Tadeusz KACZOREK, Kamil BORAWSKI
BIALYSTOK UNIVERSITY OF TECHNOLOGY, FACULTY OF ELECTRICAL ENGINEERING
ul. Wiejska 45D, 15-351 Białystok

A computer algorithm for the solution of the state equation for time-varying fractional discrete-time linear systems

Abstract

Method for finding of the solution of the state equation for time-varying fractional discrete-time linear systems is proposed and computer algorithm is presented. The effectiveness of the proposed algorithm is demonstrated on numerical examples.

Keywords: linear, time-varying, fractional, discrete-time system, state equation, solution.

1. Introduction

A dynamical system is called a fractional if it is described fractional-order differential equation. A dynamical system is called time-varying if its parameters are depending on time – for example electrical circuit's resistance, inductance and capacitance change over time.

An overview of state of the art in fractional systems theory is given in monographs [10-13]. The stability of standard and positive fractional discrete-time linear systems have been analyzed in [1-2]. The positive linear systems with different fractional orders have been addressed in [8, 9].

The Lyapunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in [3-7]. The positivity and stability of time-varying discrete-time linear systems have been addressed [13]. The switched discrete-time systems have been considered in [14-16].

In this paper method for finding of the solution of the state equation for time-varying fractional discrete-time linear systems will be proposed and a computer algorithm is presented. The effectiveness of the proposed algorithm is demonstrated on numerical examples.

The paper is organized as follows. In Section 2 method for finding of the solution of the state equation for the time-varying fractional discrete-time linear systems are proposed. A method and a procedure for computation of the state equation is given in Section 3. In Section 4 numerical examples are presented. Concluding remarks are given in Section 5. Calculations are performed in MATLAB environment using Symbolic Math Toolbox.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix.

2. Methods for finding of the solution of the state equation

Consider the time-varying fractional discrete-time linear system described by the equation

$$\Delta^\alpha x_{k+1} = A(k)x_k + B(k)u_k\ ,\ k\in Z_+ = \{0,1,...\} \tag{1}$$

where: $x_k = \Re^n$, $u_k\in\Re^m$ are the state and input vectors and $A(k)\in \Re^{n\times n}$, $B(k)\in \Re^{n\times m}$ are matrices with entries depending on $k\in Z_+$ and the fractional difference of the order α is defined by

$$\Delta^\alpha x_k = \sum_{i=0}^{k}(-1)^i\binom{\alpha}{i}x_{k-i}\ , \tag{2}$$

$$\binom{\alpha}{i} = \begin{cases} 1 & \text{for } i=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-i+1)}{i!} & \text{for } i=1,2,... \end{cases} . \tag{3}$$

Substituting (2) into (1) we obtain

$$x_{k+1} = A_\alpha(k)x_k + \sum_{i=2}^{k+1}c_i x_{k-i-1} + B(k)u_k \tag{4}$$

where

$$A_\alpha(k) = A(k) + I_n\alpha\ ,\ c_i = (-1)^{i+1}\binom{\alpha}{i}. \tag{5}$$

Coefficients c_i can be easily computed by the following algorithm suitable for computer programming [2]:

$$c_{i+1} = c_i\frac{i+1-\alpha}{i+2}\ ,\ i=2,3,... \tag{6}$$

where

$$c_2 = 0,5\alpha(1-\alpha)\ . \tag{7}$$

Theorem 1. The solution of equation (4) for known initial conditions $x_0\in\Re^n$ and input $u_k\in\Re^n$, $k\in Z_+$ is given by

$$x_k = \Phi(k)x_0 + \sum_{i=0}^{k-1}\psi(k-i-1,i)B(i)u_i \tag{8}$$

where

$$\Phi(k+1) = A_\alpha(k)\Phi(k) + \sum_{i=2}^{k+!}c_i\Phi(k-i+1)\ , \tag{9}$$

$$\psi(k+1,j) = A_\alpha(k+j+1)\psi(k,j) + \sum_{i=2}^{k+!}c_i\psi(k-i+1,j) \tag{10}$$

and

$$\Phi(0) = I_n\ ,\ \psi(0,j) = I_n\ . \tag{11}$$

Proof. Using (8) and (4) we obtain

$$\begin{aligned} &A_\alpha(k)x_k + \sum_{i=2}^{k+1}c_i x_{k-i-1} + B(k)u_k = \\ &= A_\alpha(k)\left[\Phi(k)x_0 + \sum_{i=0}^{k-1}\psi(k-i-1,i)B(i)u_i\right] + B(k)u_k = \\ &= \Phi(k+1)x_0 + \sum_{i=0}^{k}\psi(k-i,i)B(i)u_i = x_{k+1}\ .\ \square \end{aligned} \tag{12}$$

Remark 1. The solution to the equation (4) for known initial conditions $x_0\in\Re^n$ and input $u_k\in\Re^n$, $k\in Z_+$ can be also computed recurrently using the formula

$$x_k = A_\alpha(k-1)x_{k-1} + \sum_{i=2}^{k}c_i x_{k-i} + B(k-1)u_{k-1}\ . \tag{13}$$

The computer algorithm for finding of the solution of the state equation (4) proposed in Section 3 is based on (13), because of its simplicity and more efficiency than (8-11).

3. Computer algorithm for computation of the solution of the state equation

The computer algorithm which calculates the solution of the state equation for time-varying fractional discrete-time linear system based on (13) is proposed. Calculations are performed in MATLAB environment using Symbolic Math Toolbox. The diagram of the algorithm is shown in Fig. 1.

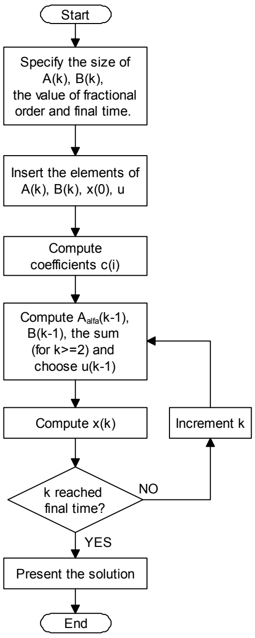

Fig. 1. Algorithm for the computation of the solution of the state equation for time-varying fractional discrete-time linear systems

The procedure of computation will be presented below.

Step 1. Specify the size of matrices $A(k)$ and $B(k)$, the value of the fractional order of the system and the final time at which the state vector will be calculated.

Step 2. Insert the elements of matrices $A(k)$ and $B(k)$, initial conditions vector x_0 and input vectors which are stored in memory in the matrix

$$u = \begin{bmatrix} u_0 & u_1 & \dots & u_{k-1} \end{bmatrix}. \tag{14}$$

Step 3. Compute coefficients c_i.

Step 4. Compute matrices $A_\alpha(k\text{-}1)$ and $B(k\text{-}1)$, the sum $\sum_{i=2}^{k} c_i x_{k-i}$ (if $k \geq 2$) and take the suitable column from matrix (14) for current value of k.

Step 5. Compute x_k using (13).

Step 6. Repeat Step 4 and Step 5 until the desirable state vector is calculated (k reaches the value set by the user).

Step 7. Present the solution in the form of a matrix

$$x = \begin{bmatrix} x_0 & x_1 & \dots & x_k \end{bmatrix}. \tag{15}$$

4. Numerical examples

Example 1.

Find the solution x_3 to the equation (4) for α=0.5 with the matrices

$$A(k) = \begin{bmatrix} 0{,}5\sin(k) & e^{-k} \\ 0{,}3\cos(k) & 0{,}1 \end{bmatrix}, \quad B(k) = \begin{bmatrix} 1 \\ \dfrac{k+1}{k+2} \end{bmatrix}, \tag{16}$$

the initial condition $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and input vectors $u_0 = 1$, $u_1 = 0$, $u_2 = 2$.

Using the algorithm we obtain:

Step 1. We specify the size of matrices n=2 and m=3, the order of the system α=0.5 and the final time k_f=3.

Step 2. We insert matrices (16). The matrix (14) has the form

$$u = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}. \tag{17}$$

Step 3. Coefficients c_i are computed: c_2=0.1250, c_3=0.0781.

Step 4. The time k=1. The matrices $A_\alpha(0)$ and $B(0)$ have the form

$$A_\alpha(0) = A(0) + 0{,}5I_n = \begin{bmatrix} 0{,}5 & 1 \\ 0{,}3 & 0{,}6 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 1 \\ 0{,}5 \end{bmatrix}. \tag{18}$$

The sum $\sum_{i=2}^{k} c_i x_{k-i}$ is not computed because k<2. The first column of matrix (17) is chosen.

Step 5. The vector x_1 is computed using (13):

$$x_1 = \begin{bmatrix} 1{,}5 \\ 0{,}8 \end{bmatrix}. \tag{19}$$

The time $k \neq k_f$ so k is incremented. Step 4 and Step 5 will be repeated for k=2 and then for k=k_f=3.

Step 6. The solution has the form

$$x = \begin{bmatrix} 1{,}0000 & 1{,}5000 & 1{,}8004 & 4{,}0822 \\ 0 & 0{,}8000 & 0{,}7231 & 1{,}8091 \end{bmatrix}. \tag{20}$$

From (15) and (20) we have

$$x_1 = \begin{bmatrix} 1{,}5000 \\ 0{,}8000 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1{,}8004 \\ 0{,}7231 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 4{,}0822 \\ 1{,}8091 \end{bmatrix}. \tag{21}$$

The solution of the state equation (4) with the matrices (16) for $\alpha = 0{,}5$, $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, zero input and $k \in \langle 0, 50 \rangle$ is shown in Fig. 2.

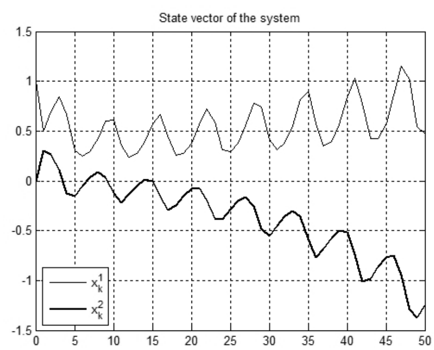

Fig. 2. State variables of the system (16)

Example 2.

Find the solution x_4 to the equation (4) for α=0.3 with the matrices

$$A(k) = \begin{bmatrix} 0{,}3\sin(2k) & 0{,}2 & 0{,}1 \\ 0{,}4 & e^{-3k}\sin(k) & e^{-2k} \\ 0{,}1e^{-k}\cos(3k) & 0{,}1 & 0{,}3 \end{bmatrix},$$

$$B(k) = \begin{bmatrix} 1 & 0,3\sin(k) \\ \frac{e^k}{k+2} & 0 \\ e^{-4k}\sin(k) & 1 \end{bmatrix}, \quad (22)$$

the initial condition $x_0 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and input vectors $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Using the algorithm we obtain the solution

$$x = \begin{bmatrix} 1,0000 & 1,7000 & 2,9136 & 1,7058 & 2,7007 \\ 2,0000 & 1,5000 & 2,3495 & 2,2167 & 5,8315 \\ 0 & 0,3000 & 2,2835 & 3,6744 & 3,6796 \end{bmatrix}. \quad (23)$$

From (15) and (23) we have

$$x_1 = \begin{bmatrix} 1,7000 \\ 1,5000 \\ 0,3000 \end{bmatrix}, x_2 = \begin{bmatrix} 2,9136 \\ 2,3495 \\ 2,2835 \end{bmatrix}, x_3 = \begin{bmatrix} 1,7058 \\ 2,2167 \\ 3,6744 \end{bmatrix}, x_4 = \begin{bmatrix} 2,7007 \\ 5,8315 \\ 3,6796 \end{bmatrix} \quad (24)$$

The solution of the state equation (4) with the matrices (22) for $\alpha = 0,3$, $x_0 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, zero input and $k \in \langle 0,50 \rangle$ is shown in Fig. 3.

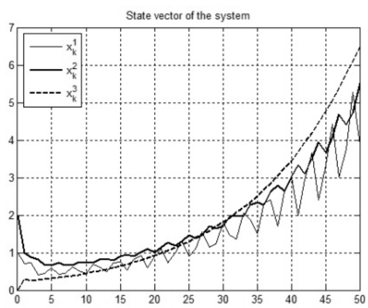

Fig. 3. State variables of the system (22)

5. Concluding remarks

The method for computing of the solution of the state equation for time-varying fractional discrete-time linear systems has been presented. An algorithm based on (13) and a procedure for computation of the state vectors have been given. The effectiveness of the algorithm has been demonstrated on numerical examples.

The presented approach can be extended to time-varying fractional continuous-time linear systems.

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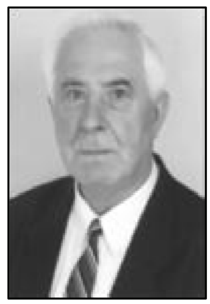
Prof. D.Sc. eng. Tadeusz KACZOREK



MSc. (1956), PhD (1962) and DSc (1964) from Electrical Engineering of Warsaw University of Technology. Since 1974 he was full professor at Warsaw University of Technology, and since 2003 he has been a professor at Bialystok University of Technology. He was awarded by ten universities by the title doctor honoris causa. His research interests cover the theory of systems and the automatic control systems. He has published 27 books (10 in English) and over 1000 scientific papers.

e-mail: kaczorek@isep.pw.edu.pl

B.Sc. eng. Kamil BORAWSKI

Born 28th March 1991 in Augustów. BSc eng. degree in Electrical Engineering received in January 2014 on Bialystok University of Technology. At the same University in March 2014 he started MSc studies in Electrical Engineering. His main scientific interests are modern control theory, especially positive and fractional-order systems.

e-mail: kbaug91@gmail.com