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ANALYSIS OF PERCOLATION CURRENTS OF SELECTED ELECTRIC NETWORKS WITH RANDOM DESTRUCTION OF BOUNDS STRUCTURE

Abstract

Introduction and aim: Analysis of percolation current on electric networks (i.e. triangular, square, and hexagonal) in matrix notation has been presented in this paper. It is a new proposition of matrix analysis for specific electric circuits. The elaborated percolation models were verified on the base of given percolation thresholds.
Material and methods: Some current characteristics for the right networks have been determined in dependence of random shorted bounds. Both analytical and numerical methods in *MathCAD* program were shown in the paper.
Results: If the number of bounds in the network is higher, than the value of percolation current is smaller. It was observed, that the curves of percolation current have the familiar forms and increasing trend. The value of percolation current for given lattice and the same number of loops is dependent from its structure of bounds.
Conclusion: For selected number of shorted bounds of all analysed lattices its decreasing means a decreasing of percolation current value.

Keywords: Percolation current, electric networks, structure of bounds, random bound destruction.
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ANALIZA PRĄDÓW PERKOLACJI WYBRANYCH SIECI ELEKTRYCZNYCH Z LOSOWYM NISZCZENIEM STRUKTURY WIĄZAŃ

Streszczenie

Wstęp i cel: *W niniejszej pracy została pokazana analiza prądu perkolacji na sieciach elektrycznych (tj. trójkątnej, kwadratowej i sześciokątnej). Jest to nowa propozycja analizy macierzowej dla wybranych sieci elektrycznych. Opracowane modele perkolacji zostały zweryfikowane na podstawie podanych progów perkolacji.*
Materiał i metody: *Pewne cechy prądu dla odpowiednich sieci zostały określone w zależności od przypadkowego zwierania oczek sieci. W artykule zostały przedstawione zarówno metoda analityczna jak i numeryczna w programie MathCAD.*
Wyniki: *Jeśli liczba w oczek sieci jest duża, to wartość prądu perkolacji jest mała. Stwierdzono, że krzywe prądu perkolacji mają podobny kształt i trend wzrastający. Zwiększenie związanego liczby powoduje pewne wygładzenie przesączania aktualnej krzywej. Wartość prąd perkolacji dla danej siatki i tej samej liczbie oczek zależy od jej struktury wiązań.*
Wniosek: *Dla wybranej liczby zwartych oczek wszystkich analizowanych sieci jej spadek oznacza obniżenie wartości bieżącej wartości prądu perkolacji.*

Słowa kluczowe: *Prąd perkolacji, sieci elektryczne, struktura wiązań, losowe zniszczenie wiązania.*
(Otrzymano: 01.03.2014; Zrecenzowano:15.10.2014; Zaakceptowano: 30.11.2014)

1. Introduction

Percolation theory *(lat. percolare – to percolate)* contains some statistical and geometrical models. It was created by mathematician J.M. Hammersley in 1957 [9]. Percolation theory is used for description of disordered systems and situations with a stochastic geometry [4].

In disordered systems with an increase of interactions, density, packing or concentration suddenly occur a long-term ranges. Sudden occurring of the long-term ranges is defined by a percolation transition.

Moreover bound occurring is defined by a probability p for $0 \le p \le 1$. During increase of p suddenly occurs a percolation threshold p_c, which means some existing of unlimited and expanded percolation cluster.

In the other hand the percolation cluster means a set of bounds stability connected with adjacent ones.

The percolation model is defined by the percolation threshold accordingly to the bounds p_c. Percolation theory in details is described in literature [10]-[12].

The authors have not found in literature the analysis of percolation currents of electric networks with the random destruction of bounds structure in matrix notation.

Thus the main aim of the paper is:

- to define some percolation models on triangular, square and hexagonal electric networks;
- to make some algorithms of random bounds destruction;
- to determinate some percolation currents of electric networks with random bounds destruction;
- to determinate some percolation thresholds for selected models on electric networks;
- to comparison of some percolation current curves.

2. Analytical model of percolation on electric networks in matrix notation

2.1. Definition of percolation threshold

Percolation threshold p_c of a model created on electric network during of bounds shorting is defined by the following formula :

$$p_c \equiv \frac{\sum_{i=1}^{m} N1_1}{Z_{w1} + \sum_{i=1}^{n} N_i} \qquad (1)$$

where $N1_i$ means the number of network sorted bounds ($1 \le i \le n$), Z_{w1} - one bound of inner impedance of voltage source, N_i - the number of network bounds ($1 \le i \le m < n$) for $n, m \in$ N.

During a bound shorting of network with applicable forced voltage suddenly occurs some percolation threshold.

The specific quality of percolation threshold (1) is sudden increase of current value, which tends to infinity.

2.2. General model of percolation on electric networks

The structure of percolation model shows the matrix equation [1]-[8]:

$$\mathbf{Z_k} \cdot \mathbf{I_k} = \mathbf{E_k} \qquad (2)$$

where the symbol $\mathbf{Z_k}$ means the matrix of mesh impedance with resistance character of percolation model, which describes the bound structure of electric network, $\mathbf{I_k}$ – one-column matrix created by the vector of mesh currents in percolation model on electric network, $\mathbf{E_k}$ – one-column matrix created by the vector of mesh electromotive forces in percolation model on electric network, Index **k** presents the name of the networks type (i.e. t – for triangular, s – for square and h – for hexagonal).

The matrices $\mathbf{Z_k}$, $\mathbf{I_k}$ and $\mathbf{E_k}$ are defined by the following formulae:

$$\mathbf{Z}_k \equiv \begin{bmatrix} Z_{1,1} & -Z_{1,2} & \cdots & -Z_{1,i} & \cdots & -Z_{1,n} \\ -Z_{2,1} & Z_{2,2} & \cdots & -Z_{2,i} & \cdots & -Z_{2,n} \\ \vdots & \vdots & & \vdots & & \\ -Z_{i,1} & -Z_{i,2} & \cdots & Z_{i,i} & \cdots & -Z_{i,n} \\ \vdots & & & \vdots & & \vdots \\ -Z_{n,1} & -Z_{n,2} & \cdots & -Z_{n,i} & \cdots & Z_{n,n} \end{bmatrix}_{n\times n} , \tag{3}$$

$$\mathbf{I}_k \equiv \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_i \\ \vdots \\ I_n \end{bmatrix}_{n\times 1} , \tag{4}$$

$$\mathbf{E}_k \equiv \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ \vdots \\ E_n \end{bmatrix}_{n\times 1} . \tag{5}$$

Left-sided multiplying the equation (2) by the inverse matrix $\mathbf{Z_k}^{-1}$ to impedance mesh matrix $\mathbf{Z_k}$ is obtained the following matrix equation:

$$(\mathbf{Z_k}^{-1})_{n\times n} \cdot (\mathbf{Z_k})_{n\times n} \cdot (\mathbf{I_k})_{n\times 1} = (\mathbf{Z_k}^{-1})_{n\times n} \cdot (\mathbf{E_k})_{n\times 1} . \tag{6}$$

Taking into account the matrix properties:

$$\mathbf{Z}_k^{-1} \cdot \mathbf{Z_k} = \mathbb{I} \text{ and } \mathbb{I} \cdot \mathbf{I_k} = \mathbf{I_k} , \tag{7}$$

where the symbol $\mathbb{I}$ means the identity matrix is obtained the one-column matrix of mesh currents in the following form:

$$(\mathbf{I_k})_{n\times 1} = (\mathbf{Z_k}^{-1})_{n\times n} \cdot (\mathbf{E_k})_{n\times 1}. \tag{8}$$

Bounds shorting of electric network (i.e. the value of resistance bound equals to zero) was done by using random method.

A random method of bounds destruction refers to assign them some numbers and their transformation into sequence of random numbers, which define the order of bound breaking. The random numbers were chosen by using the right procedure in *MathCAD* program.

The sequence of random numbers creates the vector of shorted bounds number $\mathbf{N_k}$. Mesh currents of electric network, which create a one-column matrix $\mathbf{I_k(N_k)}$, are described by the following matrix equation:

$$[\mathbf{I_k}(\mathbf{N_k})]_{n\times 1} = [\mathbf{Z_k}^{-1}(\mathbf{N_k})]_{n\times n} \cdot (\mathbf{E_k})_{n\times 1}. \tag{9}$$

Percolation current $\mathbf{I_k}$ of electric network is equal to mesh current, which refers to the first row $\mathbf{I_1}$ for vector of mesh currents $\mathbf{I_k(N_k)}$.

Percolation currents I_t, I_s and I_h for triangular (i.e. t), square (i.e. s) and hexagonal (i.e. h) lattice are shown on the figures 4, 12 and 20.

Defining the one-column matrix $\mathbf{X}$ of type

$$\mathbf{X} \equiv \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \tag{10}$$

it is possible to obtain the one-row transpose matrix $\mathbf{X^T}$ which has the following form:

$$\mathbf{X^T} \equiv [1,0,\ldots,0]_{1\times n} \,. \tag{11}$$

Left-sided multiplying the matrix equation (9) by the matrix (11) it is obtained the percolation current $\mathbf{I_k}$ in the following matrix notation:

$$(\mathbf{X^T})_{1\times n} \cdot [\mathbf{I_k}(\mathbf{N_k})]_{n\times 1} = (\mathbf{X^T})_{1\times n} \cdot [\mathbf{Z_k}^{-1}(\mathbf{N_k})]_{n\times n} \cdot (\mathbf{E_k})_{n\times 1}. \tag{12}$$

From the formula (12) we have the following form of percolation current:

$$[\mathbf{I_k}(\mathbf{N_k})]_{1\times 1} = [\mathbf{X^T} \cdot \mathbf{Z_k}^{-1}(\mathbf{N_k}) \cdot \mathbf{E_k}]_{1\times 1}. \tag{13}$$

3. Analytical and numerical selected models of percolation on electric networks

3.1. Model of percolation on triangular lattice

Percolation current $\mathbf{I_t}$ for triangular lattice was calculated from matrix equation (13), which has the following form:

$$\mathbf{I_t}(\mathbf{N_t}) = \mathbf{X^T} \cdot [\mathbf{Z_t}(\mathbf{N_t})]^{-1} \cdot \mathbf{E_t} \tag{14}$$

where $\mathbf{Z_t}$ means the mesh impedance matrix with resistance characteristic, $\mathbf{N_t}$ – number of shorted bounds after the algorithms T1, T2 and T3 (Figs. 1-3), $\mathbf{E_t}$ – vector of mesh electromotive forces for triangular lattice (14).

Percolation currents for triangular lattice obtained form the formula (14) after the algorithms T1, T2 and T3 are illustrated on the figures 5, 6 and 7. Moreover the figure 8 shows the mean value of specified percolation currents for triangular lattice.

Percolation currents obtained after the algorithms T1, T2 and T2 (Figs. 1-3), presenting the various random sequences of bounds shorting, have the familiar form and quantity (Fig. 5-7).

The value of percolation threshold obtained from the formula (1) for triangular lattice (Fig. 4) is equal to p_c = 0,3333.

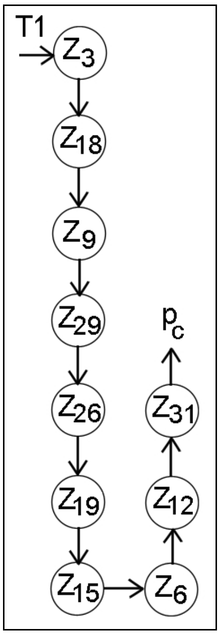

Fig. 1. Algorithm T1

Source: Elaboration of the Authors

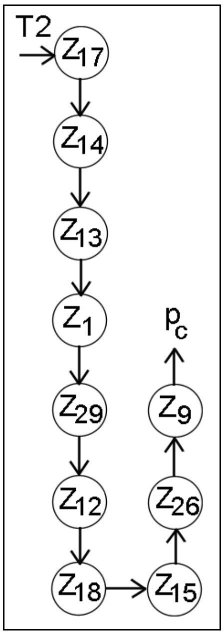

Fig. 2. Algorithm T2

Source: Elaboration of the Authors

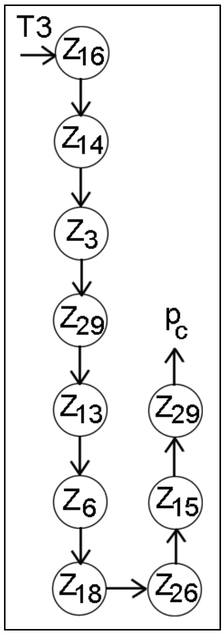

Fig. 3. Algorithm T3

Source: Elaboration of the Authors

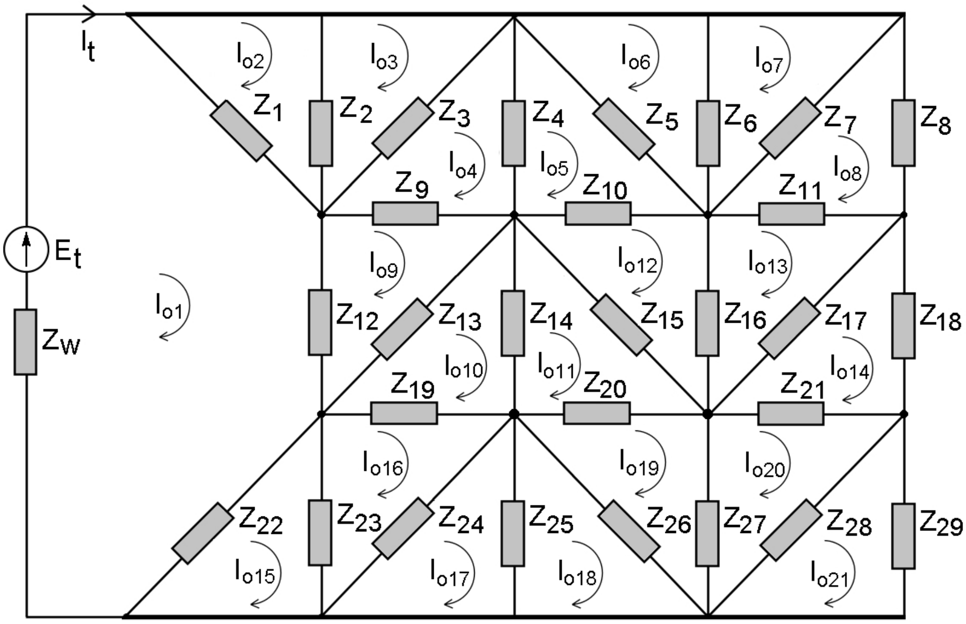

Fig. 4. Model of percolation on triangular lattice with 21 meshes (cells):
$\mathbf{E_t}$ – electromotive force, $\mathbf{I_t}$ – percolation current
Source: Elaboration of the Authors

If the number of shorted bounds in triangular lattice increases, than suddenly occurs a percolation threshold. This fact means that percolation current on triangular lattice in percolation threshold tends to infinity.

The figure 8 shows the mean value of percolation current on triangular lattice, which was calculated in accordance with the algorithms T1, T2 and T3. Form and quantity of a mean value of percolation current on triangular lattice may be compared with forms and quantities of lattices with other bound structure.

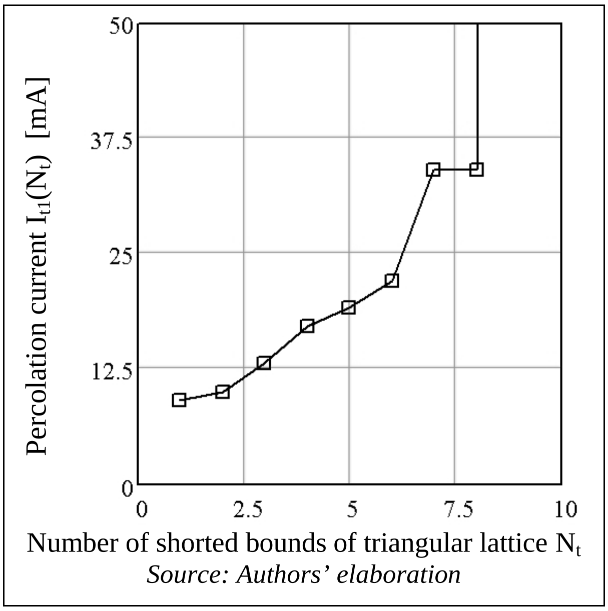

Fig. 5. Percolation current $I_{t1}(N_t)$ on triangular lattice calculated after the algorithm T1
Source: Elaboration of the Authors

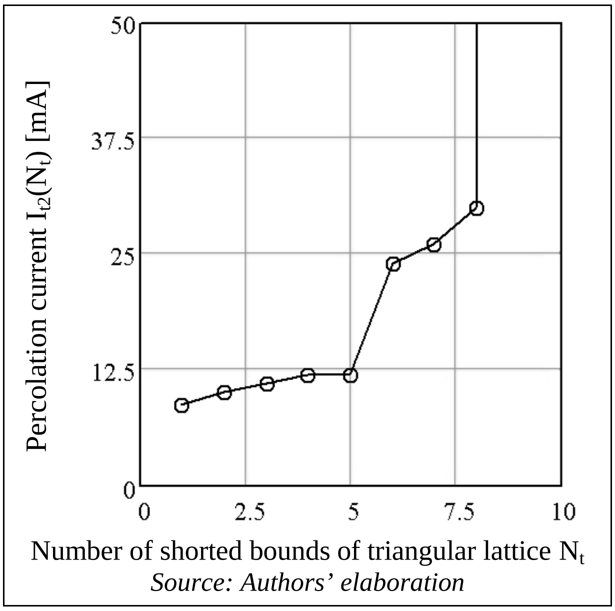

Fig. 6. Percolation current $I_{t2}(N_t)$ on triangular lattice calculated after the algorithm T2
Source: Elaboration of the Authors

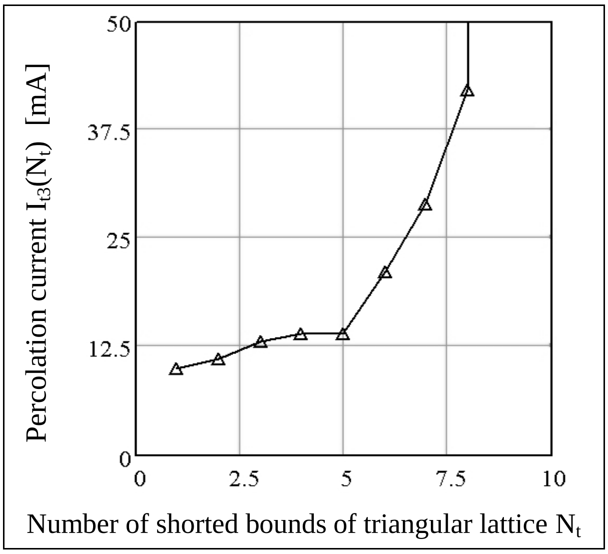

Fig. 7. Percolation current $I_{t3}(N_t)$ on triangular lattice calculated after the algorithm T3
Source: Elaboration of the Authors

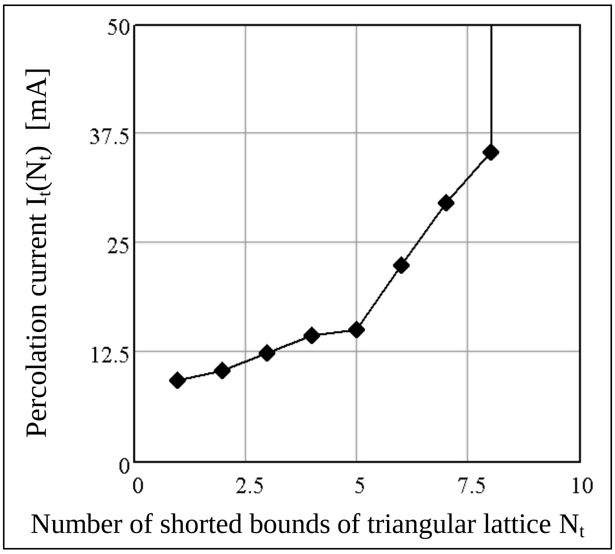

Fig. 8. Mean value $I_t(N_t)$ of percolation current on triangular lattice
Source: Elaboration of the Authors

3.2. Model of percolation on square lattice

Percolation current $\mathbf{I_s}$ for square lattice was calculated from matrix equation (13), which has the following form:

$$\mathbf{I_s(N_s)} = \mathbf{X^T} \cdot [\mathbf{Z_s(N_s)}]^{-1} \cdot \mathbf{E_s} \tag{15}$$

where $\mathbf{Z_s}$ means the mesh impedance matrix with resistance characteristic, $\mathbf{N_s}$ – number of shorted bounds after the algorithms S1, S2 and S3 (Figs. 9-11), $\mathbf{E_s}$ – vector of mesh electromotive forces for square lattice (15).

Percolation currents for square lattice obtained form the formula (15) after the algorithms S1, S2 and S3 are illustrated on the figures 13, 14 and 15. Moreover the figure 16 shows the mean value of specified percolation currents for square lattice.

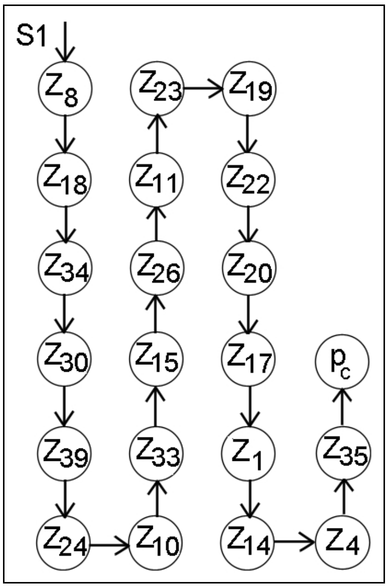

Fig. 9. Algorithm S1
Source: Elaboration of the Authors

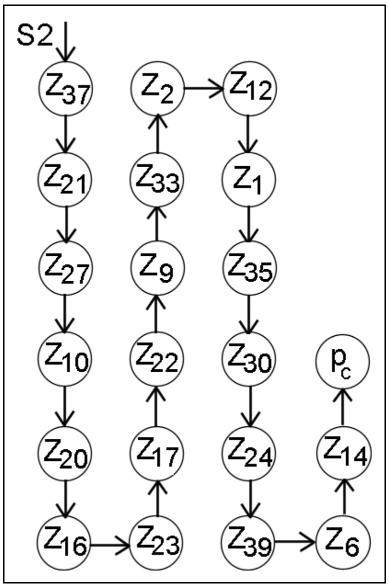

Fig. 10. Algorithm S2
Source: Elaboration of the Authors

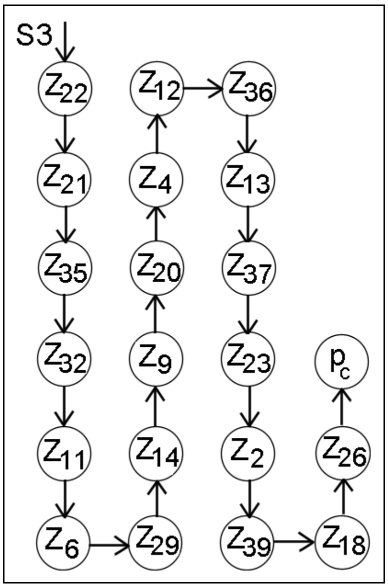

Fig. 11. Algorithm S3
Source: Elaboration of the Authors

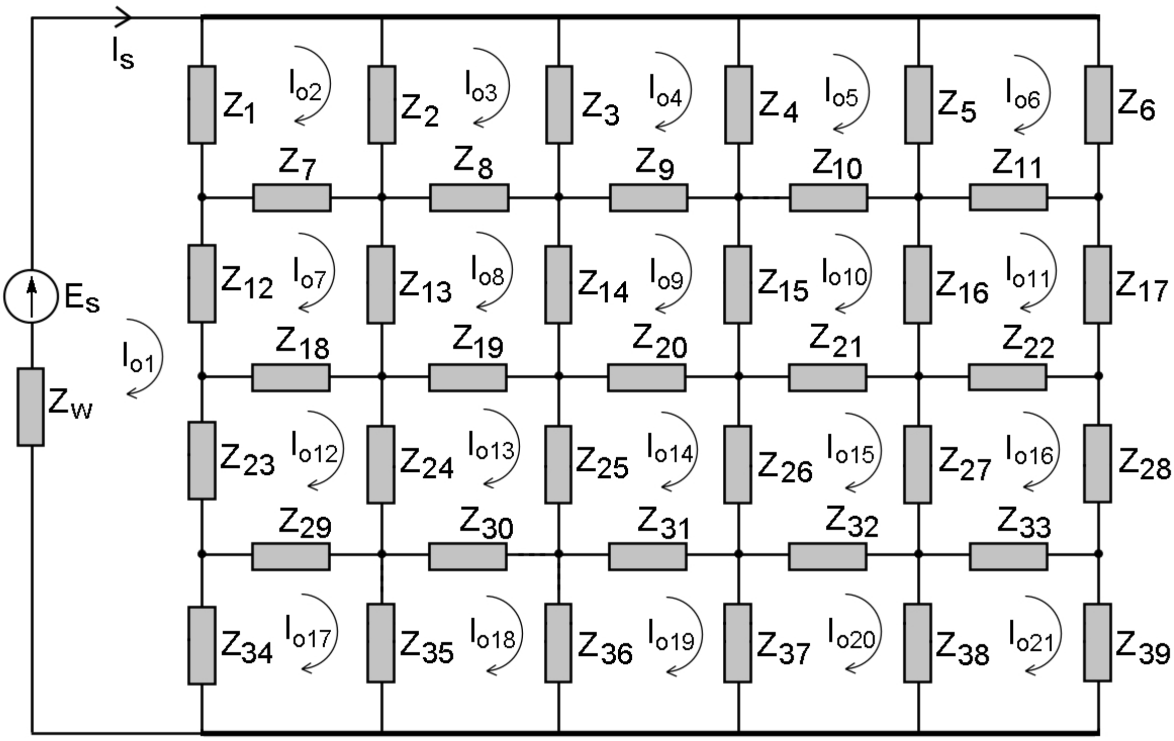

Fig. 12. Model of percolation on square lattice with 21 meshes (cells):
$\mathbf{E_s}$ – electromotive force, $\mathbf{I_s}$ – percolation current
Source: Elaboration of the Authors

Percolation currents obtained after the algorithms S1, S2 and S2 (Figs. 9-11) presenting the various random sequences of bounds shorting, have the familiar form and quantity (Figs. 13-15). The value of percolation threshold obtained from the formula (1) for square lattice (Fig. 12) is equal to p_c = 0,5.

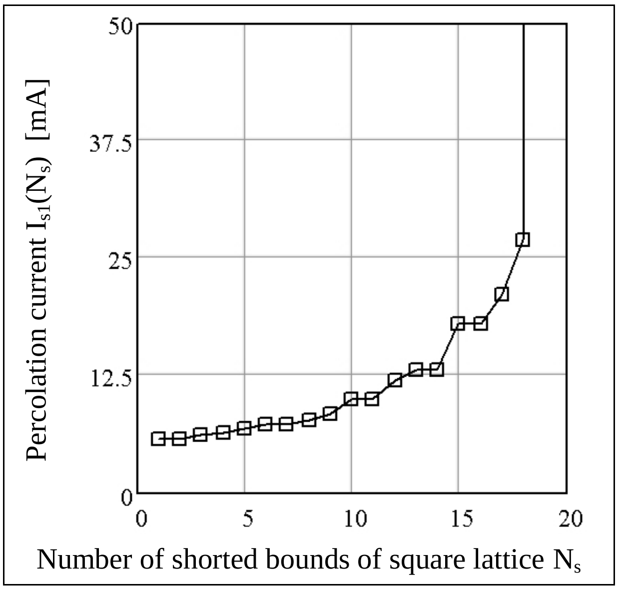

Fig. 13. Percolation current $I_{s1}(N_s)$ on square lattice calculated after the algorithm S1
Source: Elaboration of the Authors

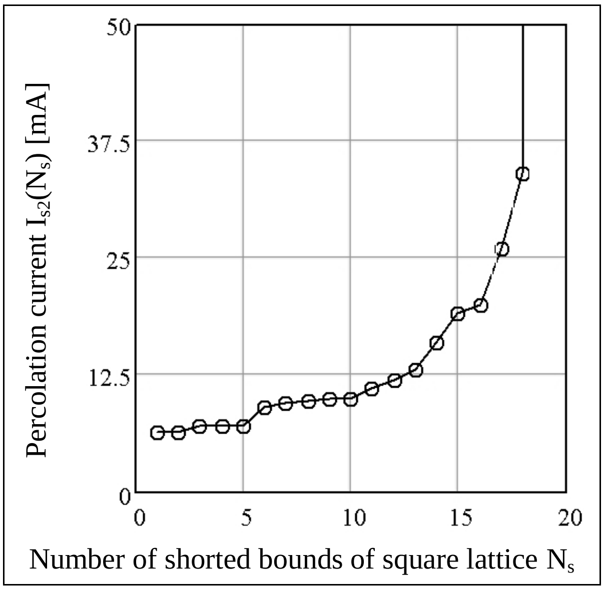

Fig. 14. Percolation current $I_{s2}(N_s)$ on square lattice calculated after the algorithm S2
Source: Elaboration of the Authors

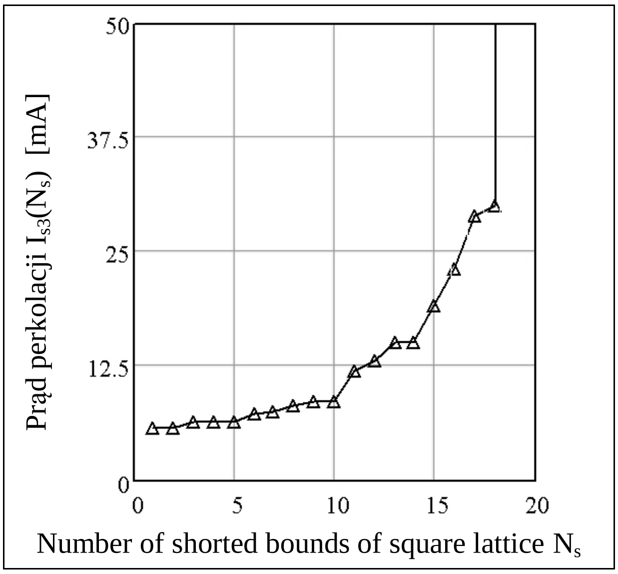

Fig. 15. Percolation current $I_{s3}(N_s)$ on square lattice calculated after the algorithm S3
Source: Elaboration of the Authors

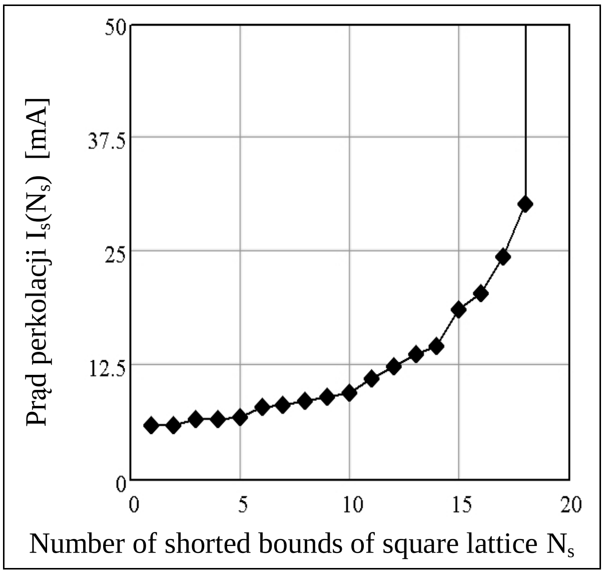

Fig. 16. Mean value $I_s(N_s)$ of percolation current on square lattice
Source: Elaboration of the Authors

If the number of shorted bounds in square lattice increases, then suddenly occurs a percolation threshold. This fact means that percolation current on square lattice in percolation threshold tends to infinity. The figure 16 shows the mean value of percolation current on square lattice, which was calculated in accordance with the algorithms S1, S2 and S3. Form and quantity of a mean value of percolation current on square lattice may be compared with forms and quantities of lattices with other bound structure.

3.3. Model of percolation on hexagonal lattice

Percolation current $\mathbf{I_h}$ for hexagonal lattice was calculated from matrix equation (13), which has the following form:

$$\mathbf{I_h}(\mathbf{N_h}) = \mathbf{X}^{\mathbf{T}} \cdot [\mathbf{Z_h}(\mathbf{N_h})]^{-1} \cdot \mathbf{E_h} \tag{16}$$

where $\mathbf{Z_h}$ means the mesh impedance matrix with resistance characteristic, $\mathbf{N_h}$ – number of shorted bounds after the algorithms H1, H2 and H3 (Figs. 17-19), $\mathbf{E_h}$ – vector of mesh electromotive forces for hexagonal lattice (16).

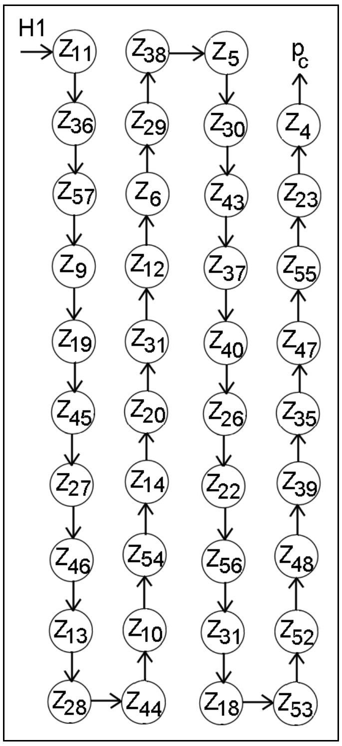

Fig. 17. Algorithm H1
Source: Elaboration of the Authors

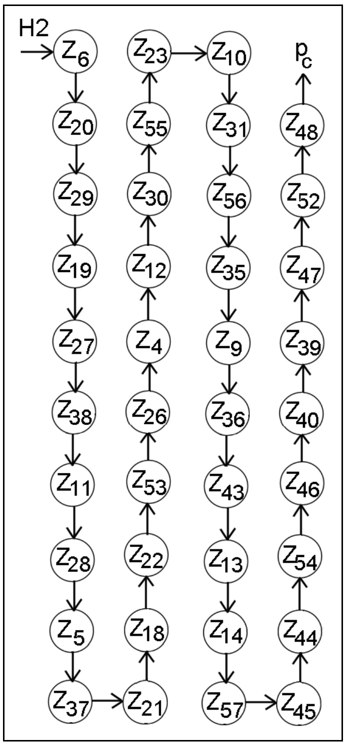

Fig. 18. Algorithm H2
Source: Elaboration of the Authors

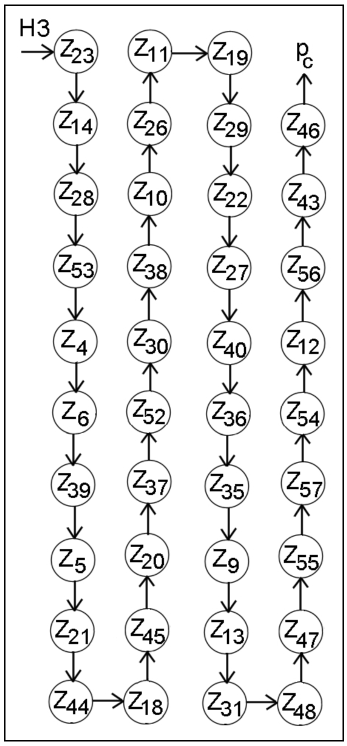

Fig. 19. Algorithm H3
Source: Elaboration of the Authors

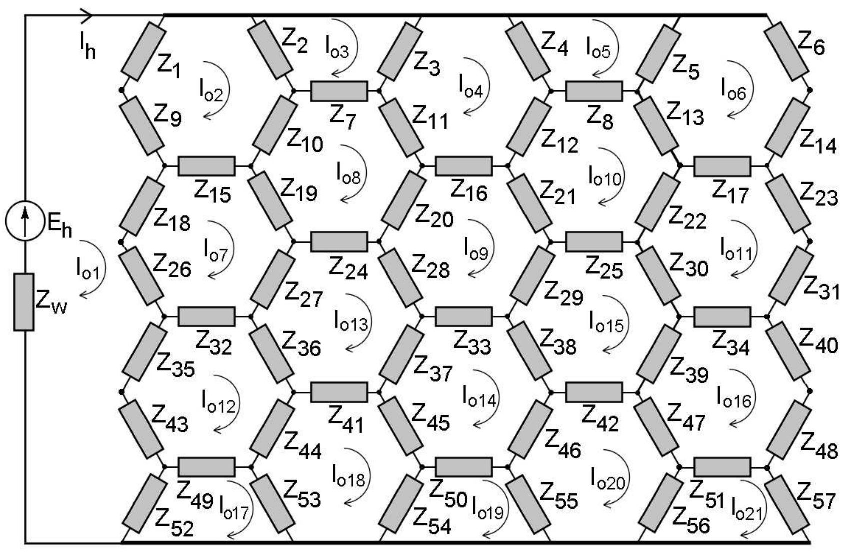

Fig. 20. Model of percolation on hexagonal lattice with 21 meshes (cells):
$\mathbf{E_{th}}$ – electromotive force, $\mathbf{I_h}$ – percolation current
Source: Elaboration of the Authors

Percolation currents for hexagonal lattice obtained form the formula (16) after the algorithms H1, H2 and H3 are illustrated on the figures 21, 22 and 23. Moreover the figure 24 shows the mean value of specified percolation currents for hexagonal lattice.

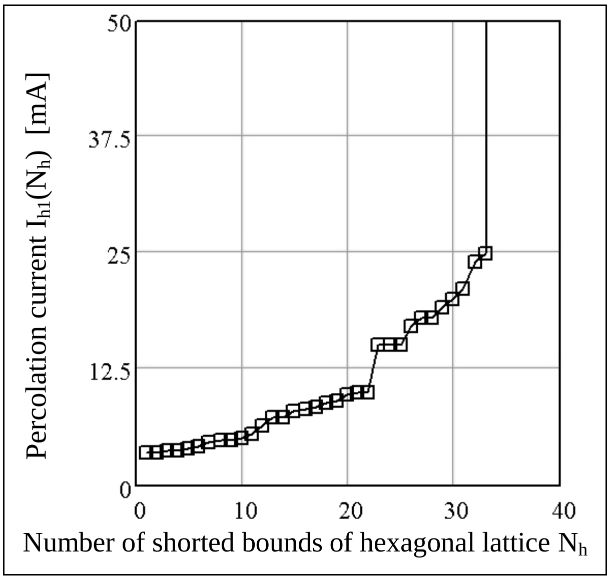

Fig. 21. Percolation current $I_{h1}(N_h)$ on hexagonal lattice calculated after the algorithm H1
Source: Elaboration of the Authors

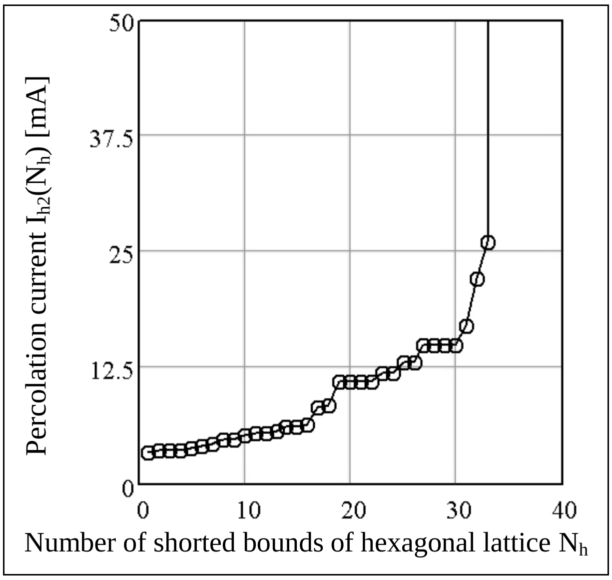

Fig. 22. Percolation current $I_{h2}(N_h)$ on hexagonal lattice calculated after the algorithm H2
Source: Elaboration of the Authors

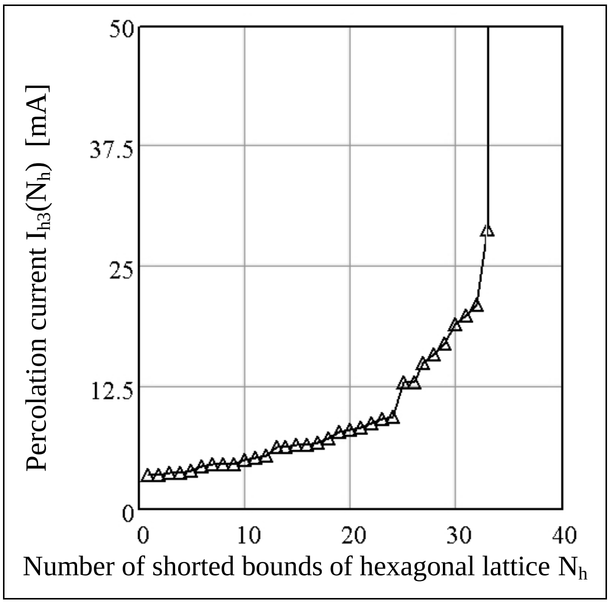

Fig. 23. Percolation current $I_{h3}(N_h)$ on hexagonal lattice calculated after the algorithm H3
Source: Elaboration of the Authors

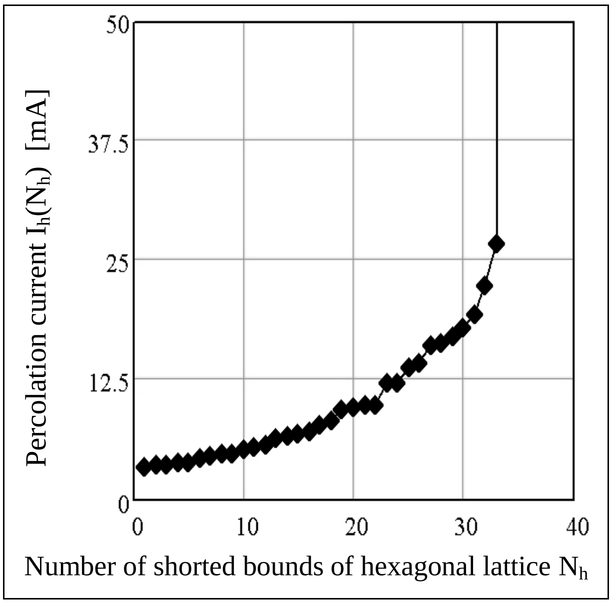

Fig. 24. Mean values $I_h(N_h)$ of percolation current on hexagonal lattice
Source: Elaboration of the Authors

Percolation currents obtained after the algorithms H1, H2 and H2 (Figs. 17-19) presenting the various random sequences of bounds shorting, have the familiar form and quantity (Figs. 21-24). The value of percolation threshold obtained from the formula (1) for hexagonal lattice (Fig. 20) is equal to $p_c = 0{,}672$.

If the number of shorted bounds in hexagonal lattice increases, than suddenly occurs a percolation threshold. This fact means that percolation current on hexagonal lattice in percolation threshold tends to infinity.

The figure 24 shows the mean value of percolation current on hexagonal lattice, which was calculated in accordance with the algorithms H1, H2 and H3.

Form and quantity of a mean value of percolation current on hexagonal lattice may be compared with forms and quantities of lattices with other bound structure.

4. Verification of simulation results

According to results for simulation of created percolation models on various lattices were calculated the percolation thresholds after the formula (1) and were shown in the table 1.

Table 1. Values of percolation thresholds calculated for selected networks according to equation (1)

Symbol	Kind of lattice	Dimension d	Co-ordinating number q	Percolation threshold p_c in accordance with	
				Reference [13]	Authors
t	triangular	2	6	0,3473	0,3333
s	squared	2	4	0,5000	0,5000
h	hexagonal	2	3	0,6527	0,6720

5. Conclusions

- Created percolation models on electric networks (i.e. triangular, square, and hexagonal) were verified according to percolation thresholds, which were analytical calculated using the formula (1) and were compared with the percolation thresholds calculated by using the other methods.
- Elaborated percolation models on electric networks (i.e. triangular, square, and hexagonal) in matrix notation with random destruction of bounds are some new proposition of matrix analysis for specific electric circuits.
- The value of percolation current for given network increases during the increase of number of shorted bounds. Moreover in the percolation threshold the value of current extremely increases tending to infinity. For the various electric networks the characteristics of percolation current have the similar form and increasing trend.
- The value of percolation current for given lattice and the same number of loops (i.e. cells) is dependent from its structure of bounds (i.e. triangular, square or hexagonal lattice). The more bounds have some lattice, the less initial percolation current characterises analysed one. For selected number of shorted bounds and for all analysed lattices, decreasing of co-ordinating number means some decreasing of current value (Table 1).

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