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"GCDALPHA" – A SEMI-ANALYTICAL METHOD FOR SOLVING FRACTIONAL STATE EQUATIONS

The paper discusses a semi-analytical method for solving systems of fractional state equations with various orders. The method bases on an expansion of the system into one where there is a single derivative order. The formulation of the matrices of the new system is explained in detail. Another characteristic feature of the method is also introduced – a consideration of forms, in which the time functions appear and the terms appearing in the solution as a result. A fractional circuit example is presented in order to test the method. The computation time for the method is also studied.

KEYWORDS: semi-analytical method, fractional derivative, circuit analysis, different orders.

1. INTRODUCTION

Fractional calculus is a field whose first concept (of a fractional order derivative) dates back to 1695 [1]. The major developments of this field have been made in the 19th century [2]. This branch of mathematics introduces definitions of both fractional integrals and fractional derivatives [3]. The presented study concerns only the fractional derivative in what is known as the Caputo definition [4] for the derivative order $\alpha \in (0,1]$:

$$ {}_0\mathrm{D}_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x^{(1)}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau. \tag{1} $$

The initial time instance of the study is assumed in $t = 0$. The mentioned range of the derivative order is applied in circuit analyses (which this study mostly concerns) when modeling various objects (e.g. supercapacitors [5] and coils with ferromagnetic cores [6]). The study can also be valuable for other fields as it concerns solutions of systems of fractional differential equations in general. Other applications of fractional calculus include:
– control analyses [7, 8] (including fractional PID controllers [9]),
– biomedical signals [10],

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– inverse kinematics [11],
– viscoelasticity [12],
– heat transfer modeling [13],
– telematics [14].

2. MOTIVATION

The following system of fractional state equations is considered:

$$\mathbf{d}^{\boldsymbol{\alpha}}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}(t), \tag{2}$$

where: $\boldsymbol{x}(t)$ is the state variable vector (length of the vector is given by N), $\boldsymbol{v}(t)$ denotes the vector of source time functions (its length is given by M), $\boldsymbol{\alpha}$ represents the vector of the fractional derivative orders α_1, α_2, … α_N, the right-hand side vector $\mathbf{d}^{\boldsymbol{\alpha}}x(t)$ consists of the fractional derivatives:

$$\mathbf{d}^{\boldsymbol{\alpha}}\boldsymbol{x}(t) = [{}_0\mathrm{D}_t^{\alpha_1}x_1(t) \quad {}_0\mathrm{D}_t^{\alpha_2}x_2(t) \quad \ldots \quad {}_0\mathrm{D}_t^{\alpha_N}x_N(t)]^{\mathrm{T}}, \tag{3}$$

$\boldsymbol{A}$ is an $N \times N$ matrix and $\boldsymbol{B}$ is an $N \times M$ matrix.
When modeling the aforementioned objects (ferromagnetic coils and supercapacitors) it has been noticed that these generally have various fractional derivative orders. Even if a circuit contains only various ferromagnetic coils or only various supercapacitors then these also have different orders [6, 15]. The well-known analytical solution of a system of fractional state equations [16] is only valid for a common fractional derivative order $\alpha = \alpha_i$, $i = 1, 2, \ldots N$. This solution is given by:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}_0(t)\boldsymbol{x}(t_0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{v}(\tau)\,\mathrm{d}\tau, \tag{4}$$

where $\boldsymbol{x}(t_0)$ is the initial condition. $\boldsymbol{\Phi}_0(t)$ can be given by the one parameter Mittag-Leffler function:

$$\boldsymbol{\Phi}_0(t) = \mathbf{E}(\boldsymbol{A}t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{5}$$

while:

$$\boldsymbol{\Phi}(t) = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}. \tag{6}$$

An analytical solution for various derivative orders does exist – it involves a complex procedure requiring formulations of matrices and N nested sums [17]. In this paper, an alternative is proposed, which could be easier to implement.

There are also various numerical methods that allow to solve systems of fractional state equations with different derivative orders. The author of this study

has developed such a numerical method [18, 19, 20, 21] – it is now known by its acronym – SubIval (for "the subinterval-based method", first appearing in [22]).

The purpose of the presented study is not to introduce a method better than the mentioned numerical method, but rather to introduce a method for the testing of new methods and their modifications.

It is a common and efficient practice to verify numerical methods through a comparison of their results with ones obtained through methods operating on a completely different basis [23, 24]. The ability to obtain referential solutions for various fractional circuit problems (through non-time stepping methods) is as given in Table 1, where this study concerns transient responses of linear fractional circuits (that can be put in the form of (2)).

Table 1. Non-time stepping methods for solving fractional circuit problems.

	periodic steady state	transient
linear	application of complex numbers [25]	for one fractional order α – solution given by (4); for various orders in [17]
nonlinear	solution using a harmonic balance approach (in the author's parallel study [26])	Unknown

3. BASIS OF THE METHOD

The base idea is to first find the greatest common divisor of the fractional derivatives:

$$\gcd(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_N) = \alpha, \tag{7}$$

hence, the acronym for the method – "gcdAlpha". In a literature review – it has been found that such an idea has also appeared as the basis of another method [27] (a numerical method). This study, however, is not dependent on that one.

The following integers are introduced:

$$N_i = \frac{\alpha_i}{\alpha}, \quad i = 1,2,\ldots N. \tag{8}$$

This allows for each equation using a derivative of order α_i to be rewritten as N_i equations of order α. An equation of the form:

$${}_0\mathrm{D}_t^{\alpha_i} x_i(t) = f_i(\boldsymbol{x}(t), \boldsymbol{v}(t)) \tag{9}$$

becomes:

$$\begin{cases} {}_0\mathrm{D}_t^{\alpha} x_i(t) = x_{i,(1)}(t), \\ {}_0\mathrm{D}_t^{\alpha} x_{i,(1)}(t) = x_{i,(2)}(t), \\ {}_0\mathrm{D}_t^{\alpha} x_{i,(2)}(t) = x_{i,(3)}(t), \\ \vdots \\ {}_0\mathrm{D}_t^{\alpha} x_{i,(N_i-1)}(t) = f_i(\boldsymbol{x}(t), \boldsymbol{v}(t)). \end{cases} \tag{10}$$

Each auxiliary variable with a notation $x_{i,(k)}(t)$ denotes a derivative of order $k\alpha$ of $x_i(t)$. One can notice that it is not the values of the auxiliary variables that are relevant but it is their derivatives, hence, one can assume zero initial conditions:

$$x_{i,(k)}(t_0), \quad k = 1,2,\ldots N_i - 1. \tag{11}$$

The newly formulated system of state equations also applies to the general form of (2). However, all derivative orders are equal to α. $\boldsymbol{A}$ then has the size $\sigma \times \sigma$, where:

$$\sigma = \sum_{i=1}^{N} N_i. \tag{12}$$

The size of $\boldsymbol{B}$ is $\sigma \times M$. The new system can be solved through the general solution given by (4).

Naturally, potential problems can arise if the derivative orders are irrational numbers or if the greatest common divisor is a relatively small number and, hence, σ is very large. This is discussed further on in Section 9.

4. MATRIX FORMULATIONS

Let the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the newly built system be denoted respectively by $\boldsymbol{C}$ and $\boldsymbol{D}$. The state vector of the newly built system is:

$$\begin{aligned} \boldsymbol{w}(t) = [\boldsymbol{x}(t)^{\mathrm{T}} \quad & x_{1,(1)} \quad x_{1,(2)} \quad \cdots \quad x_{1,(N_1-2)} \quad x_{2,(1)} \quad x_{2,(2)} \quad \cdots \quad x_{1,(N_2-2)} \\ & x_{N,(1)} \quad x_{N,(2)} \quad \cdots \quad x_{N,(N_N-2)} \\ & x_{1,(N_1-1)} \quad x_{2,(N_1-1)} \quad \cdots \quad x_{N,(N_N-1)}]^{\mathrm{T}}. \end{aligned} \tag{13}$$

The solution can be given by:

$$\boldsymbol{w}(t) = \sum_{k=0}^{\infty} \frac{\boldsymbol{C}^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \boldsymbol{w}(t_0) + \sum_{k=0}^{\infty} \frac{\boldsymbol{C}^k}{\Gamma((k+1)\alpha)} \int_{t_0}^{t} (t-\tau)^{(k+1)\alpha - 1} \boldsymbol{D}\boldsymbol{v}(t)\,\mathrm{d}\tau. \tag{14}$$

Through the ordering of the variables in (13) one can easily determine the form of $\boldsymbol{C}$:

– the old matrix $\boldsymbol{A}$ is placed in the rows from $\sigma - N + 1$ to σ at the columns from 1 to N,

– for each index i of a state variable x_i the following entries are filled:

$$\boldsymbol{C}_{i,r_i+1} = 1, \tag{15}$$

$$\boldsymbol{C}_{r_i+N_i-2,\sigma-N+i} = 1, \tag{16}$$

$$\boldsymbol{C}_{r_i+j,r_i+j+1} = 1, \quad j = 1,2,\ldots N_i - 3 \tag{17}$$

where:

$$r_i = N + \sum_{k=1}^{i-1}(N_k - 2). \tag{18}$$

$\boldsymbol{D}$ is only filled with the contents of $\boldsymbol{B}$ in the final N rows; the remaining ones are 0. Because of this – $\boldsymbol{Dv}(t)$ produces a vector with only the N final entries being filled:

$$\boldsymbol{Dv}(t) = \begin{bmatrix} 0 \\ \boldsymbol{Y}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{Bv}(t) \end{bmatrix}, \tag{19}$$

where $\boldsymbol{Y}(t)$ was introduced for further use.

5. VARIOUS SOURCE TIME FUNCTIONS

Let $S_{(k),1}$, $S_{(k),2}$, $S_{(k),N}$ denote the entries of a vector denoted by $\boldsymbol{S}_{(k)}(t)$, resulting from:

$$\boldsymbol{S}_{(k)}(t) = \int_0^t (t-\tau)^{(k+1)\alpha-1}\boldsymbol{Y}\,\mathrm{d}\tau. \tag{20}$$

By applying a notation $\mathbf{0}_k$ for a vector with k zeros one can write the solution as:

$$\boldsymbol{w}(t) = \sum_{k=0}^{\infty}\frac{\boldsymbol{C}^k t^{k\alpha}}{\Gamma(k\alpha+1)}\boldsymbol{w}(t_0) + \sum_{k=0}^{\infty}\frac{\boldsymbol{C}^k}{\Gamma((k+1)\alpha)}\begin{bmatrix} \mathbf{0}_{\sigma-N} \\ \boldsymbol{S}_{(k)}(t) \end{bmatrix}. \tag{21}$$

The entries of $\boldsymbol{S}_{(k)}(t)$ are given by:

$$S_{(k),i}(t) = \sum_{j=1}^{M} B_{i,j}a_{(k),j}(t). \quad i = 1,2,\ldots N, \tag{22}$$

where the auxiliary notation is used:

$$a_{(k),j}(t) = \int_0^t (t-\tau)^{(k+1)\alpha-1} v_j(\tau)\mathrm{d}\tau. \tag{23}$$

Some general forms can be derived for these terms when knowing the form of the sources $v_j(t)$. E.g. sources being step functions of the form:

$$v_j(t) = v_{j\max}1(t - T_j) \tag{24}$$

yield:

$$a_{(k),j}(t) = \begin{cases} 0 & t \le T_j, \\ \dfrac{v_{j\max}(t-T_j)^{(k+1)\alpha}}{(k+1)\alpha} & t > T_j. \end{cases} \tag{25}$$

For an exponential function:

$$v_j(t) = v_{j\max} \exp(p_j t) \tag{26}$$

one can obtain the following formula:

$$a_{(k),j} = (p_j)^{-\alpha(k+1)} v_{j\max} \exp(p_j t))\big(\Gamma\big(\alpha(k+1)\big) - \Gamma\big(\alpha(k+1), pt\big)\big), \tag{27}$$

where the gamma function:

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\,\mathrm{d}t, \tag{28}$$

and the incomplete gamma function:

$$\Gamma(a,z) = \int_z^\infty t^{a-1} \exp(-t)\,\mathrm{d}t. \tag{29}$$

6. COMPUTATIONAL EXAMPLE

A simple circuit problem is considered with a fractional coil and a fractional capacitor (Fig. 1). The elements are described by fractional differential equations of different order, where the coil is described by:

$$L\,{}_0\mathrm{D}_t^{\gamma} i_L(t) = u_L(t) \tag{30}$$

while the behavior of the fractional capacitor is given by:

$$C\,{}_0\mathrm{D}_t^{\beta} u_C(t) = i_C(t) \tag{31}$$

The state variable vector is:

$$\boldsymbol{x}(t) = \begin{bmatrix} i_L(t) & u_C(t) \end{bmatrix}^{\mathrm{T}}, \tag{32}$$

with the variable orders:

$$\boldsymbol{\alpha} = \begin{bmatrix} \gamma & \beta \end{bmatrix}^{\mathrm{T}}, \tag{33}$$

while the source vector:

$$\boldsymbol{v}(t) = \begin{bmatrix} E(t) & J(t) \end{bmatrix}^{\mathrm{T}}. \tag{34}$$

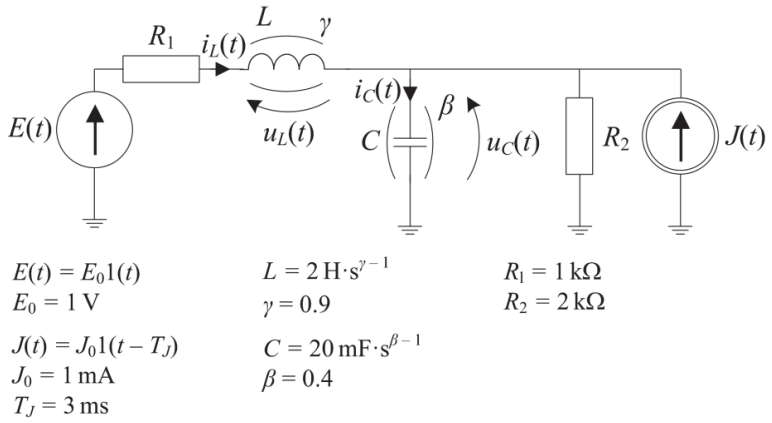

Fig. 1. Exemplary circuit with fractional elements (a fractional element is marked with parentheses and the order of the element)

The matrix $\boldsymbol{A}$ is given by:

$$\boldsymbol{A} = \begin{bmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{R_2 C} \end{bmatrix}, \tag{35}$$

while:

$$\boldsymbol{B} = \begin{bmatrix} \dfrac{1}{L} & 0 \\ 0 & \dfrac{1}{C} \end{bmatrix}. \tag{36}$$

The first step of the gcdAlpha method is to determine the α parameter according to (7). In this case the parameter is equal to 0.1, which yields $N_1 = 9$ equations of the form of (10) for the coil and $N_2 = 4$ equations for the fractional capacitor, which makes it a total of $\sigma = 13$ differential equations. The obtained time functions are presented in Fig. 2.

The sudden change in the u_C function is caused by the unit step source $J(t)$ being activated at T_J.

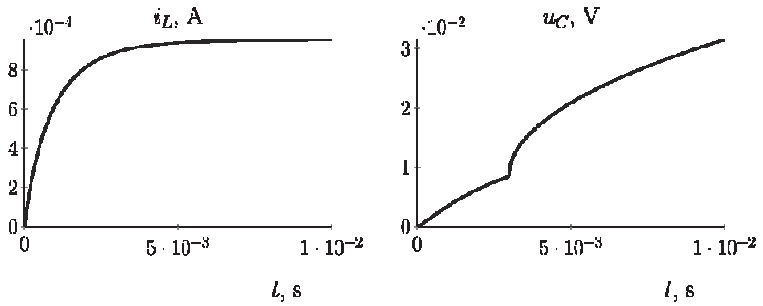

Fig. 2. Time functions of the fractional coil current and fractional capacitor voltage obtained through the gcdAlpha method

7. VERIFICATION OF RESULTS

As already mentioned – it is a common practice to verify the correctness of newly designed methods through a comparison of their results with ones operating on different bases. The results are compared with ones obtained by means of an adaptive time step size solver applying SubIval [18]. A relative difference between the results is computed through the formula:

$$e_j = 100 \cdot \frac{\left| x(t_j) - x_j \right|}{\max\limits_{j=1,2,\ldots n_t} | x(t_j) |} \%, \tag{37}$$

where x denotes the variable selected for the comparison: $x(t_j)$ is the value computed by means of gcdAlpha, while x_j is the value at t_j computed through the numerical solver applying SubIval. The time instances t_1, t_2, … t_{n_t} are those selected by the numerical solver. The options for the numerical solver are given in Table 2, where p is the maximum order of the local polynomial approximations, e_{ctrl} is the error to which the time step has been adapted, $e_{\max}$ is the maximum allowed estimated error (for which the time step was not repeated); finally – $\Delta t_{\min}$ and $\Delta t_{\max}$ are respectively the minimum and maximum allowed time step sizes.

The results obtained through the numerical solver are not visibly different than those obtained through gcdAlpha. The average of the computed e_j values is approximately equal to 0.018 % for the capacitor voltage time function $u_C(t)$, while for the coil current time function $i_L(t)$ the average value is 0.023 %.

Table 2. Non-time stepping methods for solving fractional circuit problems (T is the ending time instance of the analysis i.e. 0.01 s).

p	e_{ctrl}	$e_{\max}$	$\Delta t_{\min}$	$\Delta t_{\max}$
4	0.01 %	0.1 %	T/2000	T/20

8. DISCUSSION ON IMPLEMENTATION

The implementation of the method has been done in C#. The author's own programs have been used for the computations of this paper. The Math.NET Numerics [28] library was applied for matrix and vector operations. For the numerical solver the library SubIval.dll has been applied, which is available at [29]. The library has been compiled using a part of the code found at [30] for the computation of the gamma function. The numerical solver has been implemented in C# and also applied the Math.NET Numerics library for matrices and vectors.

There are a few factors that need to be taken into account for the implementation of the method. First off, the question of computation time.

The computation of the $\boldsymbol{C}^k$ elements is what can be noticed first. A computation of eigenvalues (put in the diagonal matrix $\boldsymbol{V}$) and eigenvectors (put as columns of the matrix $\boldsymbol{P}$) does not reduce the computation time when applying the method. This is because of the following chain of computations:

– $\boldsymbol{Dv}(t)$ is computed inside the sum for $k = 0, 1, \ldots$;
– because of the above – the substitution:

$$\boldsymbol{C}^k = \boldsymbol{P}\boldsymbol{V}^k\boldsymbol{P}^{-1} \tag{38}$$

does not later lead to $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$ being moved outside the sum,
– the above leads to $\boldsymbol{C}^k$ being computed for each k anyway – this requires a multiplication of two dense $\sigma \times \sigma$ matrices.

For k values getting larger – $\boldsymbol{C}^k$ is also a dense matrix, hence – in the simplest way one can apply $\boldsymbol{C}^k = \boldsymbol{C}^{k-1}\boldsymbol{C}$ to obtain the matrix. For a sparse $\boldsymbol{C}$ this is better in terms of numerical efficiency than when applying the eigenvalue and eigenvector approach, especially for large σ values.

The $\boldsymbol{C}^k$ matrices can be stored in memory so that one does not need to compute them again for each time instance selected for observation (e.g. when drawing time functions of the solution).

However, one still needs to take into account that $\boldsymbol{C}^k$, with larger k values, are dense $\sigma \times \sigma$ matrices. This could lead to memory issues for large σ integers. Meanwhile, taking into account terms with large k leads to errors when computing powers of the time instances and, most of all, the powers of $\boldsymbol{C}$. For the example presented in this paper and all of the computations performed – the storage of $\boldsymbol{C}^k$ matrices has not caused memory issues.

The inclusion of terms in the sum has been controlled. For a selected time instance t_j, for each $k > k_{\min}$ when a vector (denoted by $\Delta\boldsymbol{x}$) was to be added to the value finally leading to $\boldsymbol{x}(t_j)$ (called $\boldsymbol{x}$ further in this Section) then if the following condition is true:

$$e_{\mathrm{Tol}} > \max_{i=1,2,\ldots N} \left| \frac{\Delta \boldsymbol{x}_i}{\boldsymbol{x}_i} \right| \cdot 100\% \tag{39}$$

then no more terms have been taken into account. $e_{\mathrm{Tol}} = 10^{-10}$ % has been selected for the computations in the presented study.

9. DISCUSSION ON DERIVATIVE ORDERS

The smaller the greatest common divisor in comparison to the derivative orders – the greater the value of σ. In turn, the larger the $\boldsymbol{C}$ matrix, the longer the computation time. Table 3 gives the computation times for various orders of the fractional coil γ (which resulted in different α values). The computations were set up to obtain the solution for 200 equally spaced points on the time axis. The order of the fractional capacitor was always the same (i.e. $\beta = 0.4$).

Table 3. Computation times for various derivative orders, which resulted in different sizes of $\boldsymbol{C}$ (values of computation time are rounded to the nearest hundredth).

γ	α	time, s
0.90	0.100	2.78
0.95	0.05	12.80
0.86	0.02	88.35
0.91	0.01	697.99

If a certain error level is acceptable then one could consider rounding the derivative orders to obtain a less accurate solution, but the solution would be obtained much faster. A case where gcdAlpha can be applied only after rounding is when the derivative orders are given as irrational numbers. Errors arising due to approximation of the derivative orders will be discussed in a future study.

10. CONCLUSIONS

A semi-analytical method for solving systems of fractional state equations (in the form of (2)) has been presented. The basis of the method is the expansion of the system to one with a common derivative order α. This requires the greatest common divisor of the orders to be found (hence the acronym used for the method – "gcdAlpha"). The formulation of the matrices for this new system have been discussed. Another feature of the considered method is the consideration of the forms of the source time functions, which leads to general formulae not requiring any further integration.

An exemplary problem with both a fractional coil and a fractional capacitor (of different orders) has been studied. The time functions of the solution have

been obtained and compared with results obtained by means of a numerical solver (based on the numerical method SubIval). The influence of the α parameter (and, hence, the size of the system) on the computation time has also been studied.

The method will be applied in future studies regarding SubIval, where gcdAlpha allows to obtain referential solutions to problems concerning transient responses of linear fractional circuits.

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