

A SPECIAL CASE OF GENERALIZED HÖLDER FUNCTIONS

Maria Lupa

*Institute of Mathematics, Czestochowa University of Technology
Czestochowa, Poland
maria.lupa@im.pcz.pl*

Abstract. In this paper some properties of a special case of generalized Hölder functions, which belong to the space $W_\gamma[a, b]$, are considered. These functions are r -times differentiable and their r -th derivatives satisfy the generalized Hölder condition. The main result of the paper is a proof of the theorem that product of two functions belonging to the space $W_\gamma[a, b]$ also belongs to this space.

Keywords: Lipschitz condition, generalized Hölder condition, equivalency of norms

Introduction

In the paper we introduce a function space $W_\gamma[a, b]$ and consider and prove some of its properties.

In the articles [1-4] the authors discussed Nemytskii operator determined on various function spaces (cf. also [5, 6]). For instance, it is shown there that in each function space $Lip[a, b], C^r[a, b], BV[a, b]$ a generating function of this operator exists and is affine with respect to the second variable. A similar result is obtained in [7] for the $W_\gamma[a, b]$ -space. These results are then applied in [8, 9] to prove the existence and uniqueness of the solution of a certain functional equation in $W_\gamma[a, b]$.

The space W_γ and its properties

Let $[a, b]$ be a closed interval, where $a, b \in R, a < b, d := b - a$. We assume that the following condition is fulfilled

$$(I) \gamma: [0, d] \rightarrow [0, \infty) \text{ is increasing and concave, } \gamma(0) = 0, \\ \lim_{t \rightarrow 0^+} \gamma(t) = \gamma(0), \lim_{t \rightarrow d^-} \gamma(t) = \gamma(d).$$

Definition 1.

Denote by $W_\gamma[a, b]$ the set of all r -times differentiable functions, where $r \in N$, defined on the interval $[a, b]$ with values in R , such that their r -th derivatives satisfy the following condition: there exists a constant $M \geq 0$ such that

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \leq M\gamma(|x - \bar{x}|), \quad \bar{x}, x \in [a, b] \quad (1)$$

where γ fulfils condition (Γ) .

Remark 1.

It is easily seen that $W_\gamma[a, b]$ contains the class of all r -times differentiable functions $\varphi: [a, b] \rightarrow \mathbb{R}$, whose r -th derivatives satisfy the Lipschitz condition on $[a, b]$. This class is denoted by $LipC^r[a, b]$. Thus we have

$$LipC^r[a, b] \subset W_\gamma[a, b].$$

Remark 2.

Denote by $\gamma'_+(0)$ the right derivative of γ at $t = 0$. By (Γ) we have $0 \leq \gamma'_+(0) \leq +\infty$.

For $\varphi \in W_\gamma[a, b]$ and by the condition (Γ) we obtain

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \leq M\gamma(|x - \bar{x}|) \leq M\gamma'_+(0)|x - \bar{x}|, \quad \bar{x}, x \in [a, b]$$

i.e. $\varphi^{(r)}$ fulfils an ordinary *Lipschitz condition* with the constant $K = M\gamma'_+(0)$.

Thus if $\varphi \in W_\gamma[a, b]$ and $\gamma'_+(0)$ is finite then $\varphi \in LipC^r[a, b]$. In view of Remark 1 we get $LipC^r[a, b] = W_\gamma[a, b]$.

Therefore only the case $\gamma'_+(0) = +\infty$ is of interest.

Remark 3.

The functions of the form $\gamma(t) = t^\alpha$, where $0 < \alpha < 1$, $t \in [0, d]$, fulfil the assumption (Γ) and moreover $\gamma'_+(0) = +\infty$. Therefore the condition (1) is called *the generalized Hölder condition* or the γ – *Hölder condition*.

Lemma 1.

If $\varphi \in W_\gamma[a, b]$ and $\gamma'_+(0) = +\infty$, then the functions $\varphi^{(k)}$, where $k = 0, 1, \dots, r-1$, fulfil the generalized Hölder condition with a function γ_0 and the constants $L_k, k = 0, 1, \dots, r-1$.

Proof.

By the condition $\gamma'_+(0) = +\infty$, we have $\gamma(t) \geq t$ in a right neighbourhood of zero. Suppose that $\gamma(d) \geq d$. Then we have $\gamma(t) \geq t$ for $t \in [0, d]$.

Let $\gamma(d) < d$. Define the function

$$\gamma_1(t) := \frac{d}{\gamma(d)}\gamma(t), \quad t \in [0, d] \quad (2)$$

It is easily seen that γ_1 fulfils the condition (Γ) and $\gamma(t) \leq \gamma_1(t)$ for $t \in [0, d]$. Moreover, $\gamma_1(t) \geq t$ for $t \in [0, d]$. Clearly, if a function φ fulfils the generalized Hölder condition with the same function γ such that $\gamma(d) < d$ and with a constant M , then φ fulfils the generalized Hölder condition with the function γ_1 , defined by (2), and the same constant M .

Define

$$\gamma_0 = \begin{cases} \gamma & \gamma(d) \geq d. \\ \gamma_1 & \gamma(d) < d \end{cases} \quad (3)$$

where γ_1 is defined by (2). Thus the functions $\varphi^{(k)}$, where $k = 0, 1, \dots, r-1$, fulfil the generalized Hölder condition with the function γ_0 defined by (3), i.e. for $\bar{x}, x \in [a, b]$ we have

$$|\varphi^{(k)}(x) - \varphi^{(k)}(\bar{x})| \leq L_k |x - \bar{x}| \leq L_k \gamma_0(|x - \bar{x}|), k = 0, 1, \dots, r-1.$$

This completes the proof.

Remark 4.

The space $W_\gamma[a, b]$ with the norm

$$\|\varphi\| := \sum_{k=0}^r |\varphi^{(k)}(a)| + \sup \left\{ \frac{|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})|}{\gamma(|x - \bar{x}|)}; x, \bar{x} \in [a, b], x \neq \bar{x} \right\} \quad (4)$$

is a real normed vector space. Moreover, it is a Banach space ([5, 6]).

In the space $W_\gamma[a, b]$ we define the second norm by the formula

$$\begin{aligned} \|\varphi\|_0 := & \sum_{k=0}^r \sup_{x \in [a, b]} |\varphi^{(k)}(x)| + \\ & + \sup \left\{ \frac{|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})|}{\gamma(|x - \bar{x}|)}; x, \bar{x} \in [a, b], x \neq \bar{x} \right\} \end{aligned} \quad (5)$$

We will show the equivalence of norms (4) and (5).

Proposition 1.

The norms (4) and (5) are equivalent in $W_\gamma[a, b]$ and the following inequalities hold

$$\|\varphi\| \leq \|\varphi\|_0 \leq K_1 \|\varphi\| \quad (6)$$

where

$$K_1 = \max \left(\sum_{k=0}^{r+1} d^k; \sum_{k=0}^{r+1} (\gamma(d))^k \right) \quad (7)$$

Proof.

The inequality $\|\varphi\| \leq \|\varphi\|_0$ is obvious. For $\varphi \in W_\gamma[a, b]$ define the constants

$$M := \sup \left\{ \frac{|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})|}{\gamma(|x - \bar{x}|)}; x, \bar{x} \in [a, b], x \neq \bar{x} \right\},$$

$$L_k := \sup \left\{ \frac{|\varphi^{(k)}(x) - \varphi^{(k)}(\bar{x})|}{|x - \bar{x}|}; x, \bar{x} \in [a, b], x \neq \bar{x} \right\}, k = 0, 1, \dots, r - 1.$$

Since for $\varphi \in W_\gamma[a, b]$ the following inequalities hold

$$\sup_{x \in [a, b]} |\varphi^{(r)}(x)| \leq |\varphi^{(r)}(a)| + M\gamma(d)$$

$$\sup_{x \in [a, b]} |\varphi^{(k)}(x)| \leq |\varphi^{(k)}(a)| + L_k d, k = 0, 1, \dots, r - 1,$$

and by the mean value theorem we have

$$L_k \leq \sup_{x \in [a, b]} |\varphi^{(k+1)}(x)|, k = 0, 1, \dots, r - 1,$$

thus we obtain

$$\sup_{x \in [a, b]} |\varphi^{(k)}(x)| \leq |\varphi^{(k)}(a)| + |\varphi^{(k+1)}(a)|d + \dots + |\varphi^{(r)}(a)|d^{r-k} + Md^{r-k}\gamma(d)$$

In view of above inequalities we have

$$\begin{aligned} \|\varphi\|_0 &\leq \sum_{k=0}^r \sup_{x \in [a, b]} |\varphi^{(k)}(x)| + \\ &+ M \sum_{i=0}^r |\varphi^{(i)}(a)| \sum_{k=0}^i d^k + M(\gamma(d) \sum_{k=0}^r d^k + 1). \end{aligned}$$

In the case $\gamma(d) \geq d$ we obtain

$$\gamma(d) \sum_{k=0}^r d^k + 1 \leq \sum_{k=0}^{r+1} (\gamma(d))^k,$$

then

$$\|\varphi\|_0 \leq \sum_{k=0}^{r+1} (\gamma(d))^k \|\varphi\|.$$

In the case $\gamma(d) < d$ we have

$$\gamma(d) \sum_{k=0}^r d^k + 1 \leq \sum_{k=0}^{r+1} d^k,$$

therefore

$$\|\varphi\|_0 \leq \sum_{k=0}^{r+1} d^k \|\varphi\|.$$

In view of (7) we obtain inequality (6). This completes the proof.

Proposition 2.

If the function f and g belong to $W_\gamma[a, b]$, then their product $f \cdot g$ also belongs to this space and there exists a constant $K_2 > 0$ such that

$$\|f \cdot g\|_0 \leq K_2 \|f\|_0 \quad (8)$$

where

$$K_2 = \max(P_r, Q_r)$$

$$P_r = \max_{i=0, \dots, r-1} \left(\|g\|_0; \sup_{[a,b]} |g^{(i)}| + 2 \sup_{[a,b]} |g^{(i+1)}| \right)$$

$$Q_r = \left(\frac{d}{\gamma(d)} \right)^{r-1} \max_{i=0, \dots, r-1} \left(\left(\frac{d}{\gamma(d)} \right)^{r-1} \|g\|_0; \sup_{[a,b]} |g^{(i)}| + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g^{(i+1)}| \right)$$

Proof.

For $f, g \in W_\gamma[a, b]$ we have

$$\begin{aligned} & |(fg)^{(r)}(x) - (fg)^{(r)}(\bar{x})| = \\ & = \left| \sum_{i=0}^r \binom{r}{i} f^{(i)}(x) g^{(r-i)}(x) - \sum_{i=0}^r \binom{r}{i} f^{(i)}(\bar{x}) g^{(r-i)}(\bar{x}) \right| \leq \\ & \leq \sup_{x \in [a,b]} |f| |g^{(r)}(x) - g^{(r)}(\bar{x})| + \\ & + \sum_{i=1}^r \binom{r}{i} \sup_{x \in [a,b]} |f^{(i)}| |g^{(r-i)}(x) - g^{(r-i)}(\bar{x})| + \\ & + \sum_{i=0}^{r-1} \binom{r}{i} \sup_{x \in [a,b]} |g^{(r-i)}| |f^{(i)}(x) - f^{(i)}(\bar{x})| + \sup_{[a,b]} |g| |f^{(r)}(x) - f^{(r)}(\bar{x})| \end{aligned}$$

Let $L_i, K_i, i = 0, \dots, r-1$, denote Lipschitz constants of the functions $f^{(i)}$ and $g^{(i)}$ respectively, and by M_f, M_g denote γ -Hölder constants of the functions $f^{(r)}$ and $g^{(r)}$ respectively. Put

$$F_i := \sup_{[a,b]} |f^{(i)}|, \quad i = 0, \dots, r,$$

$$G_i := \sup_{[a,b]} |g^{(i)}|, \quad i = 0, \dots, r.$$

Thus by Lemma 1 we get

$$\begin{aligned} |(fg)^{(r)}(x) - (fg)^{(r)}(\bar{x})| & \leq F_0 M_g \gamma(|x - \bar{x}|) + \sum_{i=1}^r \binom{r}{i} F_i K_{r-i} \gamma_0(|x - \bar{x}|) + \\ & + \sum_{i=0}^{r-1} \binom{r}{i} G_{r-i} L_i \gamma_0(|x - \bar{x}|) + G_0 M_f \gamma(|x - \bar{x}|), \end{aligned}$$

and by definition (3) of function γ_0 there is $f \cdot g \in W_\gamma[a, b]$.

Using mathematical induction we prove the inequality (8).

For $r = 1$ we have

$$\begin{aligned} \|f \cdot g\|_0 &\leq \sup_{[a,b]} |f| \sup_{[a,b]} |g| + \sup_{[a,b]} |f| \sup_{[a,b]} |g'| + \sup_{[a,b]} |f'| \sup_{[a,b]} |g| + \\ &+ \sup_{[a,b]} \frac{|f'(x)g(x) + f(x)g'(x) - f'(\bar{x})g(\bar{x}) - f(\bar{x})g'(\bar{x})|}{\gamma(|x - \bar{x}|)} \leq \\ &\leq \sup_{[a,b]} |f| (\sup_{[a,b]} |g| + \sup_{[a,b]} |g'|) + \sup_{[a,b]} |f'| \sup_{[a,b]} |g| + \\ &+ \sup_{[a,b]} |f| \sup_{[a,b]} \frac{|g'(x) - g'(\bar{x})|}{\gamma(|x - \bar{x}|)} + \sup_{[a,b]} |g| \sup_{[a,b]} \frac{|f'(x) - f'(\bar{x})|}{\gamma(|x - \bar{x}|)} + \\ &+ \sup_{[a,b]} |f'| \sup_{[a,b]} \frac{|g(x) - g(\bar{x})|}{\gamma(|x - \bar{x}|)} + \sup_{[a,b]} |g'| \sup_{[a,b]} \frac{|f(x) - f(\bar{x})|}{\gamma(|x - \bar{x}|)} \end{aligned}$$

where supremes of the above rational expressions are for $x, \bar{x} \in [a, b], x \neq \bar{x}$. (In the next inequalities in this proof the same notations are used).

If $\gamma(d) \geq d$ then the following inequalities hold the function g :

$$\sup_{[a,b]} \frac{|g(x) - g(\bar{x})|}{\gamma(|x - \bar{x}|)} \leq \sup_{[a,b]} \frac{|g(x) - g(\bar{x})|}{|x - \bar{x}|} \leq \sup_{[a,b]} |g'|$$

and the same for function f . From the above we obtain

$$\begin{aligned} \|f \cdot g\|_0 &\leq \sup_{[a,b]} |f| \left((\sup_{[a,b]} |g| + \sup_{[a,b]} |g'|) + \sup_{[a,b]} \frac{|g'(x) - g'(\bar{x})|}{\gamma(|x - \bar{x}|)} \right) + \\ &+ \sup_{[a,b]} |f'| (\sup_{[a,b]} |g| + 2 \sup_{[a,b]} |g'|) + \sup_{[a,b]} \frac{|f'(x) - f'(\bar{x})|}{\gamma(|x - \bar{x}|)} \sup_{[a,b]} |g| \leq \\ &\leq \|f\|_0 \max \left(\|g\|_0; \sup_{[a,b]} |g| + 2 \sup_{[a,b]} |g'| \right) = P_1 \|f\|_0 \end{aligned}$$

If $\gamma(d) < d$, by (3) we have the following inequalities

$$\sup_{[a,b]} \frac{|g(x) - g(\bar{x})|}{\gamma_0(|x - \bar{x}|)} \leq \sup_{[a,b]} \frac{|g(x) - g(\bar{x})|}{|x - \bar{x}|} \leq \sup_{[a,b]} |g'|$$

and using definition (2) we obtain

$$\sup_{[a,b]} \frac{|g(x) - g(\bar{x})|}{\gamma(|x - \bar{x}|)} \leq \frac{d}{\gamma(d)} \sup_{[a,b]} |g'|$$

(similarly for the function f).

Therefore we have

$$\begin{aligned} \|f \cdot g\|_0 &\leq \sup_{[a,b]} |f| \left(\leq \sup_{[a,b]} |f| (\sup_{[a,b]} |g| + \sup_{[a,b]} |g'| + \sup \frac{|g'(x) - g'(\bar{x})|}{\gamma(|x - \bar{x}|)}) + \right. \\ &\quad \left. + \sup_{[a,b]} |f'| (\sup_{[a,b]} |g| + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g'|) + \sup \frac{|f'(x) - f'(\bar{x})|}{\gamma(|x - \bar{x}|)} \sup_{[a,b]} |g| \leq \right. \\ &\quad \left. \leq \|f\|_0 \max \left(\|g\|_0; \sup_{[a,b]} |g| + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g'| \right) = Q_1 \|f\|_0 \right) \end{aligned}$$

Putting

$$K_2 = \max(P_1, Q_1)$$

we get the inequality (8) for $r = 1$.

Let functions f, g have $(k + 1)$ -order derivatives. Denote by $\|\cdot\|_{0,k}$ the norm $\|\cdot\|_0$ for the functions k times differentiable. Suppose that the inequality (8) is true for $k \geq 1, k \in N$, i.e.

$$\|f \cdot g\|_{0,k} \leq K_2 \|f\|_{0,k}$$

where

$$K_2 = \max(P_k, Q_k)$$

$$P_k = \max_{i=0, \dots, k-1} \left(\|g\|_{0,k}; \sup_{[a,b]} |g^{(i)}| + 2 \sup_{[a,b]} |g^{(i+1)}| \right)$$

$$Q_k = \left(\frac{d}{\gamma(d)} \right)^{k-1} \max_{i=0, \dots, k-1} \left(\left(\frac{d}{\gamma(d)} \right)^{k-1} \|g\|_{0,k}; \sup_{[a,b]} |g^{(i)}| + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g^{(i+1)}| \right)$$

We prove that the inequality (8) holds for $(k + 1)$. We have

$$\|f \cdot g\|_{0,k+1} \leq \sup_{[a,b]} |f| \sup_{[a,b]} |g| + \|f \cdot g'\|_{0,k} + \|f' \cdot g\|_{0,k}$$

If $\gamma(d) \geq d$ then the following inequality holds

$$\begin{aligned} \|f \cdot g\|_{0,k+1} &\leq \sup_{[a,b]} |f| \sup_{[a,b]} |g| + \\ &\quad + \|f\|_{0,k} \max_{i=0, \dots, k-1} \left(\|g'\|_{0,k}; \sup_{[a,b]} |g^{(i+1)}| + 2 \sup_{[a,b]} |g^{(i+2)}| \right) + \\ &\quad + \|f'\|_{0,k} \max_{i=0, \dots, k-1} \left(\|g\|_{0,k}; \sup_{[a,b]} |g^{(i)}| + 2 \sup_{[a,b]} |g^{(i+1)}| \right) \end{aligned}$$

Since

$$\|f\|_{0,k} \leq \|f\|_{0,k+1} ; \|f'\|_{0,k} \leq \|f'\|_{0,k+1}$$

(the same inequalities hold for the function g) thus we have

$$\begin{aligned} \|f \cdot g\|_{0,k+1} &\leq \|f\|_{0,k+1} \max_{i=0,\dots,k} \left(\|g\|_{0,k+1}; \sup_{[a,b]} |g^{(i)}| + 2 \sup_{[a,b]} |g^{(i+1)}| \right) = \\ &= P_{k+1} \|f\|_{0,k+1} \end{aligned}$$

In the case $\gamma(d) < d$ we have

$$\|f\|_{0,k} \leq \frac{d}{\gamma(d)} \|f\|_{0,k+1}; \|f'\|_{0,k} \leq \|f'\|_{0,k+1}$$

(the same inequalities hold for the function g). Therefore

$$\begin{aligned} \|f \cdot g\|_{0,k+1} &\leq \sup_{[a,b]} |f| \sup_{[a,b]} |g| + \\ &+ \|f\|_{0,k} \left(\frac{d}{\gamma(d)} \right)^k \max_{i=0,\dots,k-1} \left(\left(\frac{d}{\gamma(d)} \right)^{k-1} \|g'\|_{0,k}; \sup_{[a,b]} |g^{(i+1)}| \right. \\ &\quad \left. + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g^{(i+2)}| \right) + \\ &+ \|f'\|_{0,k} \left(\frac{d}{\gamma(d)} \right)^{k-1} \max_{i=0,\dots,k-1} \left(\left(\frac{d}{\gamma(d)} \right)^{k-1} \|g\|_{0,k}; \sup_{[a,b]} |g^{(i)}| \right. \\ &\quad \left. + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g^{(i+1)}| \right) \leq \\ &\leq \|f\|_{0,k+1} \left(\frac{d}{\gamma(d)} \right)^k \max_{i=0,\dots,k-1} \left(\left(\frac{d}{\gamma(d)} \right)^k \|g\|_{0,k+1}; \sup_{[a,b]} |g^{(i)}| + \right. \\ &\quad \left. + 2 \frac{d}{\gamma(d)} \sup_{[a,b]} |g^{(i+1)}| \right) = Q_{k+1} \|f\|_{0,k+1} \end{aligned}$$

Putting $K_2 = \max(P_{k+1}, Q_{k+1})$ we get the inequality (8) for $(k+1)$. This completes the proof.

Conclusions

In this paper some properties of the Banach space $W_\gamma[a, b]$ have been presented. They will be applied in the forthcoming papers.

References

- [1] Matkowska A., On characterization of Lipschitzian operator of substitution in the class of Hölder functions, Scientific of Łódź Technical University 1984, 17, 81-85.
- [2] Matkowski J., Functional equations and Nemytskii operators, Funkcialaj Ekvacioj 1982, 25, 127-132.
- [3] Matkowski J., Form of Lipschitz operators of substitution in Banach space of differentiable functions, Zeszyty Naukowe Politechniki Łódzkiej, Matematyka, 1982, 25, 127-132.
- [4] Matkowski J., Miś J., On a characterization of Lipschitzian operators of substitution in the space $BV[a,b]$, Math. Nachr. 1984, 117, 155-159.
- [5] Appell J., Banaś J., Merentes N., Bounded Variation and Around, De Gruyter Studies in Non-linear Analysis and Applications, Volume 17, Würzburg 2014.
- [6] Appell J., Zabrejko P.P., Nonlinear Superposition Operators, Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sydney 1990.
- [7] Lupa M., Form of Lipschitzian operator of substitution in some class of functions, Zeszyty Naukowe Politechniki Łódzkiej, Matematyka, 1990, 21, 87-96.
- [8] Lupa M., On solutions of a functional equation in a special class of functions, Demonstratio Mathematica 1993, XXVI, 1, 137-147.
- [9] Lupa M., W_γ - solutions of linear Iterative Functional Equations, Demonstratio Mathematica 1994, XXVII, 2, 417-425.