Silesian J. Pure Appl. Math.
vol. 7, is. 1 (2017), 99–121

Barbara SMOLEŃ, Roman WITUŁA

Institute of Mathematics, Silesian University of Technology, Gliwice, Poland

TWO-PARAMETRIC QUASI-FIBONACCI NUMBERS

Abstract. This paper is devoted to the discussion on the two parametric quasi–Fibonacci numbers. The fundamental recurrence and reduction formulae for arguments and indices of these quasi–Fibonacci numbers are presented here. The matrix representations of the considered numbers are described and their applications are indicated. Moreover, a number of connections of the two parametric quasi-Fibonacci numbers with the sequences collected in the OEIS encyclopaedia are noted. Despite quite large volume of this elaboration, the Authors believe that this is just some kind of announcement, or an introduction to a definitely larger and detailed discussion including, above all, the applications of the investigated here numbers.

1. Introduction

Wituła and Słota in their papers [15, 12, 14] introduced the original systems of numbers called the quasi-Fibonacci numbers of n-th order. These numbers occurred in relation of a natural generalization of the following well known relations for the Fibonacci numbers

$$(1 + \eta^k + \eta^{4k})^n = F_{n+1} + F_n(\eta^k + \eta^{4k})$$

2010 Mathematics Subject Classification: 11B39, 11B83.
Keywords: two parametric quasi-Fibonacci numbers, δ-Fibonacci numbers.
Corresponding author: R. Wituła (Roman.Witula@polsl.pl).
Received: 10.10.2017.
While creating this paper B. Smoleń was a student of the master's degree study in mathematics at the Faculty of Applied Mathematics in Silesian University of Technology.

or

$$(1+\eta^{2k}+\eta^{3k})^n = F_{n+1} + F_n(\eta^{2k}+\eta^{3k}),$$

where $F_n,\ n \in \mathbb{N}_0$ denotes the n-th Fibonacci number, $k \in \mathbb{N} \setminus 5\mathbb{N}$ and $\eta \in \mathbb{C}$ is the primitive 5-th root of unity. Let us only notice that the above two relations are equivalent.

Considering the mentioned above systems of numbers, the most simple one is created by the δ-Fibonacci numbers discussed already in many works (see for example [3, 14, 9, 10, 11]).

Definition 1. *δ-Fibonacci numbers $a_n(\delta)$, $b_n(\delta)$ are defined by the following relations*

$$(1+\delta(\eta^k+\eta^{4k}))^n = a_n(\delta) + b_n(\delta)(\eta^k+\eta^{4k}), \tag{1}$$

where $k \in \mathbb{N} \setminus 5\mathbb{N},\ n \in \mathbb{N}_0,\ \delta \in \mathbb{C},\ \eta \in \mathbb{C}$ the primitive 5-th root of unity and $a_0(\delta) = 1,\ b_0(\delta) = 0$.

One of the most important properties of the δ-Fibonacci numbers is the fact that they represent the binomial transformation of the scaled Fibonacci numbers

$$a_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} F_{k-1}(-\delta)^k = \sum_{k=0}^{n} \binom{n}{k} F_{k+1}(1-\delta)^{n-k}\delta^k,$$
$$b_n(\delta) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} F_k \delta^k = \sum_{k=1}^{n} \binom{n}{k} F_k(1-\delta)^{n-k}\delta^k,$$

for $n \in \mathbb{N}_0$, which can be derived from Definition 1 as well as from the following recurrence relation for δ-Fibonacci numbers resulting easily from (1):

$$\begin{bmatrix} a_{n+1}(\delta) \\ b_{n+1}(\delta) \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ \delta & 1-\delta \end{bmatrix} \begin{bmatrix} a_n(\delta) \\ b_n(\delta) \end{bmatrix}, \quad n \in \mathbb{N}_0.$$

The following specific values of δ-Fibonacci numbers can be determined

$$\begin{aligned}
&a_n(1) = F_{n+1}, && b_n(1) = F_n,\\
&a_n(-1) = F_{2n-1}, && b_n(-1) = -F_{2n},\\
&a_n(2) = 5^{\lfloor n/2 \rfloor}, && b_n(2) = (1-(-1)^n)5^{\lfloor n/2 \rfloor},\\
&a_{2n+2}(-i) = (2i-1)^{n+1}F_n, && b_{2n}(-i) = (2i-1)^n F_n,
\end{aligned}$$

$$a_{n+3}(-i) = (2i-1)^{n+1}(F_n + iF_{n+1}), \quad b_{2n+1}(-i) = -(2i-1)^n(F_n + iF_{n+1}),$$
$$a_{2n}\left(\frac{1+i}{2}\right) = \frac{1}{5}(1+\frac{1}{2}i)^n L_{n+2}, \qquad b_{2n}\left(\frac{1+i}{2}\right) = (1+\frac{1}{2}i)^n F_n,$$
$$a_{2n+1}\left(\frac{1+i}{2}\right) = (1+\frac{1}{2}i)^n\left(\frac{1}{5}L_{n+2} + \frac{1+i}{2}F_n\right),$$
$$b_{2n+1}\left(\frac{1+i}{2}\right) = \frac{1}{10}(1+\frac{1}{2}i)^n(3L_{n+1} + iL_{n-2}),$$

where L_n denotes the n-th Lucas number ($L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$). Sequences $\{a_n(\delta)\}$ and $\{b_n(\delta)\}$ for the specific values of $\delta \in \mathbb{R}$ can be found in *Sloane's On-Line Encyclopedia of Integer Sequences OEIS*. For example, we have

$$\begin{array}{ll}
a_n(-3) = A188168(n+1), & \\
a_n(-2) = A015448(n), & b_n(-2) = A014445(n), \\
a_n(-1) = A001519(n), & b_n(-1) = -A001906(n), \\
a_n(1) = A000045(n+1), & b_n(1) = A000045(n), \\
a_n(2) = A074872(n), & b_{2n}(2) = A020699(n), \\
3^n a_n\left(\frac{1}{3}\right) = A081567(n), & 3^n b_n\left(\frac{1}{3}\right) = A030191(n), \\
4^n a_n\left(\frac{1}{4}\right) = A081568(n), & 4^n b_n\left(\frac{1}{4}\right) = A099453(n), \\
5^n a_n\left(\frac{1}{5}\right) = A081569(n), & 5^n b_n\left(\frac{1}{5}\right) = A081574(n), \\
5^n a_n\left(\frac{2}{5}\right) = A163073(n+1), & \\
7^n a_n\left(\frac{1}{7}\right) = A081571(n), & \\
7^n a_n\left(\frac{2}{7}\right) = A163306(n), &
\end{array}$$

where $n \in \mathbb{N}_0$. Of course, many other sequences of this type cannot be found in the OEIS, among others, the sequences of form: $\{a_n(-4)\}$, $\{b_n(-4)\}$, $\{b_n(-3)\}$, $\{a_n(3)\}$, $\{b_n(3)\}$, $\{a_n(4)\}$, $\{b_n(4)\}$ $\{a_n(5)\}$, $\{b_n(5)\}$ are not included there.

William Webb during the 13th International Conference on Fibonacci Numbers and Their Applications in Patras (July 2008) formulated the problem of finding the closed form of the following sums

$$\sum_{k=1}^{N} F_{k^r},$$

where r is a positive integer. The problem is possible to solve by using the δ-Fibonacci numbers which is shown in paper [14, Remark 6.5]. H. Prodinger in paper [5] considered the identities associated with the generating functions of the δ-Fibonacci numbers. However, the main goal of this paper is the discussion on the basic algebraic properties of two successive multi-parametric quasi-Fibonacci numbers (the multi-parametric quasi-Fibonacci numbers were defined for the first time in paper [12]).

2. Two-parametric quasi-Fibonacci numbers

It appears that it is proper to consider the quasi-Fibonacci numbers of any odd order n depending on $\frac{1}{2}\varphi(n) - 1$ parameters [12]. Obviously the symbol $\varphi(\cdot)$ denotes here the Euler function. Aim of this paper is to present exactly this multi-parameter generalization for two most simple cases of few parameters, that is the quasi-Fibonacci numbers of seventh and ninth order depending on two parameters. Let us notice that in papers [15, 12, 14, 13] only the one-parametric quasi-Fibonacci numbers are discussed, whereas the numbers of ninth order have not been yet investigated in any literature (however we suggest to see the paper [8]).

However, before presenting the appropriate definitions we discuss few important and well known facts which will be used further on in this paper.

Lemma 2. *If $\xi \in \mathbb{C}$ is the primitive 7-th root of unity, then every three numbers belonging to the set $\{1, \xi + \xi^6, \xi^2 + \xi^5, \xi^3 + \xi^4\}$ are linearly independent over $\mathbb{Q}$.*

The uncomplicated proof of this fact will be omitted here, but it can be found, for example, in paper [15] available online (see also [4]).

Remark 3. Analogical lemma can be proven for the set $\{1, \zeta + \zeta^8, \zeta^2 + \zeta^7, \zeta^3 + \zeta^6, \zeta^4 + \zeta^5\}$, where $\zeta \in \mathbb{C}$ the primitive 9-th root of unity.

Lemma 4. *Sum of all n-th complex roots of unity is equal to zero for each $n \in \mathbb{N}$, that is*

$$1 + \xi + \xi^2 + \cdots + \xi^{n-1} = 0,$$

where ξ is the primitive n-th root of unity.

Theorem 5. *Let $a_k \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$ and let $f_k, g_k \in \mathbb{Q}[\delta]$, $k = 1, 2, \ldots, n$. If for each $\delta \in \mathbb{Q}$ the equality*

$$\sum_{k=1}^{n} f_k(\delta) a_k = \sum_{k=1}^{n} g_k(\delta) a_k$$

holds true, then $f_k(\delta) = g_k(\delta)$ for each $\delta \in \mathbb{C}$ and for each $k = 1, 2, \ldots, n$.

Now we proceed to discussing the main subject of this paper, that is the definition and basic algebraic properties of two-parametric quasi-Fibonacci numbers of seventh and ninth order.

Definition 6. *Two-parametric quasi-Fibonacci numbers of seventh order are defined by means of the following relations*

$$\begin{aligned}(1+\delta(\xi^k+\xi^{6k})&+\lambda(\xi^{2k}+\xi^{5k}))^n\\&=A_{n,7}(\delta,\lambda)+B_{n,7}(\delta,\lambda)(\xi^k+\xi^{6k})+C_{n,7}(\delta,\lambda)(\xi^{2k}+\xi^{5k}),\end{aligned}\tag{2}$$

for $k\in\mathbb{N}\setminus 7\mathbb{N},\ n\in\mathbb{N}_0,$ *and* $\delta,\lambda\in\mathbb{C}$, *where* $\xi\in\mathbb{C}$ *is the primitive 7-th root of unity and* $A_{0,7}(\delta,\lambda):=1,\ B_{0,7}(\delta,\lambda):=0,\ C_{0,7}(\delta,\lambda):=0.$

Definition 7. *Two parametric quasi-Fibonacci numbers of ninth order are defined by means of the following relations*

$$\begin{aligned}(1+\delta(\zeta^k+\zeta^{8k})&+\lambda(\zeta^{2k}+\zeta^{7k}))^n\\&=A_{n,9}(\delta,\lambda)+B_{n,9}(\delta,\lambda)(\zeta^k+\zeta^{8k})+C_{n,9}(\delta,\lambda)(\zeta^{2k}+\zeta^{7k}),\end{aligned}\tag{3}$$

for $k\in\mathbb{N}\setminus 9\mathbb{N},\ n\in\mathbb{N}_0,\ \delta,\lambda\in\mathbb{C}$, *where* $\zeta\in\mathbb{C}$ *is the primitive 9-th root of unity and* $A_{0,9}(\delta,\lambda):=1,\ B_{0,9}(\delta,\lambda):=0,\ C_{0,9}(\delta,\lambda):=0.$

The special case of two-parametric quasi-Fibonacci numbers of order k, when $\lambda=0$, are the one-parametric quasi-Fibonacci numbers of order k:

$$\begin{aligned}A_{n,k}(\delta,0)&=A_{n,k}(\delta),\\B_{n,k}(\delta,0)&=B_{n,k}(\delta),\\C_{n,k}(\delta,0)&=C_{n,k}(\delta),\end{aligned}$$

where $k\in\{7,9\}$ and $n\in\mathbb{N}_0$. Some of the sequences connected to the one-parametric quasi-Fibonacci numbers of seventh order for the specific values of δ are included in OEIS, for example one can find there the sequences

$$\begin{aligned}A_{n,7}(2)&=A121442(n),\\B_{n,7}(2)&=A271944(n),\\C_{n,7}(2)&=A271945(n),\\A_{n,7}(1)&=A77998(n),\\B_{n,7}(1)&=A006054(n+1),\end{aligned}$$

$$A_{n,7}(1) + C_{n,7}(1) = (-1)^{n+1} A215112(n+2),$$
$$A_{n,7}(1) - B_{n,7}(1) = A106803(n),$$

where $n \in \mathbb{N}_0$.

Analogically like for the δ-Fibonacci numbers, the recurrence relations hold also for the defined above two-parametric quasi-Fibonacci numbers.

Theorem 8. *a) For two-parametric quasi-Fibonacci numbers of seventh order the following relations are satisfied*

$$\begin{bmatrix} A_{n+1,7}(\delta,\lambda) \\ B_{n+1,7}(\delta,\lambda) \\ C_{n+1,7}(\delta,\lambda) \end{bmatrix} = \begin{bmatrix} 1 & 2\delta-\lambda & \lambda-\delta \\ \delta & 1 & -\lambda \\ \lambda & \delta-\lambda & 1-\delta-\lambda \end{bmatrix} \begin{bmatrix} A_{n,7}(\delta,\lambda) \\ B_{n,7}(\delta,\lambda) \\ C_{n,7}(\delta,\lambda) \end{bmatrix} \tag{4}$$

for each $n \in \mathbb{N}_0$.
b) For two-parametric quasi-Fibonacci numbers of ninth order the following relations are satisfied

$$\begin{bmatrix} A_{n+1,9}(\delta,\lambda) \\ B_{n+1,9}(\delta,\lambda) \\ C_{n+1,9}(\delta,\lambda) \end{bmatrix} = \begin{bmatrix} 1 & 2\delta-\lambda & 2\lambda-\delta \\ \delta & 1+\lambda & \delta-\lambda \\ \lambda & \delta & 1-\lambda \end{bmatrix} \begin{bmatrix} A_{n,9}(\delta,\lambda) \\ B_{n,9}(\delta,\lambda) \\ C_{n,9}(\delta,\lambda) \end{bmatrix} \tag{5}$$

for each $n \in \mathbb{N}_0$.

Proof. We show the proof only for case *a)*. In case *b)* the proof runs analogically.

For $\delta, \lambda \in \mathbb{C}$ and $N \in \mathbb{N}$ we have

$$\begin{aligned}
&(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k}))^{N+1} \\
&\quad = \big(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\big)^N \big(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\big) \\
&\quad = \big[A_{N,7}(\delta,\lambda)+B_{N,7}(\delta,\lambda)(\xi^k+\xi^{6k})+C_{N,7}(\delta,\lambda)(\xi^{2k}+\xi^{5k})\big] \\
&\qquad \times \big(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\big) \\
&\quad = A_{N,7}(\delta,\lambda)+(2\delta-\lambda)B_{N,7}(\delta,\lambda)+(\lambda-\delta)C_{N,7}(\delta,\lambda) \\
&\qquad + (\xi^k+\xi^{6k})[\delta A_{N,7}(\delta,\lambda)+B_{N,7}(\delta,\lambda)-\lambda C_{N,7}(\delta,\lambda)] \\
&\qquad + (\xi^{2k}+\xi^{5k})[\lambda A_{N,7}(\delta,\lambda)+(\delta-\lambda)B_{N,7}(\delta,\lambda)+(1-\delta-\lambda)C_{N,7}(\delta,\lambda)].
\end{aligned}$$

Next, directly from the definition we get

$$\begin{aligned}(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k}))^{N+1}\\ =A_{N+1,7}(\delta,\lambda)+B_{N+1,7}(\delta,\lambda)(\xi^k+\xi^{6k})+C_{N+1,7}(\delta,\lambda)(\xi^{2k}+\xi^{5k}),\end{aligned}$$

for some $A_{N+1,7}(\delta,\lambda),\ B_{N+1,7}(\delta,\lambda),\ C_{N+1,7}(\delta,\lambda)\in\mathbb{Z}[\delta,\lambda]$. Then the linear independence of numbers $1,\xi^k+\xi^{6k}$ and $\xi^{2k}+\xi^{5k}$ over $\mathbb{Q}$ implies the identity (4) for $n=N$. $\square$

Corollary 9. *Linear systems (4) and (5) presented in Theorem 8 can be solved with respect to* $\{A_{n,7}(\delta,\lambda)\}$, $\{B_{n,7}(\delta,\lambda)\}$, *etc. In result, the elements of each sequence, from among these six sequences, are related by means of the recurrence relations of order three. What is more, we obtain from relation (4) that all three sequences* $\{A_{n,7}(\delta,\lambda)\}$, $\{B_{n,7}(\delta,\lambda)\}$, $\{C_{n,7}(\delta,\lambda)\}$ *satisfy the same recurrence relation of order three*

$$\begin{aligned}\mathbb{X}_{n+3,7}(\delta,\lambda)&=(3-\delta-\lambda)\mathbb{X}_{n+2,7}(\delta,\lambda)\\&\quad+(-3+2\delta+2\delta^2+2\lambda-3\delta\lambda+2\lambda^2)\mathbb{X}_{n+1,7}(\delta,\lambda)\\&\quad+(1-\delta-2\delta^2+\delta^3-\lambda+3\delta\lambda+3\delta^2\lambda-2\lambda^2-4\delta\lambda^2+\lambda^3)\mathbb{X}_{n,7}(\delta,\lambda)\end{aligned}\tag{6}$$

with different initial conditions

$$\begin{array}{lll}A_{0,7}(\delta,\lambda)=1, & A_{1,7}(\delta,\lambda)=1, & A_{2,7}(\delta,\lambda)=1+2\delta^2-2\delta\lambda+\lambda^2,\\ B_{0,7}(\delta,\lambda)=0, & B_{1,7}(\delta,\lambda)=\delta, & B_{2,7}(\delta,\lambda)=2\delta-\lambda^2,\\ C_{0,7}(\delta,\lambda)=0, & C_{1,7}(\delta,\lambda)=\lambda, & C_{2,7}(\delta,\lambda)=\delta^2+2\lambda-2\delta\lambda-\lambda^2.\end{array}$$

Discussion on sequences $\{A_{n,9}(\delta,\lambda)\}$, $\{B_{n,9}(\delta,\lambda)\}$ *and* $\{C_{n,9}(\delta,\lambda)\}$ *will be omitted here.*

Proof. Let $M^k=[m_{ij}(k)]_{3\times 3}$ for each $k\in\mathbb{N}$. Then by the Cayley-Hamilton theorem for each pair i,j the sequence $m_{ij}(k),\ k\in\mathbb{N}$ is a linear recurrence sequence with characteristic polynomial the same as the characteristic polynomial of M (see [2, 1.1.12] or [1]). Therefore by (7), recurrence relation (6) can be obtained computing the characteristic polynomial of the matrix used in (4). $\square$

2.1. The Jordan decomposition

Let us observe that form of the transition matrix in the recurrence relation (4) implies that

$$\begin{bmatrix} 1 & 2\delta-\lambda & \lambda-\delta \\ \delta & 1 & -\lambda \\ \lambda & \delta-\lambda & 1-\delta-\lambda \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{n,7}(\delta,\lambda) \\ B_{n,7}(\delta,\lambda) \\ C_{n,7}(\delta,\lambda) \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2\delta-\lambda & \lambda-\delta \\ \delta & 1 & -\lambda \\ \lambda & \delta-\lambda & 1-\delta-\lambda \end{bmatrix}^n \begin{bmatrix} A_{1,7}(\delta,\lambda) \\ B_{1,7}(\delta,\lambda) \\ C_{1,7}(\delta,\lambda) \end{bmatrix} = \begin{bmatrix} A_{n+1,7}(\delta,\lambda) \\ B_{n+1,7}(\delta,\lambda) \\ C_{n+1,7}(\delta,\lambda) \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 2\delta-\lambda & \lambda-\delta \\ \delta & 1 & -\lambda \\ \lambda & \delta-\lambda & 1-\delta-\lambda \end{bmatrix}^n \begin{bmatrix} A_{2,7}(\delta,\lambda) \\ B_{2,7}(\delta,\lambda) \\ C_{2,7}(\delta,\lambda) \end{bmatrix} = \begin{bmatrix} A_{n+2,7}(\delta,\lambda) \\ B_{n+2,7}(\delta,\lambda) \\ C_{n+2,7}(\delta,\lambda) \end{bmatrix}.$$

Hence we get the following formula

$$\begin{aligned} \mathcal{W}^n(\delta,\lambda) &= \begin{bmatrix} 1 & 2\delta-\lambda & \lambda-\delta \\ \delta & 1 & -\lambda \\ \lambda & \delta-\lambda & 1-\delta-\lambda \end{bmatrix}^n \\ &= \begin{bmatrix} A_n & 2B_n-C_n & C_n-B_n \\ B_n & A_n & -C_n \\ C_n & B_n-C_n & A_n-B_n-C_n \end{bmatrix}, \end{aligned} \tag{7}$$

where, in order to simplify the notation, we take

$$A_n = A_{n,7}(\delta,\lambda), \quad B_n = B_{n,7}(\delta,\lambda), \quad C_n = C_{n,7}(\delta,\lambda), \quad n \in \mathbb{N}.$$

It can be verified that the matrix $\mathcal{W}(\delta,\lambda)$ possesses the following Jordan decomposition

$$\mathcal{W}(\delta,\lambda) = \mathcal{P} \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} \mathcal{P}^{-1},$$

where

$$\begin{aligned}\Delta_1 &= 1+\delta(\xi+\xi^6)+\lambda(\xi^2+\xi^5),\\ \Delta_2 &= 1+\delta(\xi^2+\xi^5)+\lambda(\xi^3+\xi^4),\\ \Delta_3 &= 1+\delta(\xi^3+\xi^4)+\lambda(\xi+\xi^6),\end{aligned}$$

and $\mathcal{P}$ is respective similarity matrix which, because of its complicated algebraic description, is omitted here. Thus, and this is very important for the potential applications, the matrix $\mathcal{W}(\delta,\lambda)$ is a diagonalizable matrix!

3. Properties and applications of the two-parametric quasi-Fibonacci numbers

In this section we investigate the selected properties and applications of the two-parametric quasi-Fibonacci numbers defined by us. We begin with the numbers of seventh order.

From the definition of the two-parametric quasi-Fibonacci numbers of seventh order we obtain

$$\begin{aligned}&(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{3k}+\xi^{4k}))^n\\
&\qquad= (1-\lambda+(\delta-\lambda)(\xi^k+\xi^{6k})-\lambda(\xi^{2k}+\xi^{5k}))^n\\
&\qquad\overset{\lambda\neq 1}{=} (1-\lambda)^n\left[A_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)\right.\\
&\quad\left.+B_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)(\xi^k+\xi^{6k})+C_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)(\xi^{2k}+\xi^{5k})\right]\\
&\qquad= (1-\lambda)^n\left[A_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)-C_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)\right.\\
&\qquad+\left(B_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)-C_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)\right)(\xi^k+\xi^{6k})\\
&\qquad\qquad\left.-C_{n,7}\left(\frac{\delta-\lambda}{1-\lambda},\frac{\lambda}{\lambda-1}\right)(\xi^{3k}+\xi^{4k})\right].\end{aligned}\tag{8}$$

Hence we conclude as follows.

Corollary 10. *The given below identities hold true*

$$A_{n,7}(\delta,\lambda)=(1-\lambda)^n\left[A_{n,7}\left(\tfrac{\delta-\lambda}{1-\lambda},\tfrac{\lambda}{\lambda-1}\right)-C_{n,7}\left(\tfrac{\delta-\lambda}{1-\lambda},\tfrac{\lambda}{\lambda-1}\right)\right],$$

$$B_{n,7}(\delta,\lambda) = (1-\lambda)^n \left[B_{n,7}\left(\tfrac{\delta-\lambda}{1-\lambda}, \tfrac{\lambda}{\lambda-1}\right) - C_{n,7}\left(\tfrac{\delta-\lambda}{1-\lambda}, \tfrac{\lambda}{\lambda-1}\right)\right],$$
$$C_{n,7}(\delta,\lambda) = -(1-\lambda)^n C_{n,7}\left(\tfrac{\delta-\lambda}{1-\lambda}, \tfrac{\lambda}{\lambda-1}\right).$$

Remark 11. From the above formulae we get, among others, the relations

$$\begin{aligned} &C_{n,7}(\delta,2) = (-1)^{n+1} C_{n,7}(2-\delta,2), \\ &C_{n,7}(\delta,1\pm i) = -(\mp i)^n C_{n,7}\Big((\pm i)(\delta-1\mp i), \mp 1 \pm i\Big). \end{aligned}$$

Remark 12. We have one more beautiful identity

$$\begin{aligned} &\prod_{k=1}^{3}\big(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\big)^n \\ &= A_{n,7}^3(\delta,\lambda) + B_{n,7}^3(\delta,\lambda) + C_{n,7}^3(\delta,\lambda) - A_{n,7}^2(\delta,\lambda)(B_{n,7}(\delta,\lambda)+C_{n,7}(\delta,\lambda)) \\ &\quad - 2A_{n,7}(\delta,\lambda)(B_{n,7}(\delta,\lambda)^2 + C_{n,7}^2(\delta,\lambda)) + 3B_{n,7}^2(\delta,\lambda)C_{n,7}(\delta,\lambda) \\ &\quad - 4B_{n,7}(\delta,\lambda)C_{n,7}^2(\delta,\lambda)) + 3A_{n,7}(\delta,\lambda)B_{n,7}(\delta,\lambda)C_{n,7}(\delta,\lambda) \\ &\quad = \big(1-\delta-\lambda+3\delta\lambda-2\delta^2-2\lambda^2-4\delta\lambda^2+3\delta^2\lambda+\delta^3+\lambda^3\big)^n. \end{aligned} \tag{9}$$

Proof. The proof results directly from Definition 6 and Lemma 4. □

Remark 13. Lemma 3.21 in paper [15] is the particular case of Theorem 12 for $\lambda = 0$.

By using the two-parametric quasi-Fibonacci numbers of seventh order we can generate the trigonometric idenities for angles $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ i $\frac{8\pi}{7}$, which is presented in the theorems given below.

But first let us take assistantly

$$\mathcal{A}_{n,7}(\delta,\lambda) := 3A_{n,7}(\delta,\lambda) - B_{n,7}(\delta,\lambda) - C_{n,7}(\delta,\lambda), \quad n \in \mathbb{N}_0. \tag{10}$$

Theorem 14. *We obtain the following decomposition*

$$\begin{aligned} &(\mathbb{X}-(2\sin(\frac{2\pi}{7}))^{-n})(\mathbb{X}-(2\sin(\frac{4\pi}{7}))^{-n})(\mathbb{X}-(2\sin(\frac{8\pi}{7}))^{-n}) \\ &\qquad = \mathbb{X}^3 - (-\frac{\sqrt{7}}{7})^n \mathcal{A}_{n,7}(2,1)\mathbb{X}^2 + (-\frac{\sqrt{7}}{7})^n z_{n-1}\mathbb{X} - (-\frac{\sqrt{7}}{7})^n, \end{aligned} \tag{11}$$

where $z_{n-1} = (2\sin(\frac{2\pi}{7}))^n + (2\sin(\frac{4\pi}{7}))^n + (2\sin(\frac{8\pi}{7}))^n$.

Proof. We use the notation $s_k = \sin(\frac{2^k\pi}{7}),\ k = 1, 2, 3.$ Then we have

$$\begin{aligned}
&(\mathbb{X} - (2s_1)^{-n})(\mathbb{X} - (2s_2)^{-n})(\mathbb{X} - (2s_3)^{-n}) \\
&\qquad = \mathbb{X}^3 - ((2s_1)^{-n} + (2s_2)^{-n} + (2s_3)^{-n})\mathbb{X}^2 \\
&\qquad + 4^{-n}((s_1s_2)^{-n} + (s_1s_3)^{-n} + (s_2s_3)^{-n})\mathbb{X} - 8(s_1s_2s_3)^{-n} \qquad (12)
\end{aligned}$$

for each $n \in \mathbb{N}$. The identity $s_1s_2s_3 = -\frac{\sqrt{7}}{8}$ is well known. So we have

$$\begin{aligned}
4^{-n}((s_1s_2)^{-n} + (s_1s_3)^{-n} + (s_2s_3)^{-n}) &= 8^{-n}\frac{(2s_1)^n + (2s_2)^n + (2s_3)^n}{(s_1s_2s_3)^n} \\
&= \left(-\frac{\sqrt{7}}{7}\right)^n z_{n-1}
\end{aligned}$$

for each $n \in \mathbb{N}$. Furthermore we get

$$\begin{aligned}
&(2s_1)^{-n} + (2s_2)^{-n} + (2s_3)^{-n} = (-i(\xi - \xi^6))^{-n} + (-i(\xi^2 - \xi^5))^{-n} \\
&\quad + (-i(\xi^4 - \xi^3))^{-n} = (8s_1s_2s_3)^{-n}\left[(\xi^5 - \xi^2)(\xi^4 - \xi^3)\right]^n \\
&\quad \times \left[(\xi^6 - \xi)(\xi^4 - \xi^3)\right]^n \left[(\xi^6 - \xi)(\xi^2 - \xi^5)\right]^n \\
&\quad = (-\sqrt{7})^{-n}\left[\Big(1 + 2(\xi^2 + \xi^5) + \xi^3 + \xi^4\Big)^n\right. \\
&\quad \left. + \Big(1 + 2(\xi + \xi^6) + \xi^2 + \xi^5\Big)^n + \Big(1 + 2(\xi^3 + \xi^4) + \xi + \xi^6\Big)^n\right] \\
&= \left(-\frac{\sqrt{7}}{7}\right)^n\Big(3A_{n,7}(2,1) + B_{n,7}(2,1)\sum_{k=1}^{3}(\xi^k + \xi^{6k}) + C_{n,7}(2,1)\sum_{k=1}^{3}(\xi^{2k} + \xi^{5k})\Big) \\
&= \left(-\frac{\sqrt{7}}{7}\right)^n (3A_{n,7}(2,1) - B_{n,7}(2,1) - C_{n,7}(2,1)) = \left(-\frac{\sqrt{7}}{7}\right)^n \mathcal{A}_{n,7}(2,1),
\end{aligned}$$

which proves the equality of polynomials (11) and (12) and finishes the proof of theorem.

□

Remark 15. Moreover, the more general formula holds true

$$p_n(\delta, \lambda) := \prod_{k=1}^{3}\left(\mathbb{X} - \Big(1 + \delta(\xi^k + \xi^{6k}) + \lambda(\xi^{2k} + \xi^{5k})\Big)^n\right)$$

$$= \mathbb{X}^3 - \mathcal{A}_{n,7}(\delta,\lambda)\mathbb{X}^2 + (1-\delta-\lambda-\lambda^2+\lambda\delta)^n \mathcal{A}_{n,7}(\delta',\lambda')\mathbb{X} \\ - \big((\delta+\lambda)^3 - 2(\delta+\lambda)^2 - (\delta+\lambda) + 1 + 7\delta\lambda(1-\lambda)\big)^n, \quad (13)$$

where
$$\delta' := \frac{\delta^2-\lambda^2+\delta\lambda-\delta}{1-\delta-\lambda+\delta\lambda-\lambda^2}, \qquad \lambda' := \frac{\delta^2-\delta\lambda-\lambda}{1-\delta-\lambda+\delta\lambda-\lambda^2}.$$

Analyzing the presented formula one can be surprised with no symmetry of polynomials occurring in the coefficient by the power of $\mathbb{X}$ as well as in the constant term. But when we present them explicitly, it appears that the powers in these polynomials are symmetric. The explicit forms of coefficient by the power of $\mathbb{X}$ for few initial values of n are presented below

$$(1-\delta-\lambda-\lambda^2+\lambda\delta)\mathcal{A}_{1,7}(\delta',\lambda') = 3 - 2\delta - 2\delta^2 - 2\lambda + 3\delta\lambda - 2\lambda^2,$$

$$(1-\delta-\lambda-\lambda^2+\lambda\delta)^2\mathcal{A}_{2,7}(\delta',\lambda') = 3 - 4\delta + 2\delta^2 - 2\delta^3 + 6\delta^4 - 4\lambda \\ + 4\delta\lambda - 20\delta^2\lambda - 4\delta^3\lambda + 2\lambda^2 + 22\delta\lambda^2 + 15\delta^2\lambda^2 - 2\lambda^3 - 18\delta\lambda^3 + 6\lambda^4,$$

$$(1-\delta-\lambda-\lambda^2+\lambda\delta)^3\mathcal{A}_{3,7}(\delta',\lambda') = 3 - 6\delta + 12\delta^2 - 17\delta^3 + 27\delta^4 - 12\delta^5 \\ - 11\delta^6 - 6\lambda + 3\delta\lambda - 72\delta^2\lambda + 45\delta^3\lambda + 3\delta^4\lambda + 39\delta^5\lambda + 12\lambda^2 + 75\delta\lambda^2 \\ + 36\delta^2\lambda^2 - 78\delta^3\lambda^2 - 39\delta^4\lambda^2 - 17\lambda^3 - 102\delta\lambda^3 + 69\delta^2\lambda^3 \\ + 18\delta^3\lambda^3 + 27\lambda^4 + 3\delta\lambda^4 - 39\delta^2\lambda^4 - 12\lambda^5 + 39\delta\lambda^5 - 11\lambda^6,$$

as well as the constant terms of polynomials $p_1(\delta,\lambda)$, $p_2(\delta,\lambda)$ and $p_3(\delta,\lambda)$, respectively

$$-\big((\delta+\lambda)^3 - 2(\delta+\lambda)^2 - (\delta+\lambda) + 1 + 7\delta\lambda(1-\lambda)\big) \\ = -1 + \delta + 2\delta^2 - \delta^3 + \lambda - 3\delta\lambda - 3\delta^2\lambda + 2\lambda^2 + 4\delta\lambda^2 - \lambda^3,$$

$$-\big((\delta+\lambda)^3 - 2(\delta+\lambda)^2 - (\delta+\lambda) + 1 + 7\delta\lambda(1-\lambda)\big)^2 \\ = -1 + 2\delta + 3\delta^2 - 6\delta^3 - 2\delta^4 + 4\delta^5 - \delta^6 + 2\lambda - 8\delta\lambda - 4\delta^2\lambda + 20\delta^3\lambda \\ + 6\delta^4\lambda - 6\delta^5\lambda + 3\lambda^2 + 10\delta\lambda^2 - 19\delta^2\lambda^2 - 30\delta^3\lambda^2 - \delta^4\lambda^2 - 6\lambda^3 \\ + 6\delta\lambda^3 + 40\delta^2\lambda^3 + 22\delta^3\lambda^3 - 2\lambda^4 - 22\delta\lambda^4 - 22\delta^2\lambda^4 + 4\lambda^5 + 8\delta\lambda^5 - \lambda^6.$$

Furthermore, we observe that the following relation in all obtained above polynomials

$$\operatorname{coeff}(\delta^k\lambda^r) \equiv \operatorname{coeff}(\delta^r\lambda^k) (\operatorname{mod} 7)$$

holds true!

Remark 16. The following equalities are fulfilled

$$
\begin{aligned}
(\mathcal{A}_{n,7}(\delta,\lambda))^2 &= \left(\sum_{k=1}^{3}\bigl(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\bigr)^n\right)^2 && (14)\\
&= \mathcal{A}_{2n,7}(\delta,\lambda)+2(1-\delta-\lambda-\lambda^2+\lambda\delta)^n\mathcal{A}_{n,7}(\delta',\lambda'), && (15)\\
(\mathcal{A}_{n,7}(\delta,\lambda))^3 &= \left(\sum_{k=1}^{3}\bigl(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\bigr)^n\right)^3 && (16)\\
&= \mathcal{A}_{2n,7}(\delta,\lambda)\mathcal{A}_{n,7}(\delta,\lambda)-2\mathcal{A}_{3n,7}(\delta,\lambda) && (17)\\
&\quad +6\bigl(1-\delta-\lambda+3\delta\lambda-2\delta^2-2\lambda^2-4\delta\lambda^2+3\delta^2\lambda+\delta^3+\lambda^3\bigr)^n, && (18)
\end{aligned}
$$

from which we obtain the special cases

$$
\begin{gathered}
\mathcal{A}^2_{n,7}(\delta,0)-\mathcal{A}_{2n,7}(\delta,0)=2(1-\delta)^n\mathcal{A}_{n,7}\left(-\delta,\frac{\delta^2}{1-\delta}\right),\\
\mathcal{A}^3_{n,7}(\delta,0)-\mathcal{A}_{2n,7}(\delta,0)\mathcal{A}_{n,7}(\delta,0)+2\mathcal{A}_{3n,7}(\delta,0)=6\bigl(1-\delta-2\delta^2+\delta^3\bigr)^n,\\
\mathcal{A}^3_{n,7}(\delta,1)-\mathcal{A}_{2n,7}(\delta,1)\mathcal{A}_{n,7}(\delta,1)+2\mathcal{A}_{3n,7}(\delta,1)=6\bigl(-1-2\delta+\delta^2+\delta^3\bigr)^n.
\end{gathered}
$$

With reference to equality (14) we receive additionally

$$
\begin{aligned}
&\bigl(1+\delta(\xi^k+\xi^{6k})+\lambda(\xi^{2k}+\xi^{5k})\bigr)^{2n}\\
&\quad=(1+\delta^2(2+\xi^{2k}+\xi^{5k})+\lambda^2(2+\xi^{3k}+\xi^{4k})+2\delta(\xi^k+\xi^{6k})\\
&\qquad+2\lambda(\xi^{2k}+xi^{5k})+2\delta\lambda(\xi^{3k}+\xi^{4k}+\xi^k+\xi^{6k}))^n\\
&\quad=\bigl(1+2\delta^2-\lambda^2-2\delta\lambda+(\delta^2-\lambda^2+2\lambda-2\delta\lambda)(\xi^{2k}+\xi^{5k})\\
&\qquad+(2\delta-\lambda^2)(\xi^k+\xi^{6k})\bigr)^n,
\end{aligned}
$$

hence, by relation (8), we get

$$
\begin{aligned}
A_{2n,7}(\delta,\lambda)&=(1+2\delta^2-\lambda^2-2\delta\lambda)^nA_{n,7}(\delta_1,\lambda_1),\\
B_{2n,7}(\delta,\lambda)&=(1+2\delta^2-\lambda^2-2\delta\lambda)^nB_{n,7}(\delta_1,\lambda_1),\\
C_{2n,7}(\delta,\lambda)&=(1+2\delta^2-\lambda^2-2\delta\lambda)^nC_{n,7}(\delta_1,\lambda_1),
\end{aligned}
$$

where

$$
\delta_1=\frac{\delta^2-\lambda^2+2\lambda-2\delta\lambda}{1+2\delta^2-\lambda62-2\delta\lambda},\quad \lambda_1=\frac{2\delta-\lambda^2}{1+2\delta^2-\lambda62-2\delta\lambda}.
$$

It should be noted that for the sequence $\{\mathcal{A}_{n,7}(\delta,\lambda)\}$ in particular cases we have (see formula (10)):

$$\begin{aligned}
&\mathcal{A}_{n,7}(-1,-1) = A198636(n),\\
&\mathcal{A}_{n,7}(-1,1) = A215076(n),\\
&\mathcal{A}_{n,7}(-2,1) = A274663(n),\\
&\mathcal{A}_{n,7}(2,1) = A275831(n),\\
&\mathcal{A}_{n,7}(1,1) = A09675(n) = (-1)^n A094648(n),
\end{aligned}$$

where $n \in \mathbb{N}_0$.

Now we present the selected properties and applications of the two-parametric quasi-Fibonacci numbers of ninth order.

Let $c_k := \zeta^k + \overline{\zeta}^k = 2\cos\frac{2k\pi}{9}$. From the definition of the ninth order quasi-Fibonacci numbers, in view of equality

$$\sum_{k=s}^{s+2} c_{2^k} = c_1 + c_2 + c_4 = 0$$

for each $s \in \mathbb{N}_0$, the following Binet formula for numbers $A_{n,9}(\delta,\lambda)$ results easily

$$3A_{n,9}(\delta,\lambda) = \sum_{k=0}^{2}(1+\delta c_{2^k} + \lambda c_{2^{k+1}})^n. \tag{19}$$

Corollary 17. *For specific values of λ and δ we have the formulae*

$$\begin{aligned}
3A_{n,9}\left(-\frac{1}{2},-\frac{1}{2}\right) &= \sum_{k=1}^{3}\left(2\cos\left(\frac{2^k\pi}{9}\right)\right)^{2n},\\
3A_{n,9}\left(\frac{1}{2},\frac{1}{2}\right) &= \sum_{k=1}^{3}\left(2\sin\left(\frac{2^k\pi}{9}\right)\right)^{2n},\\
3A_{n,9}\left(-c_2,c_2\right) &= 1+\left(\frac{5}{2}\right)^n + \left((2\cos\frac{2\pi}{9}-1)\cos\frac{8\pi}{9}\right)^n.
\end{aligned}$$

In similar way, by using additionally the relation $c_{3\cdot 2^k} \equiv -1$, we obtain the formulae

$$9B_{n,9}(\delta,\lambda) = \sum_{k=0}^{2}(c_{2^k} - c_{2^{k+2}})(1+\delta c_{2^k} + \lambda c_{2^{k+1}})^n, \tag{20}$$

$$9C_{n,9}(\delta,\lambda) = \sum_{k=0}^{2}(c_{2^{k+1}} - c_{2^{k+2}})(1+\delta c_{2^k} + \lambda c_{2^{k+1}})^n. \tag{21}$$

Remark 18. Formulae (19), (20), (21) imply the following Binet formulae for one-parametric quasi-Fibonacci numbers

$$3A_{n,9}(\delta) = \sum_{k=0}^{2}(1+\delta c_{2^k})^n, \tag{22}$$

$$9B_{n,9}(\delta) = \sum_{k=0}^{2}(c_{2^k} - c_{2^{k+2}})(1+\delta c_{2^k})^n, \tag{23}$$

$$9C_{n,9}(\delta) = \sum_{k=0}^{2}(c_{2^{k+1}} - c_{2^{k+2}})(1+\delta c_{2^k})^n. \tag{24}$$

Additionally we define one more sequence

$$\mathcal{A}_{n,9}(\delta,\lambda) := \sum_{k=0}^{2} c_{2^{k+2}}(1+\delta c_{2^k} + \lambda c_{2^{k+1}})^n, \qquad n \in \mathbb{N},$$

for which we receive from relations (20) and (21):

$$\mathcal{A}_{n,9}(\delta,\lambda) = -3(B_{n,9}(\delta,\lambda) + C_{n,9}(\delta,\lambda)).$$

Directly from definition (6), as well as from relation (5), we get also the following reduction identities for the sums of arguments

$$\begin{aligned}2^n A_{n,9}\left(\frac{1}{2}(\delta+\tilde{\delta}), \frac{1}{2}(\lambda+\tilde{\lambda})\right) &= \sum_{m=0}^{n}\binom{n}{m}\big(A_{m,9}(\delta,\lambda)A_{n-m,9}(\tilde{\delta},\tilde{\lambda}) \\ &+ 2B_{m,9}(\delta,\lambda)B_{n-m,9}(\tilde{\delta},\tilde{\lambda}) + 2C_{m,9}(\delta,\lambda)C_{n-m,9}(\tilde{\delta},\tilde{\lambda}) + \\ &- B_{m,9}(\delta,\lambda)C_{n-m,9}(\tilde{\delta},\tilde{\lambda}) - B_{n-m,9}(\tilde{\delta},\tilde{\lambda})C_{m,9}(\delta,\lambda)\big)\,;\end{aligned} \tag{25}$$

$$\begin{aligned}2^n B_{n,9}\left(\frac{1}{2}(\delta+\tilde{\delta}), \frac{1}{2}(\lambda+\tilde{\lambda})\right) &= \sum_{m=0}^{n}\binom{n}{m}\big(A_{m,9}(\delta,\lambda)B_{n-m,9}(\tilde{\delta},\tilde{\lambda}) \\ &+ A_{n-m,9}(\tilde{\delta},\tilde{\lambda})B_{m,9}(\delta,\lambda) + B_{m,9}(\delta,\lambda)C_{n-m,9}(\tilde{\delta},\tilde{\lambda}) + \\ &+ B_{n-m,9}(\tilde{\delta},\tilde{\lambda})C_{m,9}(\delta,\lambda) - C_{m,9}(\delta,\lambda)C_{n-m,9}(\tilde{\delta},\tilde{\lambda})\big)\,;\end{aligned} \tag{26}$$

$$\begin{aligned}2^n C_{n,9}\left(\frac{1}{2}(\delta+\tilde{\delta}), \frac{1}{2}(\lambda+\tilde{\lambda})\right) &= \sum_{m=0}^{n}\binom{n}{m}\big(A_{m,9}(\delta,\lambda)C_{n-m,9}(\tilde{\delta},\tilde{\lambda}) \\ &+ A_{n-m,9}(\tilde{\delta},\tilde{\lambda})C_{m,9}(\delta,\lambda) + B_{m,9}(\delta,\lambda)B_{n-m,9}(\tilde{\delta},\tilde{\lambda}) + \\ &- C_{m,9}(\delta,\lambda)C_{n-m,9}(\tilde{\delta},\tilde{\lambda})\big)\,.\end{aligned} \tag{27}$$

Similarly like the two-parametric quasi-Fibonacci numbers of seventh order, also the two-parametric quasi-Fibonacci numbers of ninth order can be applied to generating the trigonometric identities.

Theorem 19. *The following decompositions hold*

$$\begin{aligned}
&a)\ \prod_{k=0}^{2}\Big(\mathbb{X}-(1+\delta c_{2^k}+\lambda c_{2^{k+1}})^n\Big)\\
&\quad=\mathbb{X}^3-3A_{n,9}(\delta,\lambda)\mathbb{X}^2+(1-\delta^2-\lambda^2+\delta\lambda)^n A_{n,9}(\delta',\lambda')\mathbb{X}-\Delta^n;\\
&b)\ \prod_{k=0}^{2}\Big(\mathbb{X}-c_{2^k}(1+\delta c_{2^k}+\lambda c_{2^{k+1}})^n\Big)\\
&\qquad\qquad=\mathbb{X}^3-(\mathcal{A}_{n,9}(\delta,\lambda)+9B_{n,9}(\delta,\lambda))\mathbb{X}^2\\
&\qquad\qquad\quad+(1-\delta^2-\lambda^2+\delta\lambda)^n(\mathcal{A}_{n,9}(\delta',\lambda')\\
&\qquad\qquad\qquad-3A_{n,9}(\delta',\lambda')+9B_{n,9}(\delta',\lambda')\mathbb{X}+\Delta^n;
\end{aligned}$$

where

$$\delta':=\frac{(\delta-\lambda)(\delta+1)-\delta\lambda}{1-\delta^2-\lambda^2+\delta\lambda},\qquad \lambda':=\frac{\delta+\lambda^2-2\delta\lambda}{1-\delta^2-\lambda^2+\delta\lambda},$$
$$\Delta:=1-3(\delta^2+\lambda^2)-\delta^3-\lambda^3+3\delta\lambda-3\delta\lambda^2+6\delta^2\lambda.$$

Proof. The proof runs in analogical way like the proof of Theorem 11 about decomposition of the two-parametric quasi-Fibonacci numbers of seventh order.□

Remark 20. We have also

$$\begin{aligned}
3A_{n,9}^2(\delta,\lambda)&=\left(\sum_{k=0}^{2}c_{2^{k+2}}(1+\delta c_{2^k}+\lambda c_{2^{k+1}})^n\right)^2\\
&=A_{n,9}(\delta,\lambda)+2(1-\delta^2-\lambda^2+\delta\lambda)^n A_{n,9}(\delta',\lambda'),
\end{aligned}\tag{28}$$

which implies that

$$3A_{n,9}^2(\delta,1)=A_{n,9}(\delta,1)+2(\delta-\delta^2)^n A_{n,9}\left(-1-\frac{1}{\delta-\delta^2},\frac{1}{\delta}\right).$$

In the next part of this section we present few decompositions connected to the one-parametric quasi-Fibonacci numbers of ninth order.

Let $s_k := -i(\zeta^k - \overline{\zeta}^k) = 2\sin\frac{2k\pi}{9}$ and $\sigma_n := \sum_{k=0}^{2}\big((-1)^k s_{2^k}\big)^n$, $n \in \mathbb{N}$. We formulate the theorem.

Theorem 21. *The following decompositions hold*

a)
$$\prod_{k=0}^{2}(\mathbb{X} - c_{2^k}^n) = \mathbb{X}^3 - s_n x^2 + 3(-1)^n A_{n,9}(-1)\mathbb{X} + (-1)^{n-1}. \tag{29}$$

b)
$$\prod_{k=0}^{2}(\mathbb{X} - c_{2^k}(c_{2^{k+1}})^n)$$
$$= \xi^3 - a_n\mathbb{X}^2 + 3(-1)^{n-1}(A_{n,9}(-1) - A_{n-1,9}(-1))\mathbb{X} + (-1)^n, \tag{30}$$

where
$$a_0 = 0,\ a_1 = -3,\ a_{n+1} = a_n - s_n,\ n \in \mathbb{N},$$

c)
$$\prod_{k=0}^{2}\Big(\mathbb{X} - \big((-1)^k s_{2^k}\big)^n\Big) = \mathbb{X}^3 - \sigma_n\mathbb{X}^2 + 3(-1)^n A_{n,9}(1)\mathbb{X} - (-\sqrt{3})^n.$$

Proof. The proofs run analogically like the proofs of presented before theorems about the decompositions, with the use of the following formulae in respective cases:

a)
$$\prod_{k=0}^{2} c_{2^k} = -1$$

and the easy to check formula
$$\sum_{0\le k<l\le 2}(c_{2^k}c_{2^l})^n = \sum_{k=0}^{2}(-1 + c_{2^k})^n \overset{(22)}{=} 3(-1)^n A_{n,9}(-1);$$

b)
$$\sum_{0\le k<l\le 2} c_{2^k}c_{2^l}(c_{2^{k+1}}c_{2^{l+1}})^n = \sum_{k=0}^{2}(-1 + c_{2^{k+2}})(-1 + c_{2^k})^n$$
$$= \sum_{k=0}^{2}(-1 + c_{2^{k+2}})(-1 + c_{2^k})(-1 + c_{2^k})^{n-1} = -\sum_{k=0}^{2} c_{2^k}(-1 + c_{2^k})^{n-1}$$
$$\overset{(22)}{=} 3(-1)^{n-1}(A_{n,9}(-1) - A_{n-1,9}(-1)).$$

□

Some of the sequences described by means of the two-parametric quasi-Fibonacci numbers of seventh or ninth order can be found in the *OEIS*:

$A_{n,7}(1,1) = A028495(n),$

$C_{n,7}(1,1) = A096976(n+1),$

$C_{n,7}(-1,1) = A181879(n),$

$A_{n,7}(-1,-1) = A080937(n),$

$2^n B_{n,7}(1,\tfrac{1}{2}) = A120757(n),$

$A_{n,7}(-1,-1) - C_{n,7}(-1,-1) = A052975(n).$

$A_{n,9}(1,1) = A147704(n),$

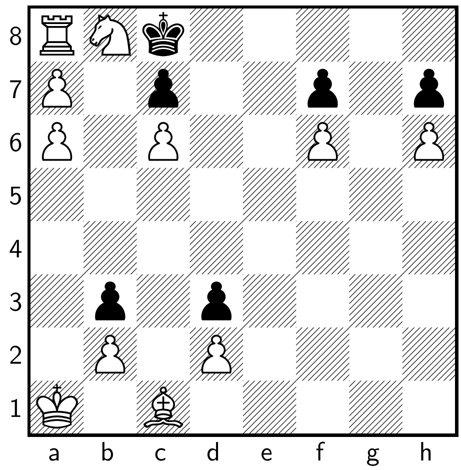
$B_{n,9}(1,1) = A123941(n),$

$B_{n,9}(-1,0) = A122100(n+2),$

$2^n A_{n,9}(-\tfrac{1}{2},0) = A124292(n+1),$

$2^n A_{n,9}(\tfrac{1}{2},0) = A094831(n+1),$

Especially interesting is the sequence $\{A_{n,7}(1,1)\}$ representing the number of ways that white can force checkmate in exactly $(n+1)$ moves, for $n \geq 0$, ignoring the fifty-move and the triple repetition rules, in the following chess position:



Another interesting sequences are:

- $\{C_{n,7}(1,1)\}$ counting the closed walks of length n at the start of graph P_3 to which a loop has been added at the other extremity.
- $C_{n,7}(1,1)$ counts the walks between the first node and the last one.
- $\{2^n A_{n,9}(-\tfrac{1}{2},0)\}$ describing the number of free generators of degree n of the symmetric polynomials in four noncommuting variables.

4. Reduction formulae

In this section we present the selected reduction formulae for the indices and the values of parameters λ and δ supplementing the set of such formulae given before.

By using the properties of multiplication of powers, we get easily the reduction formulae for indices of the quasi-Fibonacci numbers of seventh and ninth order. They are collected in the theorem presented below.

Theorem 22. *The following equalities are satisfied:*
a) for the two-parametric quasi-Fibonacci numbers of seventh order

$$\begin{aligned} A_{m+n,7}(\delta,\lambda) &= A_{m,7}(\delta,\lambda)A_{n,7}(\delta,\lambda) + B_{m,7}(\delta,\lambda)B_{n,7}(\delta,\lambda) \\ &\quad - B_{m,7}(\delta,\lambda)C_{n,7}(\delta,\lambda) + C_{m,7}(\delta,\lambda)(C_{n,7}(\delta,\lambda) - B_{n,7}(\delta,\lambda)); \end{aligned} \tag{31}$$

$$\begin{aligned} B_{m+n,7}(\delta,\lambda) &= A_{n,7}(\delta,\lambda)B_{m,7}(\delta,\lambda) + A_{m,7}(\delta,\lambda)B_{n,7}(\delta,\lambda) \\ &\quad - C_{m,7}(\delta,\lambda)C_{n,7}(\delta,\lambda); \end{aligned} \tag{32}$$

$$\begin{aligned} C_{m+n,7}(\delta,\lambda) &= A_{n,7}(\delta,\lambda)C_{m,7}(\delta,\lambda) \\ &\quad + B_{n,7}(\delta,\lambda)(B_{m,7}(\delta,\lambda) - C_{m,7}(\delta,\lambda)) \\ &\quad + C_{n,7}(\delta,\lambda)(A_{m,7}(\delta,\lambda) - B_{m,7}(\delta,\lambda) - C_{m,7}(\delta,\lambda)); \end{aligned} \tag{33}$$

b) for the two-parametric quasi-Fibonacci numbers of ninth order

$$\begin{aligned} A_{m+n,9}(\delta,\lambda) &= A_{m,9}(\delta,\lambda)A_{n,9}(\delta,\lambda) + 2B_{m,9}(\delta,\lambda)B_{n,9}(\delta,\lambda) \\ &\quad + 2C_{m,9}(\delta,\lambda)C_{n,9}(\delta,\lambda) - B_{m,9}(\delta,\lambda)C_{n,9}(\delta,\lambda) - B_{n,9}(\delta,\lambda)C_{m,9}(\delta,\lambda); \end{aligned} \tag{34}$$

$$\begin{aligned} B_{m+n,9}(\delta,\lambda) &= A_{m,9}(\delta,\lambda)B_{n,9}(\delta,\lambda) + B_{m,9}(\delta,\lambda)A_{n,9}(\delta,\lambda) \\ &\quad + B_{m,9}(\delta,\lambda)C_{n,9}(\delta,\lambda) + C_{m,9}(\delta,\lambda)B_{n,9}(\delta,\lambda) - C_{m,9}(\delta,\lambda)C_{n,9}(\delta,\lambda); \end{aligned} \tag{35}$$

$$\begin{aligned} C_{m+n,9}(\delta,\lambda) &= A_{m,9}(\delta,\lambda)C_{n,9}(\delta,\lambda) + B_{m,9}(\delta,\lambda)B_{n,9}(\delta,\lambda) \\ &\quad + A_{n,9}(\delta,\lambda)C_{m,9}(\delta,\lambda) - C_{m,9}(\delta,\lambda)C_{n,9}(\delta,\lambda). \end{aligned} \tag{36}$$

Remark 23. We have
a) for the two-parametric quasi-Fibonacci numbers of seventh order

$$\begin{aligned}
A_{2n,7}(\delta,\lambda) &= A_{n,7}^2(\delta,\lambda) + B_{n,7}^2(\delta,\lambda) + C_{n,7}^2(\delta,\lambda) - 2B_{n,7}(\delta,\lambda)C_{n,7}(\delta,\lambda),\\
B_{2n,7}(\delta,\lambda) &= 2A_{n,7}(\delta,\lambda)B_{n,7}(\delta,\lambda) - C_{n,7}^2(\delta,\lambda),\\
C_{2n,7}(\delta,\lambda) &= 2A_{n,7}(\delta,\lambda)C_{n,7}(\delta,\lambda) - 2B_{n,7}(\delta,\lambda)C_{n,7}(\delta,\lambda) + B_{n,7}^2(\delta,\lambda) - C_{n,7}^2(\delta,\lambda),
\end{aligned}$$

b) for the two-parametric quasi-Fibonacci numbers of ninth order

$$\begin{aligned}
A_{2n,9}(\delta,\lambda) &= A_{n,9}^2(\delta,\lambda) + 2B_{n,9}^2(\delta,\lambda) + 2C_{n,9}^2(\delta,\lambda) - 2B_{n,9}(\delta,\lambda)C_{n,9}(\delta,\lambda),\\
B_{2n,9}(\delta,\lambda) &= 2A_{n,9}(\delta,\lambda)B_{n,9}(\delta,\lambda) + 2B_{n,9}(\delta,\lambda)C_{n,9}(\delta,\lambda) - C_{n,9}^2(\delta,\lambda),\\
C_{2n,9}(\delta,\lambda) &= 2A_{n,9}(\delta,\lambda)C_{n,9}(\delta,\lambda) + B_{n,9}^2(\delta,\lambda) - C_{n,9}^2(\delta,\lambda).
\end{aligned}$$

While considering the reduction formulae for parameters, in case of the quasi-Fibonacci numbers of seventh order we distinguish two types of them, in two separated collections.

Theorem 24. *The following reduction formulae hold true. The first collection will be called the general reduction formulae:*

$$\begin{aligned}
(1 + a(2\delta-\lambda) + b(\lambda-\delta))^n A_{n,7}(\delta'',\lambda'') &= A_{n,7}(a,b)A_{n,7}(\delta,\lambda)\\
&\quad + B_{n,7}(a,b)(2B_{n,7}(\delta,\lambda) - C_{n,7}(\delta,\lambda)) + C_{n,7}(a,b)C_{n,7}(\delta,\lambda)\\
&\quad - C_{n,7}(a,b)B_{n,7}(\delta,\lambda)); \qquad (37)
\end{aligned}$$

$$\begin{aligned}
(1 + a(2\delta-\lambda) + b(\lambda-\delta))^n B_{n,7}(\delta'',\lambda'') &= A_{n,7}(a,b)B_{n,7}(\delta,\lambda)\\
&\quad + A_{n,7}(\delta,\lambda)B_{n,7}(a,b) - C_{n,7}(a,b)C_{n,7}(\delta,\lambda); \qquad (38)
\end{aligned}$$

$$\begin{aligned}
(1 + a(2\delta-\lambda) + b(\lambda-\delta))^n C_{n,7}(\delta'',\lambda'') &= A_{n,7}(\delta,\lambda)C_{n,7}(a,b)\\
&\quad + B_{n,7}(\delta,\lambda)(B_{n,7}(a,b) - C_{n,7}(a,b))\\
&\quad + C_{n,7}(\delta,\lambda)(A_{n,7}(a,b) - B_{n,7}(a,b) - C_{n,7}(a,b)); \qquad (39)
\end{aligned}$$

where

$$\delta'' := \frac{\delta + a - b\lambda}{1 + a(2\delta-\lambda) + b(\lambda-\delta)} \text{ and } \lambda'' := \frac{\lambda + a(\delta-\lambda) + b(1-\delta-\lambda)}{1 + a(2\delta-\lambda) + b(\lambda-\delta)}.$$

The second collection will be called the particular reduction formulae:

$$(1+2\delta\mu)^n A_{n,7}\left(\frac{\delta+\mu}{1+2\delta\mu},\frac{\delta\mu}{1+2\delta\mu}\right)=A_{n,7}(\delta)A_{n,7}(\mu)+B_{n,7}(\delta)B_{n,7}(\mu)$$
$$+\left(B_{n,7}(\delta)-C_{n,7}(\delta)\right)\left(B_{n,7}(\mu)-C_{n,7}(\mu)\right); \quad (40)$$

$$(1+2\delta\mu)^n B_{n,7}\left(\frac{\delta+\mu}{1+2\delta\mu},\frac{\delta\mu}{1+2\delta\mu}\right)=A_{n,7}(\delta)B_{n,7}(\mu)+A_{n,7}(\mu)B_{n,7}(\delta)$$
$$+C_{n,7}(\delta)C_{n,7}(\delta); \quad (41)$$

$$(1+2\delta\mu)^n C_{n,7}\left(\frac{\delta+\mu}{1+2\delta\mu},\frac{\delta\mu}{1+2\delta\mu}\right)=A_{n,7}(\delta)C_{n,7}(\mu)+B_{n,7}(\delta)B_{n,7}(\mu)$$
$$-B_{n,7}(\delta)C_{n,7}(\delta)+C_{n,7}(\delta)\left(A_{n,7}(\mu)-B_{n,7}(\mu)-C_{n,7}(\mu)\right). \quad (42)$$

Remark 25. From the presented above formulae we deduce the following specific relations

$$\begin{aligned}
&(-1)^n A_{n,7}(-2i,1)=A^2_{n,7}(i)+B^2_{n,7}(i)+\big(B_{n,7}(i)-C_{n,7}(i)\big),\\
&(-1)^n B_{n,7}(-2i,1)=2A_{n,7}(i)B_{n,7}(i)+C_{n,7}(i),\\
&(-1)^n C_{n,7}(-2i,1)=2A_{n,7}(i)C_{n,7}(i)+\big(B_{n,7}(i)-C_{n,7}(i)\big)^2,\\
&2^n A_{n,7}(\tfrac{1}{2},0)=A^2_{n,7}(1,1)+B^2_{n,7}(1,1)+\big(B_{n,7}(1,1)-C_{n,7}(1,1)\big)^2,\\
&2^n B_{n,7}(\tfrac{1}{2},0)=2A_{n,7}(1,1)B_{n,7}(1,1)-C^2_{n,7}(1,1),\\
&2^n C_{n,7}(\tfrac{1}{2},0)=2A_{n,7}(1,1)C_{n,7}(1,1)+\big(B_{n,7}(1,1)-C_{n,7}(1,1)\big)^2-C^2_{n,7}(1,1).
\end{aligned}$$

The next formulae for the quasi-Fibonacci numbers of ninth order, obtained by simple calculations, should be also noted.

Theorem 26. *The following reduction formulae are satisfied*

$$\begin{aligned}
A_{n,9}(\delta,\lambda)&=A_{n,9}(\lambda-\delta,-\delta),\\
B_{n,9}(\delta,\lambda)&=-C_{n,9}(\lambda-\delta,-\delta),\\
C_{n,9}(\delta,\lambda)&=B_{n,9}(\lambda-\delta,-\delta)-C_{n,9}(\lambda-\delta,-\delta),\\
A_{n,9}(\delta,\lambda)&=A_{n,9}(-\lambda,\delta-\lambda),\\
B_{n,9}(\delta,\lambda)&=C_{n,9}(-\lambda,\delta-\lambda)-B_{n,9}(-\lambda,\delta-\lambda),\\
C_{n,9}(\delta,\lambda)&=-B_{n,9}(-\lambda,\delta-\lambda).
\end{aligned}$$

Remark 27. The above formulae imply the next specific equalities

$$\begin{aligned}
&A_{n,9}(\delta,0) = A_{n,9}(0,\delta) = A_{n,9}(-\delta,-\delta),\\
&B_{n,9}(\delta,0) = -C_{n,9}(-\delta,-\delta) = C_{n,9}(\delta,0) + C_{n,9}(\delta,0),\\
&C_{n,9}(\delta,0) = -B_{n,9}(0,\delta) = B_{n,9}(-\delta,-\delta) + B_{n,9}(\delta,0).
\end{aligned}$$

Acknowledgment

The authors would like to thank to the reviewer for his valuable suggestions.

References

1. Elaydi S.: *An Introduction to Difference Equations*. Springer-Verlag, New York 2005.
2. Everest G., van der Poorten A., Shparlinski I., Ward T.: *Recurrence Sequences*. American Mathematical Society, Rhode Island 2003.
3. Hetmaniok E., Piątek B., Wituła R.: *Binomials transformation formulae for scaled Fibonacci numbers*. Open Math. **15** (2017), 477–485.
4. Niven I.: *Irrational Numbers*. MAA 1956 (supplement to Russian edition written by I.M. Yaglom).
5. Prodinger H.: *Representing derivatives of Chebyshev polynomials by Chebyshev polynomials and related questions*. Open Math. **15** (2017), 1156–1160.
6. Rabinowitz S.: *Algorithmic manipulation of third-order linear recurrences*. Fib. Quart. **34** (1996), 447–464.
7. Stinchcombe A. M.: *Reccurrence relations for powers of recursion sequences*. Fib. Quart. **36** (1998), 443–447.
8. Wituła R.: *Ramanujan type trigonometric formulae*. Demonstratio Math. **45**, no. 4 (2012), 779–796.
9. Wituła R.: *δ-Fibonacci numbers. Part II*. Novi Sad J. Math. **42** (2013), 9–22.
10. Wituła R.: *Binomials transformation formulae of scaled Lucas numbers*. Demonstratio Math. **46** (2013), 15–27.
11. Wituła R., Hetmaniok E., Słota D., Pleszczyński M.: *δ-Fibonacci and δ-Lucas numbers, δ-Fibonacci and δ-Lucas polynomials*. Math. Slovaca **67**, no. 1 (2017), 51–70.
12. Wituła R., Słota D.: *Quasi-Fibonacci numbers of order 11*. J. Integer Seq. **10** (2007), article 07.8.5.

13. Wituła R., Słota D.: *New Ramanujan-type Formulas and quasi-Fibonacci numbers of order 7.* J. Integer Seq. **10** (2007), article 07.5.6.

14. Wituła R., Słota D.: *δ-Fibonacci numbers.* Appl. Anal. Discrete Math. **3** (2009), 310–329.

15. Wituła R., Słota D., Warzyński A.: *Quasi-Fibonacci numbers of the seventh order.* J. Integer Seq. **9** (2006), article 06.4.3.