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ENERGY CONSUMPTION IN MECHANICAL SYSTEMS USING A CERTAIN NONLINEAR DEGENERATE MODEL

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In modern engineering materials used for creating effective ballistic shields, the issues of evaluation of their energy consumption are extremely important. The paper presents a new way of solving this problem using a certain degenerate model with dry friction. This method involves the use of specially derived identification equations which describe the decrease in potential energy of the system during its vibratory motion induced by a single pulse load. Analytical considerations have been verified using a computer simulation technique for selected examples.

Key words: composite ballistic shield, modeling, degenerated model, dissipation, impact energy

1. Introduction

The amount of the energy that is absorbed in mechanical vibrating systems is usually described by a single parameter that is related to the adopted model in form of pure viscous friction (the friction that is proportional to the rate of deformation of the material) or so-called dry friction. In the case of the ballistic impact on shields which are made of modern (lightweight) materials, this issue becomes very complicated, because of complex and time-related strains that occur during penetration of the projectile into the shield. The pierced material is subjected to varying degrees of shearing, tension and compression, which depend primarily on the impact velocity as well as on the shape and mechanical properties of the shield and the projectile. This issue was comprehensively discussed in Bourke's (2007) studies. The problem of dissipation of kinetic energy of the projectile in multilayer materials, such as laminates and composites, is currently analyzed by many researchers, such as Abrate (1998, 2010), Sanchez-Galvez *et al.* (2005), García-Castillo *et al.* (2012), Sidney *et al.* (2011), Hou *et al.* (2010), Katz *et al.* (2008). The mathematical approach to dissipation of the impact energy based on these works is an original approach to the problem. Tabiei and Nilakantan (2008) presented the in-depth description of this phenomena through the analysis of the literature on the subject matter. The authors have synthetically presented the previous areas of the research conducted by worldwide scientists. In the papers by Jamroziak and Bocian (2008), Kulisiewicz *et al.* (2008), Jamroziak *et al.* (2010), the issue of dissipation of the impact energy was presented using the degenerate models.

It is generally assumed that the work A done by the projectile during the process of piercing may be described by drop of its kinetic energy E, starting from the zero position (the impact velocity) up to the moment it leaves the shield or stops in the shield. The relations that describe piercing are derived on the basis of the *a priori* assumed models of the constitutive relations (stress–strain relations) which are highly complex for this type of materials, as it can be seen in the papers by Jach *et al.* (2004), Rusiński *et al.* (2005), Buchmayr *et al.* (2008). Some of the assumptions which are adopted for the description of these models were also included in the paper by Iluk (2012). Buchacz and his team have been conducting a long-time research aiming to develop a mathematical algorithm of analysis and synthesis of simple and complex mechanical and mechatronic systems. To realize these tasks, different categories of graphs and structural numbers were proposed by Białas (2008), Buchacz (1995), Buchacz and Wojnarowski (1995), Buchacz and Płaczek (2009). The studies included also computer-aided methods of realizing these tasks, Buchacz (2005). Vibrating mechatronic systems with piezoelectric transducers used to damp or induce vibrations were modelled and analysed in the papers by Baier and Lubczyński (2009), Buchacz and Wróbel (2010). The aim of these works was to identify the optimal (according to the adopted criteria) mathematical model of the analysed systems as well as to develop mathematical tools useful to analyse these systems using approximate methods, see Białas (2010, 2012), Buchacz and Płaczek (2010), Wróbel (2012), Żółkiewski (2010, 2011) and the paper by Kulisiewicz *et al.* (2001) presenting the balance methods. Most of the hypotheses concerning the problem of dissipation of the impact energy take as the starting point the well known law of conservation of energy. This analysis was presented in the papers by Włodarczyk (2006), Włodarczyk and Jackowski (2008), Carlucci and Jacobson (2008).

2. Formulation of the problem

The basic assumption made by the authors is that the lightweight shield acts on the piercing mass m with the resisting force S, whoose functional form is based on the mathematical analysis of the dynamic model. This paper assumes the model presented on the scheme shown in Fig. 1b.

The standard model consists of the Maxwell element in parallel configuration with a purely elastic element c and the element h that describes dry friction. It may be noted that the adoption of the constant $c_0 \to \infty$ in this system results in obtaining the widely used dynamic model which describes the vibrations of one-degree-of-freedom mechanical systems with dry friction (Fig. 1a). Similarly, if $c = 0$, the obtained model takes form of the purely Maxwell element in parallel configuration with the element h. In this sense, the system presented in Fig. 1b is the universal model, which should accurately describe the mechanical properties of many modern construction materials. The introduction of the element h of dry friction in both models has been based on the results of the previous research of the authors, which concerned the impact process. This research was presented in the papers by Bocian *et al.* (2009), Jamroziak *et al.* (2009), Jamroziak and Bocian (2010).

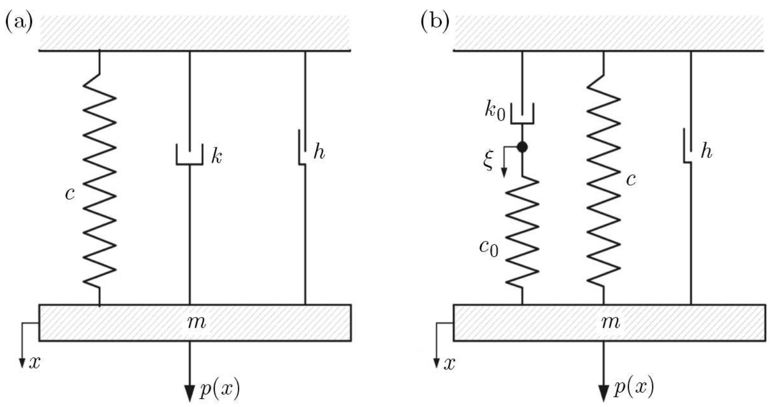

Fig. 1. The scheme of the analyzed dynamic models: (a) typical model with dry friction; (b) standard model (Zener model) with dry friction

3. Energy consumption according to the classic model with dry friction

The commonly used method for assessing the energy losses in dynamic and vibrating systems is based on the concept of damping decrement. This parameter describes the amplitude decay rate of free vibrations in linear systems with viscous damping. In the case when the damping is non-linear (as it is in the system shown in Fig. 1a), the convenient measure of the vibrations decay can take form of the decrease of potential energy of the deformed part of the system (e.g. shield), which is observed during damped vibrations. In the case of oscillations induced by a single pulse (impact), the typical shape of the system takes form shown in Fig. 2.

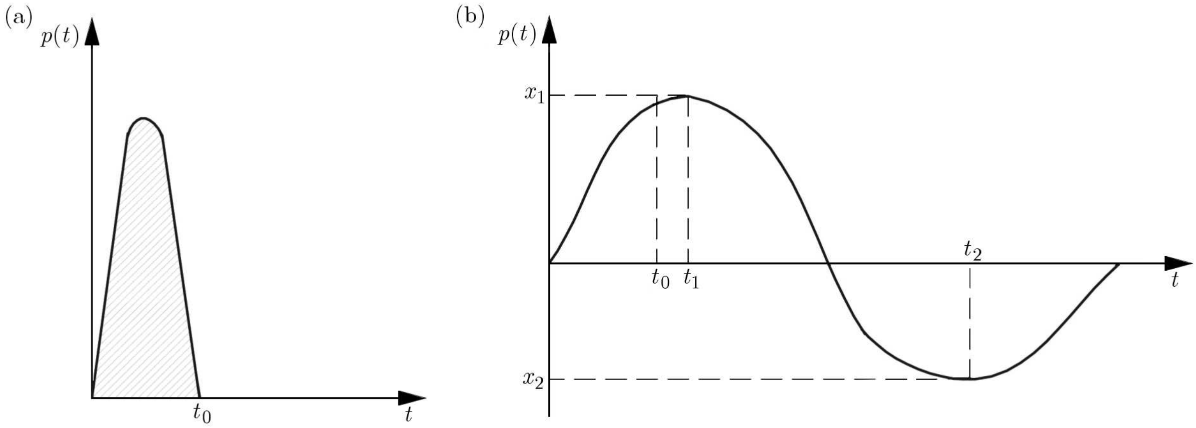

Fig. 2. The typical shape of the system response to the pulse load; (a) pulse $p = p(t)$, (b) $x(t)$ diagram

As it can be noted, at the time points t_1 and t_2, the displacement $x(t)$ reaches the extreme values. At these points of time, the velocity $v(t)$ must have the value of zero, that is

$$v(t_1) = v(t_2) = 0 \tag{3.1}$$

which follows directly from the definition of velocity $(v = dx/dt)$.

In the case of the system shown in Fig. 1a, motion of the mass m is described by the differential equation

$$m\ddot{x} + k\dot{x} + h\,\mathrm{sgn}\,\dot{x} + cx = p(t) \tag{3.2}$$

By multiplying the above equation by the elementary displacement $dx = \dot{x}dt$ and integrating it in the time period $t \in (t_1, t_2)$, we obtain the following sequence

$$\begin{aligned}
&m\int_{t_1}^{t_2} \ddot{x}\dot{x}\,dt = \int_{t_1}^{t_2} \frac{dv}{dt}v\,dt = m\frac{v^2}{2}\Big|_{v(t_2)}^{v(t_2)} = 0 \qquad && k\int_{x(t_1)}^{x(t_2)} \dot{x}\,dx = k\int_{x(t_1)}^{x(t_2)} v\,dx = k\int_{t_1}^{t_2} v^2\,dt = k\beta_x^v \\
&h\int_{x(t_1)}^{x(t_2)} \mathrm{sgn}\,\dot{x}\,dx = h(-1)\int_{x_1}^{x_2} dx = h(x_1 - x_2) && \int_{t_1}^{t_2} p(t)\dot{x}\,dt = \int_{t_1}^{t_2} 0\dot{x}\,dt = 0 \\
&c\int_{t_1}^{t_2} x\dot{x}\,dt = c\int_{x_1}^{x_2} x\,dx = c\frac{x^2}{2}\Big|_{x_1}^{x_2} = \frac{-c}{2}(x_1^2 - x_2^2)
\end{aligned} \tag{3.3}$$

It can be seen that the last integral (3.3) must be equal to zero for a pulse of any shape as long as the influence of the pulse force ends at the moment $t_0 < t_1$. Such a situation is the most common case in practice and it can be easily checked on appropriate time graphs. Taking into account results (3.3), a relation in the following form is obtained for equation (3.2)

$$k\beta_x^v + h(x_1 - x_2) = \frac{c}{2}(x_1^2 - x_2^2) \qquad \beta_x^v = \int_{t_1}^{t_2} v^2\ dt > 0 \tag{3.4}$$

It can be seen that β_x^v is equal to the field limited by the relation $v(x)$, which is the part of the phase trajectory of the pulse load system in the time interval $\Delta t = t_2 - t_1$ (Fig. 3).

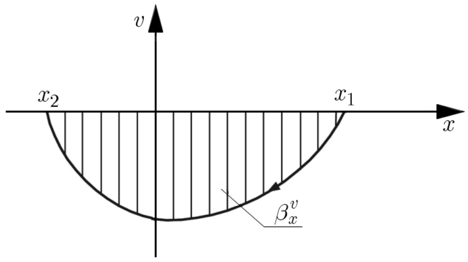

Fig. 3. Typical shape of the phase trajectory of the analyzed system in the time interval Δt

4. Energy consumption according to the Zener model with dry friction

The movement of the mass m, in the case of the model shown in Fig. 1b, can be written with a single third-order differential equation of the form

$$m\ddot{x} + cx + h\,\mathrm{sgn}\,\dot{x} + \frac{k_0}{c_0}\Big[m\dddot{x} + (c_0 + c)\dot{x} - \dot{p} + h\frac{d}{dt}(\,\mathrm{sgn}\,\dot{x})\Big] = p(t) \tag{4.1}$$

Using the same transformations as it has been done in the previous model, we obtain the following sequence

$$\begin{aligned} &\frac{k_0}{c_0}\int_{t_1}^{t_2} \dddot{x} v\ dt = \frac{k_0 m}{c_0}\int_{t_1}^{t_2}\frac{da}{dt}v\ dt = \frac{k_0 m}{c_0}\beta_a^v && \frac{k_0 h}{c_0}\int_{t_1}^{t_2}\frac{d}{dt}(\,\mathrm{sgn}\,\dot{x})\dot{x}\ dt = 0 \\ &\frac{k_0(c_0+c)}{c_0}\int_{t_1}^{t_2}\dot{x}\dot{x}\ dt = \frac{k_0(c_0+c)}{c_0}\int_{t_1}^{t_2} v^2\ dt = \frac{k_0(c_0+c)}{c_0}\beta_x^v && -\frac{k_0}{c_0}\int_{t_1}^{t_2}\dot{p}\dot{x}\ dt = 0 \end{aligned} \tag{4.2}$$

The above results, after summing up and taking into account the similar components derived from the analysis of the previous model, give us an equation of the form

$$\frac{k_0(c_0+c)}{c_0}\beta_x^v + h(x_1 - x_2) + \frac{k_0 m}{c_0}\beta_a^v = \frac{c}{2}(x_1^2 - x_2^2) \tag{4.3}$$

As it can be seen, the equation that has been derived is a bit different from the similar equation used in the previous model (cf. $(3.4)_1$). It may be noted that, although the first components are positive, the third component must be less than zero. This is because the variable β_a^v is of the form

$$\beta_a^v = \int_{a(t_1)}^{a(t_2)} v\ da = \int_{t_1}^{t_2}\dddot{x} v\ dt = 0 \tag{4.4}$$

Integrating the above integral by parts, gives

$$\beta_a^v = \ddot{x}v\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\ddot{x}\dot{v}\ dt = -\int_{t_1}^{t_2} a^2\ dt < 0 \tag{4.5}$$

In addition, for $c_0 \to \infty$, equation (4.3) is identical to equation $(3.4)_1$. Because of that, the energy losses described by equation (4.3) are more complete and, therefore, this equation should be used in practice. Some exemplary results of computer simulations are described below.

5. The simulations

Looking for solutions to differential equations (3.2) and (4.1), which describe vibrations of the analyzed models using the Mathematica software, simulation studies were performed. The following values have been substituted into equations $(3.3)_{2,3}$ and $(3.3)_5$ of model 1a and into equations $(4.2)_{1,2}$ and $(4.2)_3$ of model 1b:

- Model 1a: $k = 480\,\mathrm{kg/s}$, $c = 30000\,\mathrm{kg/s^2}$, $m = 40\,\mathrm{kg}$, $h = 5$
- Model 1b: $k = 480\,\mathrm{kg/s}$, $c = 30000\,\mathrm{kg/s^2}$, $c_0 = 20000\,\mathrm{kg/s^2}$, $m = 40\,\mathrm{kg}$, $k_0 = 406\,\mathrm{kg/s}$, $h = 5$.

Each case included simulation of the pulse load in the form of:

– for $t < 0.1\,\mathrm{s}$

– where the force $p(t)$ has been modeled by function $p(t) = A\sin(10\pi t)$.

Responses of the models for pulse loads and different parameters are illustrated in the following figures (Figs. 4–7). Examples of the applied exciting force $p(t)$ are shown in Fig. 4. The pulses were one-sided and their assumed time of duration was equal to $t_0 = 0.1\,\mathrm{s}$. Examples of the obtained responses are shown in Fig. 5 for model (a) and in Fig. 6 for model (b). The phase trajectories for both models are presented in Fig. 7.

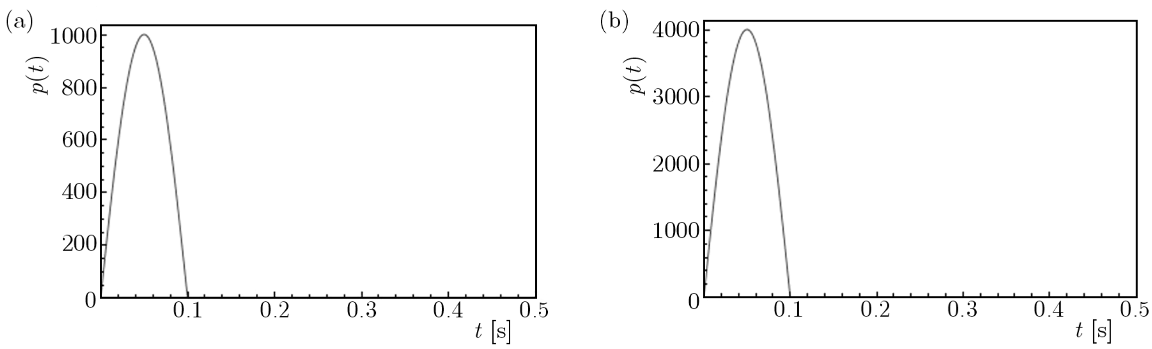

Fig. 4. Pulse loads $p(t)$: (a) for model 1a, (b) for model 1b

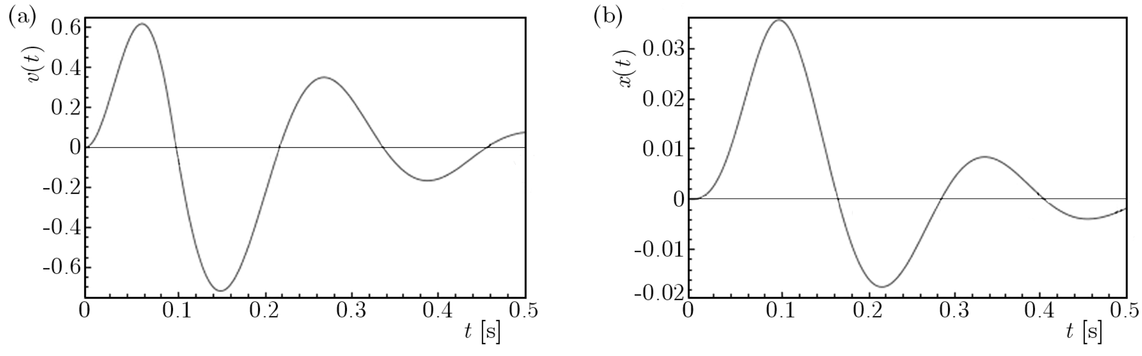

Fig. 5. Responses to the pulse loads for model 1a, (a) velocity $v = v(t)$, (b) displacement $x = x(t)$

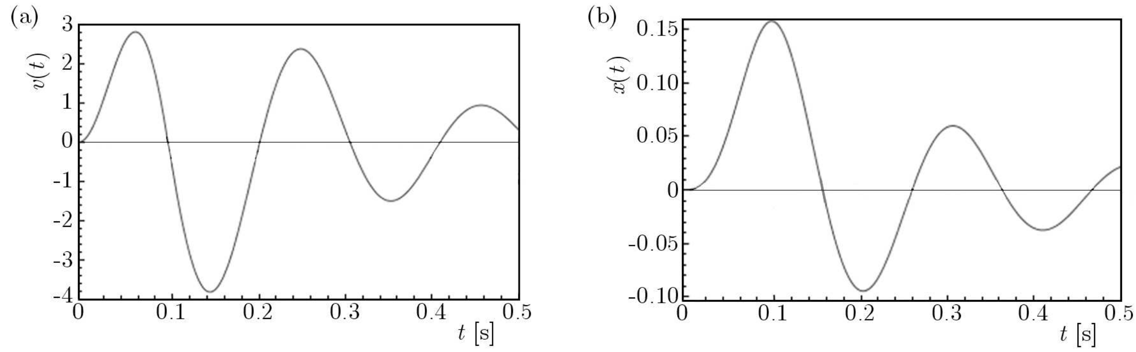

Fig. 6. Responses to the pulse loads for model 1b, (a) velocity $v = v(t)$, (b) displacement $x = x(t)$

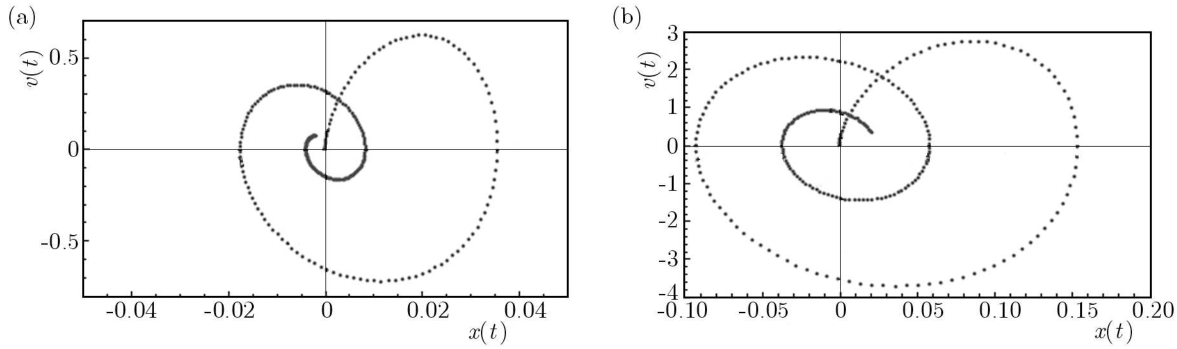

Fig. 7. Trajectories of the phase analysis for simulation in the time interval Δt: (a) for model 1a, (b) for model 1b

The loop fields have been determined on the basis of the results obtained during simulations. Then, the values of coefficients from the identification equation (Table 1) were generated using the linear regression. As it can be seen, these equations are generally satisfied, although in one case of model (b) the estimated value of the coefficient contains a big mistake. In other cases, the mistakes were no greater than 10%.

Table 1. The parameters assumed and derived from the linear regression

Model (a)		Model (b)	
Assumed	Derived	Assumed	Derived
$\frac{2k}{c} = 3.20 \cdot 10^{-2}$	$\frac{2k}{c} = 3.18 \cdot 10^{-2}$	$\frac{2k_0 m}{c_0 c} = 5.40 \cdot 10^{-5}$	$\frac{2k_0 m}{c_0 c} = 5.19 \cdot 10^{-5}$
–	–	$\frac{2k_0(c_0 + c)}{c_0 c} = 6.77 \cdot 10^{-4}$	$\frac{2k_0(c_0 + c)}{c_0 c} = 6.0 \cdot 10^{-4}$
$\frac{2h}{c} = 3.333 \cdot 10^{-4}$	$\frac{2h}{c} = 3.328 \cdot 10^{-4}$	$\frac{2h}{c} = 3.333 \cdot 10^{-4}$	$\frac{2h}{c} = 3.313 \cdot 10^{-4}$

6. Summary

The presented fragment of the research work concerns the analysis of the energy consumption of dynamic rheological models in the process of ballistic impact. Two dynamic models have been adopted for this analysis. The first model describes vibrations of mechanical systems with one degree of freedom, and the second one describes vibrations of mechanical systems with one and a half degree of freedom. This is a model from a group of the degenerate models. The analysis of the models have been conducted based on the mathematical relations which determine the dissipation of the impact energy.

The models have also been subjected to computer simulation in order to verify the theoretical assumptions. The simulations resulted in obtaining time responses for the pulse force $p(t)$. It has been assumed that at the time points t_1 and t_2, the displacement $x(t)$ should reach the extreme values for the points of time in which the velocity $v(t)$ must have the value of zero. The simulation has confirmed the expected objectives. Indeed, for the velocity $v(t)$ (Fig. 5), at moments its reaches zero, the displacement in these moments $x(t)$ reaches its maximum (Fig. 6). It can be also noted that in the case of the degenerate model, the displacement $x(t)$ takes much lower values than in model 1a. These differences occur because the responses of this model are suppressed more effectively. This also shows that the process of energy consumption is described more accurately by the degenerate model.

The obtained shapes of the phase trajectory of the analyzed systems in the time interval Δt shows also some significant differences. Drawing conclusions requires still some additional simulations at this stage. The presented diagrams provide rather the qualitative description of these phenomena. To reach a quantitative description, a number of research studies still have to be carried out, and their results will be presented in the future papers.

To sum up, the hypothesis assumed by the authors that the degenerate models can quite accurately describe the mechanical properties of modern structural materials has been confirmed and justifies the direction of the undertaken work.

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