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UPPER BOUNDS ON DISTANCE VERTEX IRREGULARITY STRENGTH OF SOME FAMILIES OF GRAPHS

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Abstract. For a graph G its distance vertex irregularity strength is the smallest integer k for which one can find a labeling $f : V(G) \to \{1, 2, \dots, k\}$ such that

$$\sum_{x \in N(v)} f(x) \neq \sum_{x \in N(u)} f(x)$$

for all vertices u, v of G, where $N(v)$ is the open neighborhood of v. In this paper we present some upper bounds on distance vertex irregularity strength of general graphs. Moreover, we give upper bounds on distance vertex irregularity strength of hypercubes and trees.

Keywords: distance vertex irregularity strength of a graph, hypercube, tree.

Mathematics Subject Classification: 05C65, 05C78.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. It is a well-known fact that in any simple graph G there are at least two vertices of the same degree. The situation changes if we consider multigraphs. Each multiple edge may be represented with some integer label and the (*weighted*) degree of any vertex x is then calculated as the sum of labels over all the edges incident to x. An assignment of positive integers weights to the edges of a simple graph G is called *irregular* if the weighted degrees of the vertices are different. The irregularity strength $s(G)$ is the maximal weight, minimized over all irregular assignments. One of the important questions in graph theory is the problem of finding $s(G)$, which was introduced by Chartrand *et al.* in [6]. In this paper we study the vertex version of this problem.

By a *k-coloring* of G we mean any function f from the set of vertices $V(G)$ to the set $\{1, 2, \dots, k\}$. Given a coloring f, consider the induced function w_f called the *weight* on the set $V(G)$ defined by the formula

$$w_f(v) = \sum_{x \in N(v)} f(x),$$

where $N(v)$ is the open neighborhood of v. The initial coloring f is called an *additive k-coloring* of G if $w_f(u) \neq w_f(v)$ for every pair of adjacent vertices u and v. The minimum number k for which there exists an additive coloring of G is denoted by $\eta(G)$. The notion of additive coloring (at first called *lucky labeling*) was introduced in [3, 9] as a vertex version of the 1-2-3-conjecture of Karoński *et al.* [12].

On the other hand Miller *et al.* [13] defined a *distance magic labeling* of a graph G of order n as a bijective n-coloring ℓ so that there exists a positive integer μ such that the *weight* $w_\ell(v) = \mu$ for all $v \in V(G)$, whereas a *distance antimagic labeling* of G is such bijective n-coloring of G that different vertices have distinct weights. A graph that admits a distance magic (antimagic) labeling is called a *distance magic (antimagic) graph*.

Slamin [14], inspired by distance magic labeling [13] and irregular labeling [6], introduced the concept of a distance vertex irregular labeling of graphs. A k-coloring is called a *distance vertex irregular labeling* if the set of vertex weights consists of distinct numbers.

The minimum k for which a graph G has a distance vertex irregular labeling is called the *distance vertex irregularity strength* of G, denoted by $\mathrm{dis}(G)$. If such k does not exist we say that $\mathrm{dis}(G) = \infty$.

In [15] a lower bound of distance irregularity strength of graphs was introduced and provided its sharpness for some graphs with pendant vertices.

Theorem 1.1 ([15])**.** *Let G be a graph with maximum degree Δ and minimum degree δ. Let n_i be the number of vertices of degree i in G for every $i = \delta, \delta + 1, \dots, \Delta$. Then*

$$\mathrm{dis}(G) \geq \max_{\delta \leq i \leq \Delta} \left\{ \left\lceil \frac{\delta + \sum_{j=\delta}^{i} n_j - 1}{i} \right\rceil \right\}.$$

Bong *et al.* [5] generalized the concept to inclusive vertex irregular labeling. In inclusive labeling, the label of each vertex is included in it weights.

So far the exact value of distance vertex irregularity strength has been found only for a few classes of graphs [4, 14, 15].

In this paper we will present several upper bounds for $\mathrm{dis}(G)$. We start on bounds for general graphs. Later, we present the bound for hypercubes and in the last section for trees.

2. GENERAL UPPER BOUNDS ON dis(G)

We start this section with a result analogous to one presented in [2] for inclusive distance vertex irregularity strength. The following theorem gives a sufficient condition for $\mathrm{dis}(G) = \infty$. In this case the graph G is not necessarily connected.

Theorem 2.1. *For a graph G, if there exist two distinct vertices $u, v \in V(G)$ such that $N_G(u) = N_G(v)$, then* $\mathrm{dis}(G) = \infty$.

Proof. Let us consider that in G there exist two distinct vertices $u, v \in V(G)$ such that $N_G(u) = N_G(v)$. Let f be any vertex labeling of G, $f : V(G) \to \{1, 2, \dots, k\}$. For the weights of vertices u and v under the vertex labeling f we get

$$w_f(u) = \sum_{x \in N_G(u)} f(x) = \sum_{x \in N_G(v)} f(x) = w_f(v).$$

Thus we get $w_f(u) = w_f(v)$ which evidently means that f can not be an vertex irregular distance labeling. □

Now, let us suppose that for all vertices $u, v \in V(G)$ it holds that $N_G(u) \neq N_G(v)$. Let $S = \{1, 2, \dots, \Delta(G)(|V(G)| - 1) + 1\}$. Below we present a construction of a vertex irregular distance labeling $f : V(G) \to S$, analogous to that in [8] for inclusive distance vertex irregularity strength.

Theorem 2.2. *If $N_G(u) \neq N_G(v)$ for every two distinct vertices $u, v \in V(G)$, then* $\mathrm{dis}(G) \leq \Delta(G)(|V(G)| - 1) + 1$.

Proof. Let $v_1, v_2, \dots, v_n$ for $n = |V(G)|$ be vertices in G. We start with putting 1 on v_1 and 0 on all of remaining vertices. After that all of its neighbours have temporary weights 1. Then we will label the remaining vertices of G with elements of S in $n - 1$ stages. Let V_i denote the set of vertices with non-zero (temporary) weights in G after the stage i and let $G_i = G[V_i]$ for $i \in \{1, 2, \dots n\}$. In the stage $i + 1$ we will change the label for v_{i+1} avoiding sum conflicts between vertices belonging to the set $V_i \setminus N_G(v_{i+1})$ and vertices from $N_G(v_{i+1})$ for $i < n$.

By $w(v_t)$, for each $t \in \{1, 2, \dots, n\}$, we mean the temporary weight at v_t, obeying the following rule:

$$\begin{aligned} &\text{for every } j\text{: } w(v_j) \neq w(v_t) \text{ unless} \\ &N_G(v_j) \cap \{v_1, \dots, v_j\} = N_G(v_t) \cap \{v_1, \dots, v_j\}, \text{ where } v_j, v_t \in V_i. \end{aligned} \tag{2.1}$$

Note that if (2.1) holds after every stage, then the labeling f of G obtained at the end of our construction will be vertex irregular distance vertex labeling, as desired. Then the fact that $w(v_j) \neq w(v_t)$ for every $v_j \neq v_t$ in $V(G)$ follows directly from (1°), as $G_n = G$.

So assume we are about to perform step i of the construction for some $i \in \{2, 3, \dots, n\}$ and thus far all our requirements have been fulfilled. Let $v_{i_1}, v_{i_2}, \dots, v_{i_b}$ be all neighbors of v_i (hence $b \leq \Delta(G)$), and set $b = 0$ if there are none. If $b > 0$, we choose a label for v_i consistently with our rule. Since we are changing the weights of vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_b}$ for each vertex, we are not allowed to use at most $|V_{i-1}|$ labels to avoid forbidden sum-conflicts. As we have used thus altogether at most $\Delta(G)(|V(G)|-1)$ forbidden labels, we are left with at least one available option in S to label v_i. Therefore, after step n of the construction we obtain a desired vertex labeling f of G. □

3. CARTESIAN PRODUCT $G\Box K_2$

We use the definition of Cartesian product given in [11]. Given two graphs G and H, the *Cartesian product* of G and H, denoted $G\Box H$, is the graph with the vertex set $V(G)\times V(H)$, where two vertices (g,h) and (g',h') are adjacent if and only if $g=g'$ and h is adjacent to h' in H, or $h=h'$ and g is adjacent to g' in G.

We will show some upper bound on distance vertex irregularity strength for Cartesian product $G\Box K_2$. Recall that for a graph G of order n, a bijection $\ell'\colon V(G)\to\{1,2,\ldots,n\}$ such that $w_{\ell'}(u)\neq w_{\ell'}(v)$ for every pair of vertices $u,v\in V(G)$ is called the distance antimagic labeling.

Theorem 3.1. *Let $d>2$ and G be a d-regular graph of order n such that there exists a distance antimagic labeling ℓ' of G. Let $M=\max\{0,n-\min_{v\in V(G)} w_{\ell'}(v)\}$, then* $\operatorname{dis}(G\Box K_2)\leq n+2\left\lceil\frac{M}{d}\right\rceil+2$.

Proof. Let $V(K_2)=\{u_1,u_2\}$ and $v\in V(G)$. Define

$$f(v,u)=\begin{cases}1 & if\ \ u=u_1,\\ \ell'(v)+2\left\lceil\frac{M}{d}\right\rceil+2 & if\ \ u=u_2.\end{cases}$$

Let $x=(v,u)$. Thus for $u=u_1$ we have

$$\begin{aligned}w_f(x)=\sum_{y\in N_{G\Box K_2}(x)} f(y)&=d+\ell'(v)+2\left\lceil\frac{M}{d}\right\rceil+2\\ &\leq d+\ell'(v)+2\left\lceil\frac{M}{2}\right\rceil+2\leq d+n+M+3.\end{aligned}$$

Whereas for $u=u_2$ there is

$$\begin{aligned}w_f(x)&=\sum_{y\in N_{G\Box K_2}(x)} f(y)=\sum_{z\in N_G(v)}\ell'(z)+\left(2\left\lceil\frac{M}{d}\right\rceil+2\right)d+1\\ &\geq \min_{v\in V(G)} w_{\ell'}(v)+2M+2d+1\geq n+M+2d+1\geq n+M+d+4.\end{aligned}$$

Therefore $w_f(a)\neq w_f(b)$ for $a\neq b$, $a,b\in V(G\Box K_2)$. □

The n-dimensional hypercube, denoted $\mathcal{Q}_n$, is a graph the vertices of which are binary n-tuples and two of them are adjacent if and only if the corresponding n-tuples differ precisely at one position. The hypercube $\mathcal{Q}_n$ can be also defined recursively in terms of the Cartesian product of two graphs as follows: $\mathcal{Q}_1=K_2$, $\mathcal{Q}_n=\mathcal{Q}_{n-1}\Box K_2$ for $n\geq 2$.

Recall that vertices of $\mathcal{Q}_n$ can be written in the n-dimensional vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$, i.e., in their binary representation. Let $\oplus$ denote the addition in $\mathbb{F}_2^n$ and let $e_1, \dots, e_n$ be the standard basis of $\mathbb{F}_2^n$. By $\tilde{v}$ we denote the *dual* string of $v \in \mathbb{F}_2^n$, that is, the string $v \oplus (1, 1, \dots, 1)$ (i.e., 0 in v becomes 1 in $\tilde{v}$ and 1 in v becomes 0 in $\tilde{v}$), thus $\tilde{\tilde{v}} = v$. For a vector $a = (a_0, \dots, a_{n-1}) \in \mathbb{F}_2^n$, we denote by $\zeta(a)$ the element of $\mathbb{R}$ such that

$$\zeta(a) = \sum_{i=0}^{n-1} a_i 2^i.$$

Obviously ζ is a bijection. Observe that for any $v \in \mathbb{F}_2^n$ we have $\zeta(v) + \zeta(\tilde{v}) = 2^n - 1$.

For $v \in \mathbb{F}_2^n$, we define

$$N(v) = \{v \oplus e_i \mid i \in \{1, \dots, n\}\}.$$

Gregor and Kovář proved the following theorem:

Theorem 3.2 ([10]). *Let $n \equiv 2 \pmod 4$, then there exists a distance magic labeling ℓ of a hypercube $\mathcal{Q}_n$.*

For a finite Abelian group Γ a Γ-distance antimagic labeling was defined in [7], where instead of numbers we are using elements of Γ. Taking similar arguments as in [1], we show that for $n \geq 3$ the graph $\mathcal{Q}_n$ is distance antimagic.

Theorem 3.3. *Let $n \geq 3$, then there exists a distance antimagic labeling ℓ' of hypercube $\mathcal{Q}_n$ such that $w_{\ell'}(v) = (n-2) \cdot \zeta(v) + 2^n - 1 + n$ for each $v \in V(\mathcal{Q}_n)$.*

Proof. Let $\ell' \colon V(\mathcal{Q}_n) \to \{1, 2, \dots, 2^n\}$ for $v = (v_0, \dots, v_{n-1}) \in \mathbb{F}_2^n$ be defined as $\ell'(v) = \zeta(v) + 1$. Note that

$$\begin{aligned} \sum_{i=1}^{n} v \oplus e_i &= (n \cdot v) \oplus (1, 1, \dots, 1) \\ &= ((n-1) \cdot v) \oplus v \oplus (1, 1, \dots, 1) \\ &= ((n-1) \cdot v) \oplus \tilde{v}. \end{aligned}$$

Moreover, $\zeta(((n-1) \cdot v) \oplus \tilde{v}) = (n-1) \cdot \zeta(v) + \zeta(\tilde{v})$. Hence for $v \in \mathbb{F}_2^n$, we obtain

$$\begin{aligned} w_{\ell'}(v) &= \sum_{u \in N(v)} \ell'(u) = \sum_{u \in N(v)} \zeta(u) + n \\ &= \sum_{i=1}^{n} \zeta(v \oplus e_i) + n \\ &= (n-1) \cdot \zeta(v) + \zeta(\tilde{v}) + n \\ &= (n-2) \cdot \zeta(v) + \zeta(v) + \zeta(\tilde{v}) + n \\ &= (n-2) \cdot \zeta(v) + 2^n - 1 + n. \end{aligned}$$ □

By Theorem 1.1 $\operatorname{dis}(\mathcal{Q}_n) \geq \frac{2^n-1}{n} + 1$. Using Theorems 3.1 and 3.3 we obtain the following upper bound.

Corollary 3.4. $\operatorname{dis}(\mathcal{Q}_n) \leq 2^{n-1} + 2$ *for* $n \geq 4$.

Proof. There exists a distance antimagic labeling ℓ' of $\mathcal{Q}_{n-1}$ for $n > 3$ by Theorem 3.3 such that

$$\min_{v \in V(\mathcal{Q}_{n-1})} w_{\ell'}(v) = 2^{n-1} + n - 2 \geq 2^{n-1}.$$

Thus

$$M = \max \left\{0, 2^{n-1} - \min_{v \in V(\mathcal{Q}_{n-1})} w_{\ell'}(v)\right\} = 0.$$

Therefore $\operatorname{dis}(\mathcal{Q}_n) \leq 2^{n-1} + 2$ by Theorem 3.1. □

We will show that this bound can be improved for n large enough. Before doing that we need to give some terminology. We say that a graph G has a $(2, k)$-*partition* if there exists a partition of the set $V(G)$ into V_1, V_2 (that is, $V(G) = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$) such that for every $x \in V_1$ and for every $y \in V_2$

$$|N(x) \cap V_2| = |N(y) \cap V_1| = k.$$

Lemma 3.5. *For any positive integer* $k < n$*, hypercube* $\mathcal{Q}_n$ *has a* $(2, k)$*-partition.*

Proof. Recall that the set of vertices of $\mathcal{Q}_n$ is the set of binary strings of length n, that is, $V(\mathcal{Q}_n) = \{(v_1, v_2, \ldots, v_n),\ v_i \in \{0, 1\}\}$. Two vertices are adjacent if and only if the corresponding strings differ at exactly one position. For positive $k < n$, let

$$V_1 = \{(v_1, \ldots, v_n) : v_1 + \ldots + v_k \equiv 0 \pmod 2\},$$
$$V_2 = \{(v_1, \ldots, v_n) : v_1 + \ldots + v_k \equiv 1 \pmod 2\}.$$

Notice that $|N(x) \cap V_2| = |N(y) \cap V_1| = k$ for every $x \in V_1$ and for every $y \in V_2$. □

Theorem 3.6. *Let* $G = \mathcal{Q}_n$*, for* $n \geq 2$*. If there exist* $d \equiv 2 \pmod 4$, $k > 0$ *such that* $k \leq \frac{d}{2}$, $\gcd(n - d, n - 2k - 2) = h$ *and* $h \cdot 2^d + 2 \leq n - 2k$*, then*

$$\operatorname{dis}(\mathcal{Q}_n) \leq 2^{n-d} + 2^d - 1.$$

Proof. Let V_1, V_2 be a $(2, k)$-partition for $\mathcal{Q}_d$ which exists by Lemma 3.5.

Let ℓ be the distance magic labeling of $\mathcal{Q}_d$ defined in Theorem 3.2. Denote by μ the weight $w_\ell(u)$ for any $u \in V(\mathcal{Q}_d)$.

Let $v = (v_0, \ldots, v_{n-1}) \in V(\mathcal{Q}_n)$ and for the graph $\mathcal{Q}_n$ set

$$f(v) = \begin{cases} \zeta(v_0, \ldots, v_{n-d-1}) + \ell(v_{n-d}, \ldots, v_{n-1}) & if \ \ (v_{n-d}, \ldots, v_{n-1}) \in V_1, \\ \zeta(\widetilde{v_0, \ldots, v_{n-d-1}}) + \ell(v_{n-d}, \ldots, v_{n-1}) & if \ \ (v_{n-d}, \ldots, v_{n-1}) \in V_2. \end{cases}$$

First we observe that

$$\zeta(x_0,\dots,x_{n-d-1}) = (n-d-2)\zeta(v_0,\dots,v_{n-d-1}) + 2^{n-d} - 1, \tag{3.1}$$

$$\begin{aligned} &k\zeta(\widetilde{v_0,\dots,v_{n-d-1}}) + (d-k)\zeta(v_0,\dots,v_{n-d-1}) \\ &= (d-2k)\zeta(v_0,\dots,v_{n-d-1}) + k(2^{n-d}-1) \end{aligned} \tag{3.2}$$

and

$$\sum_{(y_{n-d},\dots,y_{n-1})\in N_{\mathcal{Q}_d}(v_{n-d},\dots,v_{n-1})} \ell(y_{n-d},\dots,y_{n-1}) = \mu. \tag{3.3}$$

Then

$$\begin{aligned} w_f(v) &= \sum_{t\in N(v)} f(t) \\ &= \sum_{(x_0,\dots,x_{n-d-1})\in N_{\mathcal{Q}_{n-d}}(v_0,\dots,v_{n-d-1})} f(x_0,\dots,x_{n-d-1},v_{n-d},\dots,v_{n-1}) \\ &\quad+ \sum_{(y_{n-d},\dots,y_{n-1})\in N_{\mathcal{Q}_d}(v_{n-d},\dots,v_{n-1})} f(v_0,\dots,v_{n-d-1},y_{n-d},\dots,y_{n-1}) \\ &= \sum_{(x_0,\dots,x_{n-d-1})\in N_{\mathcal{Q}_{n-d}}(v_0,\dots,v_{n-d-1})} \zeta(x_0,\dots,x_{n-d-1}) + (n-d)\ell(v_{n-d},\dots,v_{n-1}) \\ &\quad+ k\cdot\zeta(\widetilde{v_0,\dots,v_{n-d-1}}) + (d-k)\zeta(v_0,\dots,v_{n-d-1}) \\ &\quad+ \sum_{(y_{n-d},\dots,y_{n-1})\in N_{\mathcal{Q}_d}(v_{n-d},\dots,v_{n-1})} \ell(y_{n-d},\dots,y_{n-1}) \end{aligned}$$

By substituting (3.1), (3.2) and (3.3) we obtain

$$\begin{aligned} w_f(v) =& (n-d-2)\zeta(v_0,\dots,v_{n-d-1}) + 2^{n-d} - 1 + (n-d)\ell(v_{n-d},\dots,v_{n-1}) \\ &+ (d-2k)\zeta(v_0,\dots,v_{n-d-1}) + k\cdot(2^{n-d}-1) + \mu \\ =& (n-2k-2)\zeta(v_0,\dots,v_{n-d-1}) + (k+1)\cdot(2^{n-d}-1) + \mu \\ &+ (n-d)\ell(v_{n-d},\dots,v_{n-1}). \end{aligned}$$

Similarly for $(v_{n-d},\dots,v_{n-1}) \in V_2$ we have

$$\begin{aligned} w_f(v) &= \sum_{t\in N(v)} f(t) \\ &= \sum_{(x_0,\dots,x_{n-d-1})\in N_{\mathcal{Q}_{n-d}}(v_0,\dots,v_{n-d-1})} f(x_0,\dots,x_{n-d-1},v_{n-d},\dots,v_{n-1}) \\ &\quad+ \sum_{(y_{n-d},\dots,y_{n-1})\in N_{\mathcal{Q}_d}(v_{n-d},\dots,v_{n-1})} f(v_0,\dots,v_{n-d-1},y_{n-d},\dots,y_{n-1}) \end{aligned}$$

$$
\begin{aligned}
=& \sum_{(x_0,\ldots,x_{n-d-1})\in N_{\mathcal{Q}_{n-d}}(v_0,\ldots,v_{n-d-1})} \zeta(\widetilde{x_0,\ldots,x_{n-d-1}})\\
&+ (n-d)\ell(v_{n-d},\ldots,v_{n-1})\\
&+ k\cdot\zeta(v_0,\ldots,v_{n-d-1}) + (d-k)\zeta(\widetilde{v_0,\ldots,v_{n-d-1}})\\
&+ \sum_{(y_{n-d},\ldots,y_{n-1})\in N_{\mathcal{Q}_d}(v_{n-d},\ldots,v_{n-1})} \ell(y_{n-d},\ldots,y_{n-1})\\
=&(n-d-2)\zeta(\widetilde{v_0,\ldots,v_{n-d-1}}) + 2^{n-d} - 1 + (n-d)\ell(v_{n-d},\ldots,v_{n-1})\\
&+ (d-2k)\zeta(\widetilde{v_0,\ldots,v_{n-d-1}}) + k\cdot(2^{n-d}-1) + \mu\\
=&(n-2k-2)\zeta(\widetilde{v_0,\ldots,v_{n-d-1}}) + (k+1)\cdot(2^{n-d}-1) + \mu\\
&+ (n-d)\ell(v_{n-d},\ldots,v_{n-1}).
\end{aligned}
$$

Let $u=(u_0,\ldots,u_{n-1})$, $v=(v_0,\ldots,v_{n-1})\in V(\mathcal{Q}_n)$, we will show that $w_f(u)\neq w_f(v)$. Suppose first that $(u_{n-d},\ldots,u_{n-1})=(v_{n-d},\ldots,v_{n-1})$, then $w_f(u)\neq w_f(v)$ because ζ is a bijection. Assume now that $(u_{n-d},\ldots,u_{n-1})\neq(v_{n-d},\ldots,v_{n-1})$. Since $\gcd(n-d,n-2k-2)=h$, the group generated by $(n-d)\in\mathbb{Z}_{n-2k-2}$ is isomorphic to $\mathbb{Z}_{(n-2k-2)/h}$ (i.e., $\langle n-d\rangle\cong\mathbb{Z}_{(n-2k-2)/h}$), therefore because

$$\ell(v_{n-d},\ldots,v_{n-1})\leq 2^d-1<(n-2k-2)/h$$

and ℓ is a bijection, we obtain $w_f(u)\not\equiv w_f(v) \pmod{(n-2k-2)}$. □

As a corollary we obtain the following.

Corollary 3.7. *Let $d\equiv 2 \pmod 4$ and $n\geq 2^d+d+2$ for odd n or $n\geq 2^{d+1}+d+2$ for even n, then*

$$\operatorname{dis}(\mathcal{Q}_n)\leq 2^{n-d}+2^d-1.$$

Proof. Let $d\equiv 2 \pmod 4$ and $k=d/2$. Then $k>0$ and

$$\gcd(n-d,n-2k-2)=\gcd(n-d,n-d-2)=1$$

for odd n and

$$\gcd(n-d,n-2k-2)=\gcd(n-d,n-d-2)=2$$

for even n. Applying Theorem 3.6 we are done. □

4. TREES

In the case of trees are able to give some upper bound.

Theorem 4.1. *Let T be a tree of order $n\geq 4$ such that $N_T(u)\neq N_T(v)$ for any distinct vertices $u,v\in V(T)$. Then* $\operatorname{dis}(T)\leq 2n-5$.

Proof. Note that if in a tree there exist vertices $u \neq v$ such that $N(u) = N(v)$, then because there is no cycle $\deg(u) = \deg(v)$. The proof will be by induction on n. For $n \in \{4, 5\}$, the desired labelings are given in Figure 1.

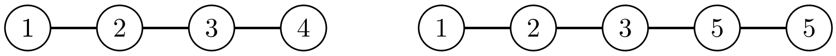

Fig. 1. Vertex irregular distance vertex labelings for paths P_4 and P_5

Assume $n \geq 6$. Observe that since $N_T(u) \neq N_T(v)$ for any distinct vertices $u, v \in V(T)$, there exists a vertex $v \in V(T)$ such that $\deg(v) = 1$ and for $u \in N_T(v)$ we have $\deg(u) = 2$. Let $T' = T - \{u, v\}$ and $a \neq v$ be the second neighbor of u. Assume first that $N_{T'}(x) \neq N_{T'}(y)$ for any $x, y \in V(T')$, then there exists a vertex irregular distance vertex labeling $f'\colon V(T') \to \{1, 2, \dots, 2n-9\}$. We will extend this labeling to a vertex irregular distance vertex labeling $f\colon V(T) \to \{1, 2, \dots, 2n-5\}$. Let $f(x) = f'(x)$ for any $x \in V(T')$. We will first label the vertex u. Note that we must have $w_f(a) = w_{f'}(a) + f(u) \neq w_f(x)$ and $w_f(v) = f(u) \neq w_f(x)$ for any $x \in V(T')$, $x \neq a$ therefore we are not allowed to use at most $2n-6$ labels from the set $\{1, 2, \dots, 2n-5\}$. Thus we can label the vertex u in such a way that $w_f(x) \neq w_f(y)$ for any distinct vertices $x, y \in V(T)$ and $u \notin \{x, y\}$. In the second step we label the vertex v, since we must have $w_f(u) \neq w_f(x)$ for any $x \in V(T)$, at most $n-1$ labels are prohibited.

If there exist $x, y \in V(T')$ such that $N_{T'}(x) = N_{T'}(y)$, then $a \in \{x, y\}$ and $\deg_{T'}(a) = 1 = \deg_T(a) - 1$. Let $b \neq u$ be the second neighbor of a. Observe that $\deg_T(b) \geq 3$ and $n \geq 7$. Let $T'' = T - \{u, v, a\}$. Notice that $N_{T''}(x) \neq N_{T''}(y)$ for any $x, y \in V(T'')$, therefore there exists a vertex irregular distance vertex labeling $f''\colon V(T'') \to \{1, 2, \dots, 2n-11\}$. We will extend this labeling to a vertex irregular distance vertex labeling $f\colon V(T) \to \{1, 2, \dots, 2n-5\}$. Let $f(x) = f''(x)$ for any $x \in V(T'')$. We will first label the vertex a. Note that we must have $w_f(b) = w_{f''}(b) + f(a) \neq w_f(x)$ for $x \in V(T'')$, $x \neq b$ therefore we are not allowed to use at most $n-4$ labels from the set $\{1, 2, \dots, 2n-5\}$. In the second step we label the vertex u. Since $w_f(a) = f(b) + f(u) \neq w_f(x)$ and $w_f(v) = f(u) \neq w_f(x)$ for any $x \in V(T'')$, at most $2n-6$ labels are prohibited. In the last step we label the vertex v, since $w_f(u) = f(v) + f(a) \neq w(x)$ for any $x \in V(T)$, at most $n-1$ labels are prohibited. □

Recently the following lower bound was given.

Theorem 4.2 ([15]). *Let T be a tree with the maximum degree Δ such that $N_T(u) \neq N_T(v)$ for any distinct vertices $u, v \in V(T)$. Let n_i be the number of vertices of degree i for every $i = 1, 2, \dots, \Delta$. Then* $\operatorname{dis}(T) \geq \max \left\lceil \frac{n_1+n_2}{2} \right\rceil$.

Therefore we believe that the upper bound can be improved and thus we state following conjecture.

Conjecture 4.3. *Let T be a tree of order n such that $N_T(u) \neq N_T(v)$ for any distinct vertices $u, v \in V(T)$. Then* $\operatorname{dis}(T) \leq n$.

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REFERENCES

[1] M. Anholcer, S. Cichacz, D. Froncek, R. Simanjuntak, J. Qiu, *Group distance magic and antimagic hypercubes*, Discrete Math. **344** (2021), 112625.

[2] M. Bača, A. Semaničová-Feňovčíková, Slamin, K.A. Sugeng, *On inclusive distance vertex irregular labelings*, Electron. J. Graph Theory Appl. **6** (2018), no. 1, 61–83.

[3] T. Bartnicki, B. Bosek, S. Czerwiński, J. Grytczuk, G. Matecki, W. Żelazny, *Additive coloring of planar graphs*, Graphs Combin. **30** (2014), 1087–1098.

[4] N.H. Bong, Y. Lin, Slamin, *On distance irregular labelings of cycles and wheels*, Australas. J. Comb. **69** (2017), no. 3, 315–322.

[5] N.H. Bong, Y. Lin, Slamin, *On inclusive and non-inclusive vertex irregular d-distance vertex labelings*, J. Combin. Math. Combin. Comput. **113** (2020), 233–247.

[6] G. Chartrand, M.S. Jacobson, J. Lehel, O. Oellermann, S. Ruiz, F. Saba, *Irregular networks*, Congr. Numer. **64** (1988), 187–192.

[7] S. Cichacz, D. Froncek, K. Sugeng, S. Zhou, *Group distance magic and antimagic graphs*, Acta Math. Sin. (Engl. Ser.) **32** (2016), 1159–1176.

[8] S. Cichacz, A. Görlich, A. Semaničová-Feňovčíková, *Upper bounds on inclusive distance vertex irregularity strength*, Graphs Combin. **37** (2021), 2713–2721.

[9] S. Czerwiński, J. Grytczuk, W. Żelazny, *Lucky labelings of graphs*, Inform. Process. Lett. **109** (2009), 1078–1081.

[10] P. Gregor, P. Kovář, *Distance magic labelings of hypercubes*, Electronic Notes in Discrete Math. **40** (2013), 145–149.

[11] R. Hammack, W. Imrich, S. Klavžar, *Handbook of product graphs*, 2nd ed., Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2011, with a foreword by Peter Winkler.

[12] M. Karoński, T. Łuczak, A. Thomason, *Edge weights and vertex colours*, J. Comb. Theory Ser. B **91** (2004), 151–157.

[13] M. Miller, C. Rodger, R. Simanjuntak, *Distance magic labelings of graphs*, Australas. J. Comb. **28** (2003), 305–315.

[14] Slamin, *On distance irregular labelings of graphs,* Far East J. Math. Sci. **102** (2017), no. 5, 919–932.

[15] F. Susanto, K. Wijaya, Slamin, A. Semaničová-Feňovčíková, *Distance irregularity strength of graphs with pendant vertices*, Opuscula Math. **42** (2022), no. 3, 439–458.

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