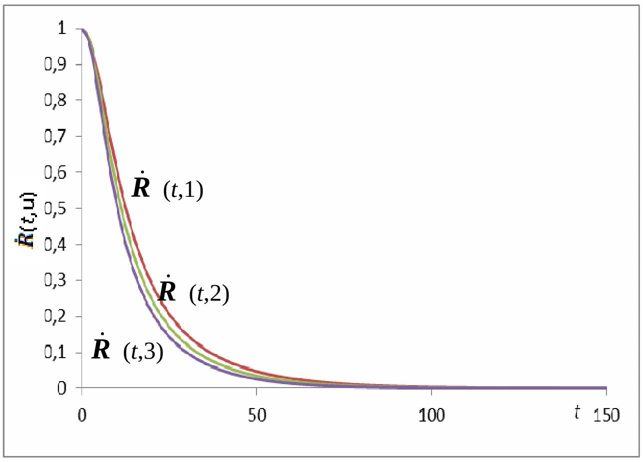

Figure 6. The graph of the exemplary system optimal reliability function $\dot{\boldsymbol{R}}(t,\cdot)$ coordinates

As the critical reliability state is r = 2, then the exemplary system optimal system risk function, according to (65), is given by

$$\dot{\boldsymbol{r}}(t) = 1 - \dot{\boldsymbol{R}}(t,2) \text{ for } t \geq 0, \quad (89)$$

where $\dot{\boldsymbol{R}}(t,2)$ is given by (88).

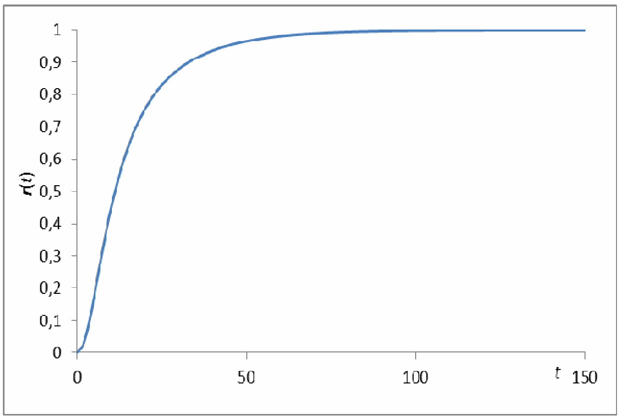

Figure 7. The graph of the exemplary system optimal risk function $\dot{\boldsymbol{r}}(t)$

Hence and considering (66), the moment when the optimal system risk function exceeds a permitted level, for instance δ = 0.025, is

$$\dot{\tau} = \dot{\boldsymbol{r}}^{-1}(\delta) \cong 2.55. \quad (90)$$

It can be seen that the optimal system reliability characteristics given by (87)-(88), (83)-(85), (86), (89) and (90) are better than that before optimization given respectively by (40)-(41), (42), (43), (44) and (45).

Substituting the exemplary operation process optimal transient probabilities at operation states

$$\dot{p}_1 = 0.351,\ \dot{p}_2 = 0.095,\ \dot{p}_3 = 0.245,\ \dot{p}_4 = 0.309,$$

determined by (82) and the steady probabilities

$$\pi_1 \cong 0.236,\ \pi_2 \cong 0.169,\ \pi_3 \cong 0.234,\ \pi_4 \cong 0.361,$$

determined by (9) into (68), we get the following system of equations with the unknown optimal mean values $\dot{M}_b$ of the exemplary system operation process unconditional sojourn times at the operation states we are looking for

$$\begin{aligned}
&-0.153164\dot{M}_1 + 0.059319\dot{M}_2 + 0.082134\dot{M}_3 \\
&+0.126711\dot{M}_4 = 0 \\
&0.02242\dot{M}_1 - 0.152945\dot{M}_2 + 0.02223\dot{M}_3 \\
&+0.034295\dot{M}_4 = 0 \\
&0.05782\dot{M}_1 + 0.041405\dot{M}_2 - 0.17667\dot{M}_3 \\
&+0.088445\dot{M}_4 = 0 \\
&0.072924\dot{M}_1 + 0.052221\dot{M}_2 + 0.072306\dot{M}_3 \\
&-0.249451\dot{M}_4 = 0.
\end{aligned} \quad (91)$$

The determinant of the main matrix of the above homogeneous system of equations is equal to zero and therefore there are non-zero solutions of this system of equations that are ambiguous and dependent on one or more parameters. Thus, we may fix some of them and determine the remaining ones. To show the way of solving this system of equations, we may suppose that we are arbitrarily interested in fixing the value of $\dot{M}_4$ and we put $\dot{M}_4 = 400.$ Further, substituting this value into the system of equations (91), we get

$$\begin{aligned}
&-0.153164\dot{M}_1 + 0.059319\dot{M}_2 + 0.082134\dot{M}_3 \\
&= -50.6844 \\
&0.02242\dot{M}_1 - 0.152945\dot{M}_2 + 0.02223\dot{M}_3 \\
&= -13.7180 \\
&0.05782\dot{M}_1 + 0.041405\dot{M}_2 - 0.17667\dot{M}_3 \\
&= -35.3780 \\
&0.072924\dot{M}_1 + 0.052221\dot{M}_2 + 0.072306\dot{M}_3 \\
&= 99.7804
\end{aligned}$$

and we solve it with respect to $\dot{M}_1$, $\dot{M}_2$ and $\dot{M}_3$, after omitting its last equation. This way obtained solutions that satisfy (91), are

$$\begin{aligned}
&\dot{M}_1 \cong 689,\ \dot{M}_2 \cong 261, \\
&\dot{M}_3 \cong 487,\ \dot{M}_4 = 400.
\end{aligned} \quad (92)$$

It can be seen that these solution differ much from the values M_1, M_2, M_3 and M_4 estimated in Section 2 by (8).

Having these solutions, it is also possible to look for the optimal values $\dot{M}_{bl}$ of the mean values M_{bl} of the exemplary system operation process conditional sojourn times at operation states. Namely, substituting the values $\dot{M}_b$ instead of M_b, the probabilities p_{bl} of the exemplary system operation process transitions between the operation states given in the matrix $[p_{bl}]$ defined by (6) and replacing M_{bl} by $\dot{M}_{bl}$ in (69), we get the following system of equations

$$\begin{aligned}
&0.22\dot{M}_{12} + 0.32\dot{M}_{13} + 0.46\dot{M}_{14} = 689 \\
&0.20\dot{M}_{21} + 0.30\dot{M}_{23} + 0.50\dot{M}_{24} = 261 \\
&0.12\dot{M}_{31} + 0.16\dot{M}_{32} + 0.72\dot{M}_{34} = 487 \\
&0.48\dot{M}_{41} + 0.22\dot{M}_{42} + 0.30\dot{M}_{14} = 400
\end{aligned}$$

with the unknown optimal values $\dot{M}_{bl}$ we want to find.

As the solutions of the above system of equations are ambiguous, then we fix some of them, say that because of practically important reasons, and we find the remaining ones. For instance:

- we fix in the first equation $\dot{M}_{12} = 200,\ \dot{M}_{13} = 500$ and we find $\dot{M}_{14} \cong 1054;$
- we fix in the second equation $\dot{M}_{21} = 100,$ $\dot{M}_{23} = 100$ and we find $\dot{M}_{24} \cong 422;$
- we fix in the third equation $\dot{M}_{31} = 900,\ \dot{M}_{32} = 500$ and we find $\dot{M}_{34} \cong 415;$
- we fix in the fourth equation $\dot{M}_{41} = 300,\ \dot{M}_{42} = 500$ and we find $\dot{M}_{43} \cong 487.$ (93)

It can be seen that these solutions differ greatly from the mean values of the exemplary system conditional sojourn times at the particular operation states before its operation process optimization given by (7).

Another very useful set of tools, which are much more easily applied in practice and which can help in planning the operation process of the system are the system operation process optimal mean values of the total sojourn times at the particular operation states during the system operation time that by the same assumpion as in Section 2 is equel to $\theta = 1$ year = 365 days. Under this assumption, after aplying (70), we get the optimal values of the exemplary system operation process total sojourn times at the particular operation states during 1 year

$$\dot{\hat{M}}_1 = \dot{E}[\hat{\theta}_1] = \dot{p}_1\theta = 0.341 \cdot 365 \cong 124.5,$$
$$\dot{\hat{M}}_2 = \dot{E}[\hat{\theta}_2] = \dot{p}_2\theta = 0.105 \cdot 365 \cong 38.3,$$
$$\dot{\hat{M}}_3 = \dot{E}[\hat{\theta}_3] = \dot{p}_3\theta = 0.245 \cdot 365 \cong 89.4,$$
$$\dot{\hat{M}}_4 = \dot{E}[\hat{\theta}_4] = \dot{p}_4\theta = 0.309 \cdot 365 \cong 112.8 \text{ days}, \quad (94)$$

that differ much from the values of $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$, $\hat{M}_4$, determined by (11).

In practice, the knowledge of the optimal values of $\dot{M}_b$, $\dot{M}_{bl}$, $\dot{\hat{M}}_b$, given respectively by (92), (93), (94), can be very important and helpful for the system operation process planning and improving in order to make the system operation more reliable.

The comparison of the values of the exemplary system reliability characteristics before the system operation process optimization given by (42)-(43) and (45) with their values after the system operation process optimization respectively given by (83)-(86) and (90) justifies the sensibility of the performed system operation process optimization.

From the analysis of the results of the exemplary system operation process optimization it can be suggested to organize the system operation process in the way that causes the replacing (or the approaching/convergence to) the conditional mean sojourn times M_{bl} of the system at the particular operation states before the optimization given by (7) by their optimal values $\dot{M}_{bl}$ after the optimization given by (93). However, the suggested change of the parameters of the system operation process very often is not easy to perform in practice.

An easier way might be to change the operation process characteristics that results in replacing (or the approaching/convergence to) the unconditional mean sojourn times M_b of the system at the particular operation states before the optimization given by (8) by their optimal values $\dot{M}_b$ after the optimization given by (92).

Practically, the easiest way of the system operation process reorganizing might be to replace (or to approach/converge to) the total sojourn times $\hat{M}_b$ of the system operation process at the particular operation states during the operation time $\theta = 1$ year before the optimization given by (11) by their optimal values $\dot{\hat{M}}_b$ after the optimization given by (94).

The evaluation and optimization of the exemplary system operating at the varying operation conditions reliability are based on the arbitrary assumed input data. Therefore, the achieved results may only be considered as an illustration of the possibilities of applications of the proposed methods and procedures to this system operation and reliability analysis, prediction and optimization. However, the obtained evaluation may be a very useful example in real complex technical systems reliability optimization, especially during the design and when planning and improving the effectiveness of their operation processes.

6. Conclusion

The constructed general model of complex systems' reliability, linking their reliability models and their operation processes models and considering variable at different operation states their reliability structures and their components reliability parameters was applied to the reliability evaluation of the exemplary system composed of a series-parallel and a series-"*m* out of *l*" subsystems linked in series. Next, the results of this model and the linear programming were applied to the optimization reliability and operation process of the considered exemplary system.

Presented in this paper tool is useful in reliability and operation optimization of a very wide class of real technical systems operating at the varying conditions that have an influence on changing their reliability structures and their components reliability parameters. The results can be interesting for reliability practitioners from various industrial sectors.

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