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On discounted LQR control problem for disturbanced singular system

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The LQR (linear quadratic regulator) control problem subject to singular system constitutes a optimization problem in which one must be find an optimal control that satisfy the singular system and simultaneously to optimize the quadratic objective functional. In this paper we establish a sufficient condition to obtain the optimal control of discounted LQR optimization problem subject to disturbanced singular system where the disturbance is time varying. The considered problem is solved by transforming the discounted LQR control problem subject to disturbanced singular system into the normal LQR control problem. Some available results in literatures of the normal LQR control problem be used to find the sufficient conditions for the existence of the optimal control for discounted LQR control problem subject to disturbanced singular system. The final result of this paper is in the form a method to find the optimal control of discounted LQR optimization problem subject to disturbanced singular system. The result shows that the disturbance is vanish with the passage of time.

Key words: discounted LQR control, singular system, disturbance

1. Introduction

In some literatures, e.g. Cobb [1] and Duan [2], the LQR control problem subject to time invariant singular system is formulated as follows:

$$\min_{\mathbf{u}} \Im(\mathbf{u}) = \int_0^\infty \left(\mathbf{y}^\top \mathbf{y} + \mathbf{u}^\top R\mathbf{u}\right) \mathrm{d}t, \tag{1}$$

$$\begin{aligned} \text{s.t.}\ \ (L\triangle - A)\,\mathbf{x} = B\mathbf{u},\ \ \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y} = C\mathbf{x}, \end{aligned} \tag{2}$$

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where $R_{\mathrm{m\times m}}$ is a definite positive matrix, $L_{\mathrm{n\times n}}$, $A_{\mathrm{n\times n}}$, $B_{\mathrm{n\times m}}$, $C_{\mathrm{r\times n}}$, $\mathbf{x} = \mathbf{x}(t)$ is state vector, $\mathbf{u} = \mathbf{u}(t)$ is control vector, $\mathbf{y} = \mathbf{y}(t)$ is output vector, $\triangle = \dfrac{\mathrm{d}}{\mathrm{d}t}$ and rank $(L) = \mathrm{p} < \mathrm{n}$. If $\mathrm{p} = \mathrm{n}$ then the system (2) constitutes a normal system. The couple equation (2) is known as the time invariant singular system. Different to the normal differential equation that always have a solution, the system (2) may be not have a solution for some situation. The system (2) has a unique solution if for some admissible initial state $\mathbf{x}_0$ there exists a complex number λ such that $\det(\lambda L - A) \neq 0$. Note that an initial state $\mathbf{x}_0$ is said admissible for the system (2) if the system (2) has at least one solution satisfying the initial conditon $\mathbf{x}(0) = \mathbf{x}_0$ [3]. In case in which $\det(\lambda L - A) \neq 0$ for some complex number λ it is called regular.

In general, the problem to be solved in the LQR control problem subject to singular system is to find the optimal control $\mathbf{u}$ satisfying (2) and to minimize the quadratic objective functional (1).

Along with development of the singular system (2) in various fields, formulation of the optimization problem (1) and (2) has also been expanded. Chen [4] inserted the disturbance vector in system (2) such that the system (2) become

$$\begin{aligned} (L\triangle - A)\mathbf{x} &= B\mathbf{u} + E\boldsymbol{\omega}, \qquad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y} &= C\mathbf{x} + F\boldsymbol{\omega}, \end{aligned} \tag{3}$$

where $E_{\mathrm{n\times q}}$, $F_{\mathrm{r\times q}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ is disturbance vector.

In many applications always come about disparity between output of the process (factual output) and the desired output. Zulakmal et al. [5] modified the objective functional (1) by including the disparity of output term and assumed that the disturbance $\boldsymbol{\omega}$ is a constant vector.

In this paper, the problem in Zulakmal et al. [5] is expanded by inserting the discount factor β with $0 < \beta < 1$ into the objective functional such that the optimization problem becomes

$$\min_{\mathbf{u}} \Im(\mathbf{u}) = \int_0^\infty e^{-\beta t} \left(\boldsymbol{\varepsilon}^\top S \boldsymbol{\varepsilon} + \triangle\mathbf{u}^\top R \triangle\mathbf{u}\right) \mathrm{d}t, \tag{4}$$

$$\text{s.t. } \begin{aligned} (L\triangle - A)\ \mathbf{x} &= B\mathbf{u} + E\boldsymbol{\omega}, \qquad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{y} &= C\mathbf{x} + F\boldsymbol{\omega}, \end{aligned} \tag{5}$$

where $\boldsymbol{\varepsilon} = \mathbf{y} - \mathbf{y}_\mathrm{d}$ denotes disparity of output with $\mathbf{y}_\mathrm{d}$ is the desired output. The objective functional of this kind always occur in economic field, see Benigno et al. [6]. Different to the assumption in Zulakmal et al. [5], it is assumed here that the disturbance $\boldsymbol{\omega}$ is time varying, that is it satisfies the following differential equation:

$$\triangle\boldsymbol{\omega} = G\boldsymbol{\omega}, \tag{6}$$

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where $G_{q\times q}$. Moreover, it is also assumed that (5) is regular. The addressed problem in this paper is to construct a sufficient condition for existence of the optimal control $\mathbf{u}$ satisfying (5) such that the objective functional (4) is minimized and $\boldsymbol{\omega} \to \mathbf{0}$ when $t \to 0$. Let say the optimal control as $\mathbf{u}_{\mathrm{opt}}$.

2. Results

In this section, some sufficient condition for existence of the optimal control $\mathbf{u}_{\mathrm{opt}}$ of the addressed problem were constructed. By defining a new variable $\mathbf{z}$ such that $\triangle\mathbf{z} = \boldsymbol{\varepsilon}$, the system (5) can be written as:

$$\left(\mathrm{diag}(L, I_{\mathrm{r}}, I_{\mathrm{q}})\begin{bmatrix}\triangle\mathbf{x}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\omega}\end{bmatrix} - \begin{bmatrix}A & O & E\\ C & O & F\\ O & O & I_{\mathrm{q}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\\ \boldsymbol{\omega}\end{bmatrix}\right) = \begin{bmatrix}B\\ O\\ O\end{bmatrix}\mathbf{u} - \begin{bmatrix}O\\ I_{\mathrm{r}}\\ O\end{bmatrix}\mathbf{y}_{\mathrm{d}}, \tag{7}$$

where O denotes zero matrix of suitable size. Since the desire output $\mathbf{y}_{\mathrm{d}}$ are constant vectors, differentiation for t of (7) yields

$$\left(\overline{L}\triangle - \overline{A}\right)\begin{bmatrix}\triangle\mathbf{x}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\omega}\end{bmatrix} = \overline{B}\triangle\mathbf{u},$$

where

$$\overline{L} = \mathrm{diag}(L, I_{\mathrm{r}}, I_{\mathrm{q}}), \qquad \overline{A} = \begin{bmatrix}A & O & E\\ C & O & F\\ O & O & G\end{bmatrix}, \qquad \overline{B} = \begin{bmatrix}B\\ O\\ O\end{bmatrix}.$$

By defining

$$\boldsymbol{\Pi} = e^{-\frac{1}{2}\beta t}\triangle\mathbf{u} \qquad \text{and} \qquad \boldsymbol{\xi} = e^{-\frac{1}{2}\beta t}\begin{bmatrix}\triangle\mathbf{x}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\omega}\end{bmatrix}, \tag{8}$$

the problem under consideration is equivalent to the following optimization problem:

$$\min_{\Pi} \mathfrak{I}_{\varepsilon}(\Pi) = \int_0^{\infty}\left(\boldsymbol{\xi}^{\top}Q\boldsymbol{\xi} + \Pi^{\top}R\Pi\right)\mathrm{d}t, \tag{9}$$

$$\text{s.t.}\left(\overline{L}\triangle - \overline{A} + \frac{1}{2}\beta\overline{L}\right)\boldsymbol{\xi} = \overline{B}\Pi, \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \tag{10}$$

where

$$Q = \begin{bmatrix}O & O & O\\ O & S & O\\ O & O & O\end{bmatrix}.$$

One can observe that $\text{rank}(\overline{L}) = \text{p}+\text{r}+\text{q} < \text{n}+\text{r}+\text{q}$. Regularity assumption of system (5) implies regularity of system (10), i.e. $\det\left(\overline{\lambda L} - \overline{A} + \frac{1}{2}\beta\overline{L}\right) \neq 0$ for some complex number $\overline{\lambda}$. Note that (9) and (10) constitute a LQR control problem subject to singular system without disturbance with control Π and state $\boldsymbol{\xi}$.

In Duan [2] is mentioned that the optimal control for problem (9) and (10) exists if and only if the system (9) is impulse controllable and stabilizable. A review of the impulse controllable concept is referred to Ishihara [7] and Yan [8], and the stabilizing concept is referred to Duan [2]. Using the results in [7] and Duan [2], the system (10) is impulse controllable if

$$\text{rank}\begin{bmatrix} \overline{L} & O & O \\ \overline{A} - \frac{1}{2}\beta\overline{L} & \overline{L} & \overline{B} \end{bmatrix} = (\text{n}+\text{r}+\text{q}) + \text{rank}(\overline{L}),$$

and it is stabilizable if $\text{rank}\left[\overline{\lambda L} - (\overline{A} - \frac{1}{2}\beta\overline{L}) \ \ \overline{B}\right] = \text{n}+\text{r}+\text{q}$ for every complex number $\overline{\lambda}$ with $\text{Re}\left(\overline{\lambda}\right) \geqslant 0$.

Since the system (10) is associated to the original system (5), it is needed a condition in order to the system (10) is impulse controllable and stabilizable in term the matrices L, A and B.

Theorem 1 *If*

$$\text{rank}\begin{bmatrix} L & O & O \\ A & L & B \end{bmatrix} = \text{n} + \text{rank}(L) \tag{11}$$

then the system (10) *is impulse controllable.*

Proof. Under the assumption (11) we have

$$\text{rank}\begin{bmatrix} \overline{L} & O & O \\ \overline{A} - \frac{1}{2}\beta\overline{L} & \overline{L} & \overline{B} \end{bmatrix} = \text{rank}\begin{bmatrix} L & O & O & O & O & O & O \\ O & I_\text{r} & O & O & O & O & O \\ O & O & I_\text{q} & O & O & O & O \\ A - \frac{1}{2}\beta L & O & O & L & O & O & B \\ C & -\frac{1}{2}\beta I_\text{r} & F & O & I_r & O & O \\ O & O & G - \frac{1}{2}\beta I_\text{q} & O & O & I_\text{q} & O \end{bmatrix}$$

$$
= \operatorname{rank}\begin{bmatrix} L & O & O & O & O & O & O \\ O & O & O & I_r & O & O & O \\ O & O & O & O & O & O & I_q \\ A-\frac{1}{2}\beta L & B & O & L & O & O & O \\ C & O & O & -\frac{1}{2}\beta I_r & I_r & O & F \\ O & O & O & O & O & I_q & G-\frac{1}{2}\beta I_q \end{bmatrix}
$$

$$
= \operatorname{rank}\begin{bmatrix} L & O & O & O & O & O & O \\ O & O & O & I_r & O & O & O \\ O & O & O & O & O & O & I_q \\ A-\frac{1}{2}\beta L & B & O & O & O & O & O \\ C & O & O & O & I_r & O & O \\ O & O & O & O & O & I_q & O \end{bmatrix}
$$

$$
= \operatorname{rank}\begin{bmatrix} L & O & O \\ O & O & O \\ A-\frac{1}{2}\beta L & L & B \end{bmatrix} + 2r + 2q
$$

$$
= \operatorname{rank}\begin{bmatrix} L & O & O \\ A & L & B \end{bmatrix} + 2r + 2q
$$

$$
= n + \operatorname{rank}(L) + 2r + 2q
$$

$$
= n + p + 2r + 2q
$$

$$
= (n + r + q) + (p + r + q)
$$

$$
= (n + r + q) + \operatorname{rank}(\overline{L}).
$$

This proves that the system (10) is impulse controllable. □

Theorem 2 *If*

$$
\operatorname{rank}\,[\lambda L - A \quad B] = n \tag{12}
$$

for every complex number λ *with* $\operatorname{Re}(s) \geqslant 0$ *and* $\lambda = \overline{\lambda} + \frac{1}{2}\beta$, *then the system* (10) *is stabilizable.*

Proof. Under the assumption (12) and $\lambda = \overline{\lambda} + \frac{1}{2}\beta$, we have

$$\operatorname{rank}\begin{bmatrix} \overline{\lambda L} - \overline{A} + \frac{1}{2}\beta\overline{L} & \overline{B} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \left(\overline{\lambda}+\frac{1}{2}\beta\right)L - A & O & O & B \\ -C & \left(\overline{\lambda}+\frac{1}{2}\beta\right)I_r & -F & O \\ O & O & \left(\overline{\lambda}+\frac{1}{2}\beta\right)I_q - G & O \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \left(\overline{\lambda}+\frac{1}{2}\beta\right)L - A & B & O & O \\ -C & O & -F & \left(\overline{\lambda}+\frac{1}{2}\beta\right)I_r \\ O & O & \left(\overline{\lambda}+\frac{1}{2}\beta\right)I_q - G & O \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \lambda L - A & B \end{bmatrix} + \mathrm{r} + \mathrm{q}$$

$$= \mathrm{n} + \mathrm{r} + \mathrm{q}.$$

This proves that the system (10) is stabilizable. □

Impulse controllability of (10) implies that there exists a matrix K of size $\mathrm{m}\times(\mathrm{n}+\mathrm{q}+\mathrm{r})$ such that

$$\det\left(\overline{\lambda L} - \left(\overline{A} - \frac{1}{2}\beta\overline{L} + \overline{B}K\right)\right) = \operatorname{rank}\left(\overline{L}\right). \tag{13}$$

By choosing a feedback control $\Pi = K\boldsymbol{\xi} + \boldsymbol{\eta}$ for some new control vector $\boldsymbol{\eta}$ and apply it to system (10), one get

$$\left(\overline{L}\triangle - \left(\overline{A} - \frac{1}{2}\beta\overline{L} + \overline{B}K\right)\right)\boldsymbol{\xi} = \overline{B}\boldsymbol{\eta}, \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0. \tag{14}$$

Thus by using the Singular Value Decomposition [9], one can get some nonsingular matrices Q_1 and P_1 of size $(\mathrm{n}+\mathrm{q}+\mathrm{r})$ by $(\mathrm{n}+\mathrm{q}+\mathrm{r})$, respectively, such that

$$Q_1\overline{L}P_1 = \operatorname{diag}\left(I_{\mathrm{p+q+r}}, O\right),$$

$$Q_1(\overline{A} - \frac{1}{2}\beta\overline{L} + \overline{B}K)P_1 = \operatorname{diag}(\overline{A}_1, I_{\mathrm{n-p}}),$$

$$Q_1\overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix},$$

where $\overline{A}_1$ of size (p+q+r) by (p+q+r), $\overline{B}_1$ of size (p+q+r) by m and $\overline{B}_2$ of size (n−p) by m, see [9]. By defining

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = P_1^{-1}\xi, \tag{15}$$

where ξ_1 is a (p+q+r) vector and ξ_2 is a (n−p) vector, the system (14) is equivalent to

$$(\triangle - \overline{A}_1)\xi_1 = \overline{B}_1\eta, \qquad \xi_1(0) = \xi_{10}, \tag{16}$$

$$\xi_2 = -\overline{B}_2\eta, \qquad \xi_2(0) = \xi_{20}. \tag{17}$$

Using (17), the objective functional (9) becomes

$$\Im_\varepsilon(\eta) = \int_0^\infty \left(\begin{bmatrix} \xi_1 \\ \eta \end{bmatrix}^\top \begin{bmatrix} \widehat{Q} & H \\ H^\top & \widetilde{R} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \eta \end{bmatrix} \right) \mathrm{d}t \tag{18}$$

where

$$\begin{bmatrix} \widehat{Q} & H \\ H^\top & \widetilde{R} \end{bmatrix} = \begin{bmatrix} I & O \\ O & -\overline{B}_2 \\ O & I \end{bmatrix}^\top \Gamma \begin{bmatrix} I & O \\ O & -\overline{B}_2 \\ O & I \end{bmatrix}, \tag{19}$$

$$\Gamma = \begin{bmatrix} P_1 & O \\ KP_1 & I \end{bmatrix}^\top \begin{bmatrix} Q & O \\ O & R \end{bmatrix} \begin{bmatrix} P_1 & O \\ KP_1 & I \end{bmatrix} \tag{20}$$

By using the substitution

$$\eta = \mathbf{w} - \widetilde{R}^{-1}H^\top\xi_1, \tag{21}$$

such that

$$\mathbf{w} = \mathbf{\Pi} - K\xi + \widetilde{R}_1^{-1}H^\top\xi_1, \tag{22}$$

the optimization problem (9) and (10) be equivalent to the following normal LQR control problem:

$$\min_{\mathbf{w}} \Im_\varepsilon(\mathbf{w}) = \int_0^\infty \left(\xi_1^\top \widetilde{Q}\xi_1 + \mathbf{w}^\top \widetilde{R}\mathbf{w} \right) \mathrm{d}t, \tag{23}$$

$$\text{s.t.} \ \left(\triangle - \widetilde{A}\right)\xi_1 = \overline{B}_1\mathbf{w}, \qquad \xi_1(0) = \xi_{10}, \tag{24}$$

where

$$\widetilde{Q} = \widehat{Q} - H\widetilde{R}^{-1}H^\top, \tag{25}$$

and

$$\widetilde{A}_1 = \widetilde{A}_1 - \overline{B}_1 \widetilde{R}^{-1} H^\top . \tag{26}$$

It is obvious that (23)–(26) constitute the normal LQR control problem with control $\mathbf{w}$ and state $\boldsymbol{\xi}_1$. We can solve it using the theory of normal LQR control problem that mention the optimal control for (23) and (24) exists and unique if the system (24) is stabilizable [10].

Note that the Theorem 2 implies the system (16) is stabilizable. Since (21) is the closed-loop system resulted in by applying a state feedback with gain matrix $-\widetilde{R}^{-1} H^\top$ to the system (16) and the fact that the state feedback does not change stabilizability, we get the system (24) is stabilizable.

It follows that solution of the LQR control problem (23) and (24) be given by

$$\mathbf{w} = -\widetilde{R}^{-1} \overline{B}_1^\top P \boldsymbol{\xi}_1 , \tag{27}$$

where $\boldsymbol{\xi}_1$ satisfies the following initial value problem

$$\left(\triangle - \overline{A}_1 + \overline{B}_1 \widetilde{R}^{-1} \left(\overline{B}_1^\top + H^\top \right) \right) \boldsymbol{\xi}_1 = \mathbf{0}, \qquad \boldsymbol{\xi}_1(0) = \boldsymbol{\xi}_{10}, \tag{28}$$

and P is the unique symmetric positive definite solution of the algebraic Riccati equation

$$\left(\overline{A}_1 - \overline{B}_1 \widetilde{R}^{-1} H^\top \right)^\top P + P \left(\overline{A}_1 - \overline{B}_1 \widetilde{R}^{-1} H^\top \right) - P \overline{B}_1 \widetilde{R}^{-1} \overline{B}_1^\top P + \widehat{Q} - H \widetilde{R}^{-1} H^\top = O \tag{29}$$

with the minimum value is $\boldsymbol{\xi}_{10}^\top P \boldsymbol{\xi}_{10}$. Moreover, the solution of initial value problem (28) is stable, i.e. $\boldsymbol{\xi}_1(t) \to 0$ if $t \to \infty$. Using (21) it is obtained that

$$\boldsymbol{\eta} = -\widetilde{R}^{-1} \left(\overline{B}_1^\top P + H^\top \right) \boldsymbol{\xi}_1 \tag{30}$$

and using (8), (22) and (15) it is obtain

$$\triangle \mathbf{u} = e^{\frac{1}{2}\beta t} \left(K P_1 \begin{bmatrix} I_{\mathrm{p+q+r}} \\ \overline{B}_2 \widetilde{R}^{-1} \left(\overline{B}_1^\top P + H^\top \right) \end{bmatrix} - \widetilde{R}^{-1} \left(\overline{B}_1^\top P - H^\top \right) \right) \boldsymbol{\xi}_1 . \tag{31}$$

Moreover, from (8) and (15) one get

$$\begin{bmatrix} \triangle \mathbf{x} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\omega} \end{bmatrix} = e^{\frac{1}{2}\beta t} P_1 \begin{bmatrix} I_{\mathrm{p+q+r}} \\ \overline{B}_2 \widetilde{R}^{-1} \left(\overline{B}_1^\top P + H^\top \right) \end{bmatrix} \boldsymbol{\xi}_1 . \tag{32}$$

Therefore, the optimal control $\mathbf{u}_{\mathrm{opt}}$ of the addressed problem exists and unique, and satisfy (31). Moreover, we also obtain $\boldsymbol{\varepsilon} \to \mathbf{0}$ when $t \to \infty$ due to $\boldsymbol{\xi}_1(t) \to 0$

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when $t \to \infty$. One can see that the disturbance is vanish with the passage of time. Thus we have proved the following Theorem that constitutes the sufficient condition for existence of optimal control.

Theorem 3 *The optimal control* $\mathbf{u}_{\text{opt}}$ *for the LQR control problem (4) and (5) exists and unique if*

1. $\text{rank}\begin{bmatrix} L & O & O \\ A & L & B \end{bmatrix} = \text{n+rank}(L)$

2. $\text{rank}\,[\lambda L - A \quad B] = \text{n},$

   for every complex number λ *with* $\text{Re}(s) > 0$ *and* $\lambda = \overline{\lambda} + \frac{1}{2}\beta$, *where* $\mathbf{u}_{\text{opt}}$ *satisfies the equation (31). Moreover,* $\mathbf{y} \to \mathbf{y}_{\text{d}}$ *when* $t \to \infty$.

One can see that the process of constructing the Theorem 3 constitutes a method to find the optimal control $\mathbf{u}_{\text{opt}}$ for the LQR control problem (4) and (5). One can also see that the varying disturbance in this paper is vanishing with the passage of time. This result is different from the result in Zulakmal et al. [5], in which the constant disturbance does not effect the system.

3. Conclusion

A sufficient for existence of the optimal control for the discounted LQR control problem subject to disturbanced singular system has been established. The result shows that the disturbance is vanish with the passage of time.

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